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LYAPUNOV EXPONENTS AS A NONPARAMETRIC DIAGNOSTIC FOR STABILITY ANALYSIS

W. D. DECHERT

Department of Economics, University of Houston, 4800 Calhoun, Houston, TX 77204-5882, USA

AND

R. GENCAY

Department of Economics, University of Windsor, 401 Sunset, Windsor, Ontario N9B 3P4, Canada

SUMMARY

The common observation made in the empirical nonlinear dynamics literature is the constraints imposed by the availability of a limited number of observations in the implementation of the existing algorithms of Lyapunov exponents. The algorithm discussed here can estimate all n Lyapunov exponents of an unknown n-dimensional dynamical system accurately with limited number of observations. This makes the algorithm attractive for applications to economic as well as financial time-series data. The implementation of the algorithm is carried out by multilayer feedforward networks which are capable of approximating any function and its derivatives to any degree of accuracy.

1. INTRODUCTION

Lyapunov exponents measure the rate of divergence or convergence of two nearby initial points of a dynamical system. They are used in the determination of the local stability properties of nonlinear and linear systems. One property of Lyapunov exponents is their multi-dimensional nature. They provide information not only on the local stability properties of the data-generating process of the observations but also on its dimensionality. This is useful information for structural model building. The other property of Lyapunov exponents is their nonparametric nature. They do not require imposition of a parametric specification on the data and hence can be used as a complementary diagnostic to parametric stability techniques.

The common observation made in the empirical nonlinear dynamics literature is the constraints imposed by the availability of a limited number of observations in the implementation of the existing algorithms of Lyapunov exponents. Brock and Sayers (1988) found that the Wolf *et al.* (1985) algorithm of Lyapunov exponents may not provide a correct characterization of Lyapunov exponents of a time series with limited number of observations which an applied econometrician normally faces; also the performance of this algorithm is very sensitive to the degree of noise in the data. Frank and Stengos (1988) indicated that the algorithm of Lyapunov exponents due to Kurths and Herzel (1987) had a positive bias and Frank *et al.* (1988) reported the presence of positive bias in various GNP series.

In this paper we will discuss the algorithm of Lyapunov exponents proposed by Dechert and Gencay (1990) with an application to the spot exchange rate series. When a dynamical system is n-dimensional and the observations are embedded in an m-dimensional space such that

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$m \geqslant 2n+1$, then, by a theorem of Takens (1981), the observations can be used to reconstruct the local dynamics on the attractor of the dynamical system. Dechert and Gencay (1990) show that the n largest Lyapunov exponents of the reconstructed dynamics are the Lyapunov exponents of the unknown system. In simulation studies we show that the algorithm performs satisfactorily with limited number of observations. This property of the algorithm makes it attractive as compared to the other recent algorithms such as those of Brown *et al.* (1991), which require a large number of observations for correct approximations. Our results are robust to the presence of system as well as measurement noise.

The implementation of the algorithm is carried out by multilayer feedforward networks. Subject to mild regularity conditions, Hornik *et al.* (1990) have shown that these networks can approximate any function and its derivatives to any degree of accuracy. They have the capability to approximate a function and its derivatives with as few as a hundred observations.

The organization of this paper is in the following manner. In Section 2 we will define the Lyapunov exponents and describe the algorithm. Multilayer feedforward networks will be explained in Section 3. Simulation experiments as well as applications to the spot exchange rate data are the content of Section 4. We will conclude thereafter.

2. LYAPUNOV EXPONENTS

The Lyapunov exponents for a dynamical system, $f: R^n \to R^n$, with the trajectory

$$x_{t+1} = f(x_t) \qquad t = 0, 1, 2, \ldots \tag{1}$$

are measures of the average rate of divergence or convergence of a typical trajectory.[1] For an n-dimensional system as above, there are n exponents which are customarily ranked from largest to smallest:

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \tag{2}$$

Associated with each exponent, $j = 1, 2, \ldots, n$, there are nested subspaces $V^j \subset R^n$ of dimension $n+1-j$ and with the property that

$$\lambda_j = \lim_{t\to\infty} t^{-1} \ln \| (Df^t)_{x_0} v \| \tag{3}$$

for all $v \in V^j \backslash V^{j+1}$. (See Guckenheimer and Holmes, 1983, p. 256, for additional properties of Lyapunov exponents and for a formal definition). Note that for $j \geqslant 2$ the subspaces V^j are sets of Lebesgue measure zero, and so for almost all $v \in R^n$ the limit in equation (3) equals λ_1. This is the basis for the computational algorithm of Wolf *et al.* (1985), which is a method for calculating the largest Lyapunov exponent.

Equation (3) suggests a more direct approach to calculating the exponents. Since

$$(Df^t)_{x_0} = (Df)_{x_t}(Df)_{x_{t-1}} \cdots (Df)_{x_0} \tag{4}$$

all the Lyapunov exponents can be calculated by evaluating the Jacobian of the function f along a trajectory, $\{v_t\}$. Eckmann and Ruelle (1985) propose the use of the QR algorithm for extracting the eigenvalues from $(Df^t)_{x_0}$, which is the method that we use here.

An attractor is a set of points towards which the trajectories of f converge. More precisely, Λ is an attractor if there is an open $U \subset R^n$ with

$$\Lambda = \bigcap_{t \geqslant 0} f^t(\bar{U}) \tag{5}$$

[1] The trajectory is also written in terms of the iterates of f. With the convention that f^0 is the identity map, and $f^{t+1} = f \circ f^t$, then we also write, $x_t = f^t(x_0)$. A trajectory is also called an orbit in the dynamical system literature.

Table I. Lyapunov exponents of n-dimensional non-chaotic attractors

	Lyapunov exponent
Stable equilibrium point	$\lambda_i < 0$ for $i = 1, 2, \ldots, n$
Limit cycle	$\lambda_1 = 0$, $\lambda_i < 0$ for $i = 2, 3, \ldots, n$
Two-torus	$\lambda_1 = \lambda_2 = 0$ and $\lambda_i < 0$ for $i = 3, 4, \ldots, n$
K-torus	$\lambda_2 = \lambda_2 = \cdots = \lambda_k = 0$ and $\lambda_i < 0$ for $i = k + 1, \ldots, n$

where $\bar{U}$ is the closure of U. The attractor Λ is said to be indecomposable if there is no proper subset of Λ which is also an attractor. It is a consequence of Oseledec's Theorem (1968) that the limit in equation (3) exists for a broad class of functions.[2]

An attractor can be chaotic or ordinary (which we will refer to as non-chaotic). There is more than one definition of a chaotic attractor in the literature. In practice, the presence of a positive Lyapunov exponent is taken as a signal that the attractor is chaotic. The Lyapunov exponents of non-chaotic attractors are classified in Table I. Note that no Lyapunov exponent of a non-chaotic attractor is positive.

In practice, one rarely has the advantage of observing the state of the system, x_t, let alone knowing the actual functional form, f, that generate the dynamics. The model that is widely used is the following: associated with the dynamical system in equation (1) there is an observer function $h: R^n \to R$ which generates observations

$$y_t = h(x_t). \tag{6}$$

It is assumed that all that is available to the researcher is the sequence $\{y_t\}$. Under general conditions, Takens (1981) has shown that if the set $\bar{U}$ is compact manifold then for $m \geqslant 2n + 1$

$$J^m(x) = (h(x), h(f(x)), \ldots, h(f^{m-1}(x))) \tag{7}$$

is generically an embedding. For a function $g: R^m \to R^m$ for which $J^m \circ f = g \circ J^m$ on an indecomposable attractor, Dechert and Gencay (1990) show that n largest Lyapunov exponents of g are the Lyapunov exponents of f. This is the basis of our approach: estimate the function g based on the data sequence $\{J^m(x_t)\}$ and calculate the Lyapunov exponents of g.[3] As m increases there is a value between n and $2n + 1$ at which the n largest exponents remain constant as m increases and the remaining $m - n$ exponents diverge to $-\infty$ as the number of observations increases.

The mapping, g, which is to be estimated may be taken to be[4]

$$g: \begin{bmatrix} y_{t+m-1} \\ y_{t+m-2} \\ \vdots \\ y_t \end{bmatrix} \to \begin{bmatrix} v(y_{t+m-1}, y_{t+m-2}, \ldots, y_t) \\ y_{t+m-1} \\ \vdots \\ y_{t+1} \end{bmatrix} \tag{8}$$

and this reduces to estimating

$$y_{t+m} = v(y_{t+m-1}, y_{t+m-2}, \ldots, y_t). \tag{9}$$

[2] See Raghunathan (1979), Ruelle (1979) and Cohen *et al.* (1986) for precise conditions and the proofs of the theorem.
[3] Dechert and Gencay (1990) work was done independently from the works of McCaffrey *et al.* (1991) and Nychka *et al.* (1992). The departure for the computational methodology used in Dechert and Gencay (1990) and the ones used in McCaffrey *et al.* (1991) and Nychka *et al.* (1992) are basically the same, except that they calculate only the largest Lyapunov exponent, whereas we show that it is possible to calculate all Lyapunov exponents of an unknown dynamical system. The theoretical results for our approach are given in detail in Dechert and Gencay (1990).
[4] All we can hope to calculate is a generic function g.

3. MULTILAYER FEEDFORWARD NETWORKS

In this paper we use a single layer feedforward network,

$$v_{N,m}(z;\beta,w,b)=\sum_{j=1}^{N}\beta_j\phi\left(\sum_{i=1}^{m}w_{ij}z_i+b_j\right) \tag{10}$$

where $z\in R^m$ is the input, the parameters to be estimated are β, w, b, and ϕ is a known hidden unit activation function. $\beta\in R^N$ represents hidden to output unit weights, $w\in R^{N\times m}$ and $b\in R^N$ represent input to hidden unit weights. N is the number of hidden units.

Hornik *et al.* (1990) have given conditions under which the set of single-layer feedforward networks are dense in Sobolev space of functions. The important part of their result of which we make use is that both a function and its derivatives can be approximated to any arbitrary degree of accuracy with a single-layer feedforward network. Gallant and White (1992) have shown that feedforward networks can be used to consistently estimate a function and its derivatives. They show that the least squares estimates are strongly consistent in Sobolev norm, provided that the number of hidden units increases with the size of the data set.

For notational purposes, let

$$y_t^m=(y_{t+m-1},y_{t+m-2},\ldots,y_t). \tag{11}$$

For a single-layer network the least squares criterion for a data set of length T is:

$$L(\beta,w,b)=\sum_{t=0}^{T-m-1}[y_{t+m}-v_{N,m}(y_t^m;\beta,w,b)]^2. \tag{12}$$

This is a straightforward multivariate minimization problem. In our work we used the sigmoid function

$$\phi(u)=\frac{1}{1+\exp(-u)} \tag{13}$$

as the hidden-layer activation function. $v_{N,m}(z;\beta,w,b)$ has many local minima and the fitting procedure must pass these to a global minimum. We used the conjugant gradient routines in Press *et al.* (1986) and chose the starting values from a uniform pseudo-random number generator. We found that this procedure worked very well for this problem.

4. ILLUSTRATIVE EXAMPLES

In this section we present some results on two examples. One is the attractor of the Henon map,

$$x_{t+1}=1-1\cdot 4x_t^2+y_t \qquad y_{t+1}=0\cdot 3x_t \tag{14}$$

which is a widely used example. We use it in part as a benchmark to test our methods against the results of others. The matrix of derivatives of the Henon map is

$$\begin{bmatrix}-2\cdot 4x_t & 1\\ 0\cdot 3 & 0\end{bmatrix} \tag{15}$$

Since the determinant of this matrix is constant, the Lyapunov exponents for this map satisfy

$$\lambda_1+\lambda_2=\ln(0\cdot 3)\approx -1\cdot 2. \tag{16}$$

We estimated the Henon map and the first column of the matrix of partial derivatives based on 200 observations. In the estimation of this map, six hidden-layer activation functions are used in a single-layer feedforward network. We have compared the quality of the fit with seven and eight hidden-unit activation functions as well, but did not observe any improvement in the quality of the fit.[5] The results are given in Figures 1–3, along with the mean squared error of

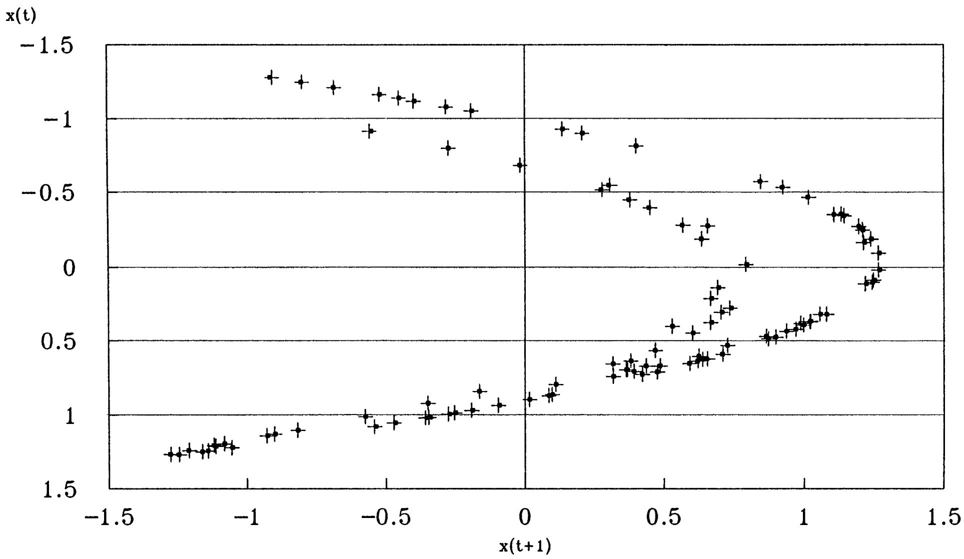

Figure 1. Henon map. · actual; + estimated (mean square error = 0·000063)

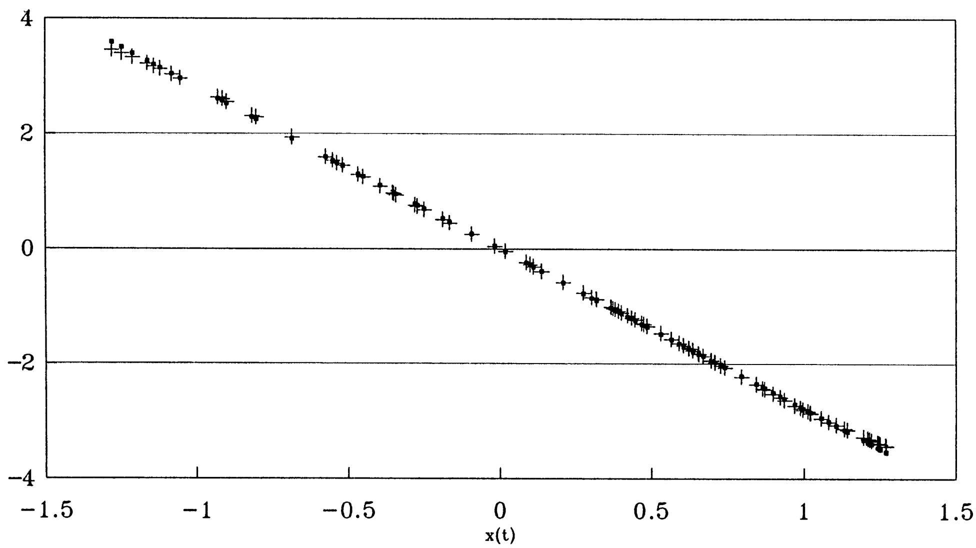

Figure 2. Henon map: derivative of $x(t+1)$ w.r.t $x(t)$. · actual; + estimated (mean square error = 0·004564)

[5] Gallant and White (1992) pointed out that the number of activation functions should grow with the size of the data set at just the right rate to ensure good approximation without overfitting. In simulation experiments, they also reported that the quality of the fit was not very sensitive in slight variations to this rate.

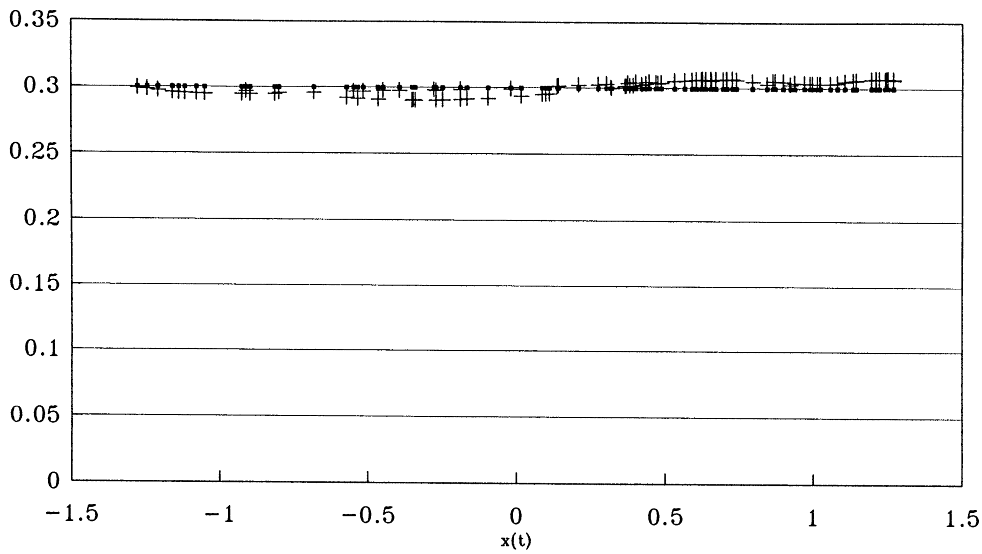

Figure 3. Henon map: derivative of $y(t+1)$ w.r.t $x(t)$. · actual: + estimated (mean square error = 0·000304)

the approximations. This error is calculated by:

$$\mathrm{mse}_{N,m} = \frac{1}{\sqrt{T-m}} \sum_{t=0}^{T-m-1} [y_{t+m} - \bar{v}_{N,m,t+m}]^2 \tag{17}$$

where $\bar{v}_{N,m,t+m}$ is the fit from a single hidden-layer feedforward network. It can be seen that the single hidden-layer network approximates the Henon map as well as its derivatives closely.

In Table II we have summarized the estimates for the Lyapunov exponents of the Henon map. Two points worth noting are that for embedding dimensions of 2 to 4 the first two Lyapunov exponents are quite stable at approximately their true values. The spurious Lyapunov exponents at embedding dimensions 3 and 4 are quite unstable. On larger data sets, these spurious exponents converge to $-\infty$ and are more easily identified.

Table II. The Lyapunov exponents of the Henon map, number of observations: 200

Embedding dimension	Estimated Lyapunov exponents	True Lyapunov exponents
$m=1$	0·3912	
$m=2$	0·4054	0·408
	−1·6252	−1·620
$m=3$	0·4397	
	−1·6280	
	−3·7971	
$m=4$	0·4398	
	−1·6458	
	−2·7047	
	−2·9969	

It is worth comparing our results with those of Brown *et al.* (1991). Their method is to use a Taylor series in estimating the dynamics. Since the Henon map is in fact a polynomial of degree two, their methods should have a strong advantage in this case. However, they reported that when they used a quadratic polynomial in two variables, the estimated Lyapunov exponents were

$$\lambda_1 = 0\cdot 44707 \qquad \lambda_2 = 1\cdot 5096 \tag{18}$$

These estimates were based on 11,000 observations. The relative errors of these estimated exponents are 9·6 per cent and 6·8 per cent, respectively. Compare these with the relative errors of our estimates (which are based on 200 observations) of 0·64 per cent and 0·32 per cent.

In the presence of noise the estimates deteriorate.[6] In Table III we report the results of estimating the Lyapunov exponents for two types of noise. For measurement noise we use the system equation (1) along with the observation equation

$$y_t = x_{1,t} + \sigma\varepsilon_t \tag{19}$$

where $x_{1,t}$ is the first component of x_t and $\{\varepsilon_t\}$ is an independent and identically distributed normal random variable with zero mean and standard deviation σ_ε. If σ_x is the standard deviation of the system data, then the signal-to-noise ratio in the data $\{y_t\}$ is

$$\frac{\sigma_x}{\sigma\sigma_\varepsilon}. \tag{20}$$

The estimated Lyapunov exponents are given in terms of the reciprocal of the signal-to-noise ratio.

Table III. Henon map with noise, number of observations: 200

Type of noise	%	λ_1	λ_2
Measurement noise	0·01	0·3899	−1·7251
		(0·0573)	(0·6114)
	0·05	0·3612	−1·7961
		(0·0633)	(0·6114)
	0·10	0·3591	−2·2514
		(0·0909)	(0·5369)
System noise	0·005	0·3398	−1·8109
		(0·0879)	(0·6869)
	0·007	0·2937	−2·1902
		(0·0925)	(0·7522)

Notes:
(1) λ_1 and λ_2 are the Lyapunov exponent estimates from largest to smallest, respectively. Each Lyapunov exponent is an average of 100 simulations. Numbers in parentheses are the standard deviations.
(2) The series y_t with measurement noise is generated by $x_{t+1} = f(x_t)$, $y_t = x_{1,t} + \sigma\varepsilon_t$ where $\sigma = \%\sigma_x/\sigma_\varepsilon$, σ_x is the standard deviation of the data and σ_ε is the standard deviation of ε_t series. ε_t is drawn from $N(0, 1)$.
(3) The series x_t with system noise is generated by $x_{t+1} = f(x_t) + \sigma\varepsilon_t$, $y_t = x_{1,t}$, where σ is defined above. ε_t is drawn from $N(0, 1)$.

[6] There are two reasons for this. First, the noise itself reduces the extent to which the deterministic relationship can be uncovered. Second, as pointed out by Ellner *et al.* (1991), the QR algorithm also deteriorates due to noise.

For system noise we used the system model

$$x_{t+1} = f(x_t) + \sigma\varepsilon_t \qquad y_{t+1} = x_{1,t} \tag{21}$$

and again the results are reported in terms of the reciprocal of the signal-to-noise ratio.[7] The maximum value of 0·007 is due to the fact that for any larger value of the noise term the system gets knocked out of the basin of attraction and diverges.

Now we will focus on economic time-series data. We will calculate the Lyapunov exponents of the Canadian, German, Italian and Japanese monthly average spot exchange rates and end-of-month spot exchange rates for embedding dimensions, $m = 1, 2$ and 3. We have data for two different time periods. The first data set is the monthly fixed and flexible spot exchange rate data from January 1960 through December 1990, a total of 372 observations. The second data set is the monthly flexible exchange rate data from January 1973 through December 1990, a total of 216 observations.[8] The data are obtained from the OECD Main Economic Indicators tape. The purpose of calculating the Lyapunov exponents of the flexible exchange period is to isolate the dynamics of this period from the fixed exchange-rate period. The reason for calculating the Lyapunov exponents with average and end-of-period data is to observe any potential differences introduced to the dynamics of the series from averaging.

Table IV. Lyapunov exponent estimates of the spot exchange data

	Monthly average					
	Fixed and flexible exchange-rate period			Flexible exchange-rate period		
	$m=1$	$m=2$	$m=3$	$m=1$	$m=2$	$m=3$
Canada	$\lambda_1 = 0.017$	$\lambda_1 = -0.011$	$\lambda_1 = -0.022$	$\lambda_1 = -0.011$	$\lambda_1 = -0.002$	$\lambda_1 = -0.011$
	$b = 1.020$	$\lambda_2 = -0.421$	$\lambda_2 = -0.705$	$b = 0.983$	$\lambda_2 = -1.224$	$\lambda_2 = -1.131$
	$ssr = 0.000$	$ssr = 0.000$	$\lambda_3 = -0.893$	$ssr = 0.000$	$ssr = 0.000$	$\lambda_3 = -1.376$
			$ssr = 0.000$			$ssr = 0.000$
Italy	$\lambda_1 = -0.027$	$\lambda_1 = -0.061$	$\lambda_1 = -0.051$	$\lambda_1 = -0.007$	$\lambda_1 = 0.003$	$\lambda_1 = -0.004$
	$b = 0.974$	$\lambda_2 = -0.996$	$\lambda_2 = -1.121$	$b = 0.993$	$\lambda_2 = -1.005$	$\lambda_2 = -0.769$
	$ssr = 0.000$	$ssr = 0.005$	$\lambda_3 = -1.246$	$ssr = 0.000$	$ssr = 0.000$	$\lambda_3 = -1.786$
			$ssr = 0.001$			$ssr = 0.001$
Japan	$\lambda_1 = -0.001$	$\lambda_1 = -0.003$	$\lambda_1 = -0.005$	$\lambda_1 = 0.008$	$\lambda_1 = -0.033$	$\lambda_1 = -0.008$
	$b = 0.982$	$\lambda_2 = -0.670$	$\lambda_2 = -0.954$	$b = 1.008$	$\lambda_2 = -1.271$	$\lambda_2 = -1.070$
	$ssr = 0.000$	$ssr = 0.000$	$\lambda_3 = -1.120$	$ssr = 0.000$	$ssr = 0.001$	$\lambda_3 = -1.749$
			$ssr = 0.000$			$ssr = 0.001$
Germany	$\lambda_1 = -0.030$	$\lambda_1 = -0.099$	$\lambda_1 = -0.081$	$\lambda_1 = 0.005$	$\lambda_1 = -0.010$	$\lambda_1 = 0.011$
	$b = 0.971$	$\lambda_2 = -1.106$	$\lambda_2 = -1.211$	$b = 0.996$	$\lambda_2 = -1.222$	$\lambda_2 = -1.329$
	$ssr = 0.001$	$ssr = 0.001$	$\lambda_3 = -1.291$	$ssr = 0.000$	$ssr = 0.001$	$\lambda_3 = -2.025$
			$ssr = 0.001$			$ssr = 0.001$

Notes:
(1) m refers to the embedding dimension. For each column of m, the Lyapunov exponent estimates are ordered from largest to smallest.
(2) The monthly fixed and flexible spot exchange-rate data are from January 1960 through December 1990, a total of 372 observations. The monthly spot flexible exchange-rate data are from January 1973 through December 1990, a total of 216 observations.

[7] Ellner *et al.* (1991) show that Jacobian-based estimates of the largest Lyapunov exponent are asymptotically consistent in the presence of system noise.
[8] The first data set is estimated with seven hidden-unit activation functions. The second data set is estimated with six hidden-unit activation functions.

Table V. Lyapunov exponent estimates of the spot exchange data

	End of month					
	Fixed and flexible exchange-rate period			Flexible exchange-rate period		
	$m=1$	$m=2$	$m=3$	$m=1$	$m=2$	$m=3$
Canada	$\lambda_1=-0.034$ $b=0\cdot971$ $ssr=0\cdot000$	$\lambda_1=-0\cdot044$ $\lambda_2=-0\cdot678$ $ssr=0\cdot000$	$\lambda_1=-0\cdot012$ $\lambda_2=-1\cdot256$ $\lambda_3=-1\cdot261$ $ssr=0\cdot000$	$\lambda_1=-0\cdot010$ $b=0\cdot991$ $ssr=0\cdot000$	$\lambda_1=0\cdot019$ $\lambda_2=-1\cdot545$ $ssr=0\cdot000$	$\lambda_1=-0\cdot008$ $\lambda_2=-1\cdot199$ $\lambda_3=-1\cdot208$ $ssr=0\cdot000$
Italy	$\lambda_1=-0\cdot001$ $b=1\cdot000$ $ssr=0\cdot001$	$\lambda_1=0\cdot010$ $\lambda_2=-0\cdot650$ $ssr=0\cdot001$	$\lambda_1=-0\cdot002$ $\lambda_2=-1\cdot089$ $\lambda_3=-1\cdot198$ $ssr=0\cdot000$	$\lambda_1=-0\cdot010$ $b=0\cdot990$ $ssr=0\cdot000$	$\lambda_1=-0\cdot012$ $\lambda_2=-2\cdot309$ $ssr=0\cdot001$	$\lambda_1=0\cdot001$ $\lambda_2=-0\cdot966$ $\lambda_3=-1\cdot089$ $ssr=0\cdot001$
Japan	$\lambda_1=0\cdot022$ $b=1\cdot023$ $ssr=0\cdot001$	$\lambda_1=-0\cdot074$ $\lambda_2=-1\cdot068$ $ssr=0\cdot001$	$\lambda_1=0\cdot026$ $\lambda_2=-1\cdot100$ $\lambda_3=-1\cdot928$ $ssr=0\cdot000$	$\lambda_1=-0\cdot005$ $b=1\cdot000$ $ssr=0\cdot001$	$\lambda_1=-0\cdot029$ $\lambda_2=-2\cdot371$ $ssr=0\cdot001$	$\lambda_1=-0\cdot006$ $\lambda_2=-1\cdot602$ $\lambda_3=-1\cdot927$ $ssr=0\cdot001$
Germany	$\lambda_1=0\cdot002$ $b=1\cdot002$ $ssr=0\cdot001$	$\lambda_1=0\cdot076$ $\lambda_2=-0\cdot880$ $ssr=0\cdot000$	$\lambda_1=-0\cdot004$ $\lambda_2=-0\cdot836$ $\lambda_3=-0\cdot974$ $ssr=0\cdot000$	$\lambda_1=-0\cdot006$ $b=0\cdot996$ $ssr=0\cdot001$	$\lambda_1=-0\cdot033$ $\lambda_2=-2\cdot531$ $ssr=0\cdot001$	$\lambda_1=-0\cdot039$ $\lambda_2=-0\cdot872$ $\lambda_3=-1\cdot070$ $ssr=0\cdot001$

Notes:
(1) m refers to the embedding dimension. For each column of m, the Lyapunov exponent estimates are ordered from largest to smallest.
(2) The monthly fixed and flexible spot exchange-rate data are from January 1960 through December 1990, a total of 372 observations. The monthly spot flexible exchange-rate data are from January 1973 through December 1990, a total of 216 observations.

These estimates for embedding dimension $m=1$ are given in the first and fourth columns of Tables IV and V. For $m=1$, there are three entries in each cell. The first is the Lyapunov exponent estimate, the second entry is the average derivative (parameter) estimate of the regression of x_{t+1} on x_t by single hidden-layer feedforward networks and the last entry is the average sum of squared residuals from the same network.

Note that all series have a parameter value very close to one or a unit root. The performance of the single hidden-layer feedforward networks (NNet) with a unit root process is presented in Figures 4–7. In these figures the fitted value of the unit root process is plotted against its lags and compared with the actual process and its lags for 1000 observations. It is clear that the fit is excellent and the persistence of the process in its lags is captured. We have also plotted the phase diagrams for the monthly end-of-period spot exchange rate data.[9] In Figures 8–19, phase diagrams up to lag four are plotted for the German, Japanese and Canadian currencies. These figures give a clear indication of the presence of a unit root process when they are compared to Figures 4–7.

The presence of the unit root is captured in the calculated Lyapunov exponent values which lie close to zero.[10] To compare this performance, we also calculated the Dickey–Fuller unit

[9] Each exchange series is scaled down from its level.

[10] For higher-order integrated processes, the determinant of the Jacobian is always constant and equal to one. Hence, the sum of the Lyapunov exponents is equal to the logarithm of the determinant which is zero. In simulation experiments, we observed that all d Lyapunov exponents of an $I(d)$ process are zero. This is due to the fact that the distance between two trajectories generated from two nearby initial starting points is a constant for integrated processes.

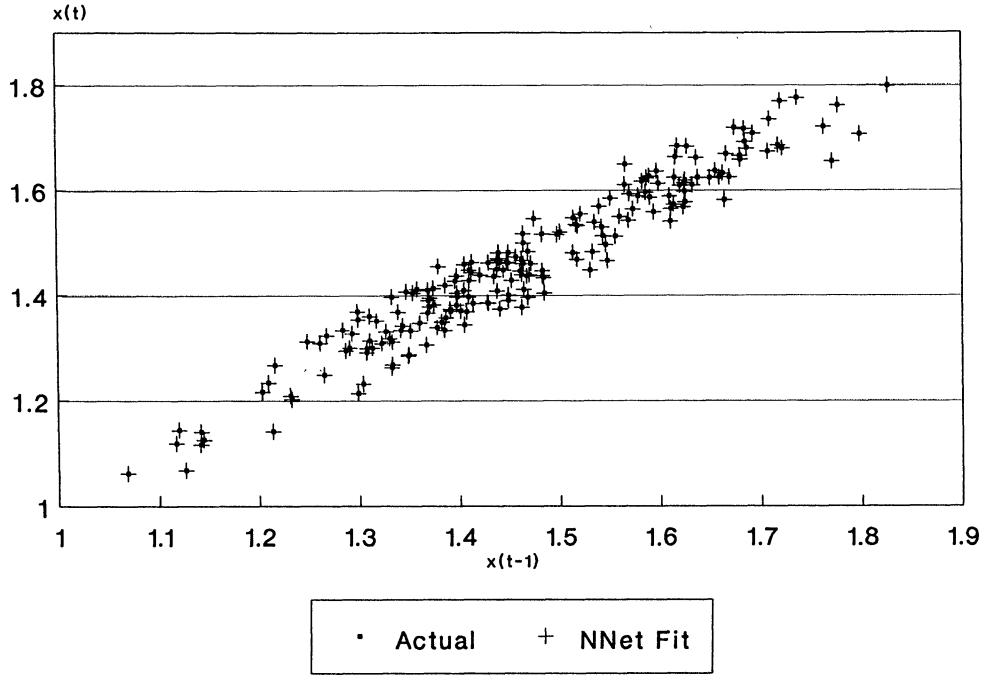

Figure 4. Unit root process (x_t against x_{t-1})

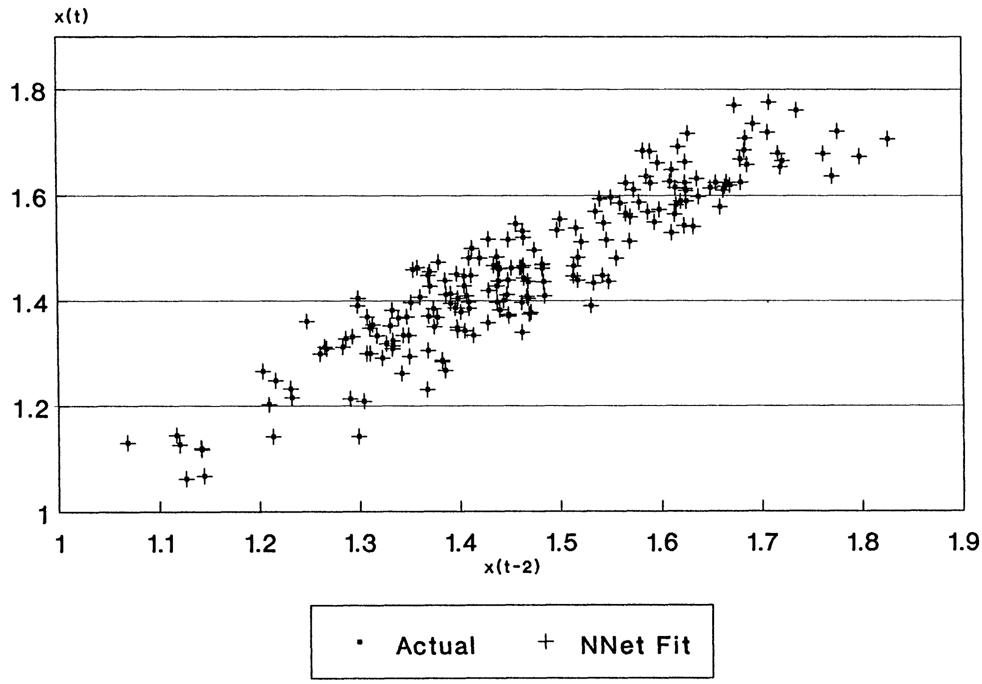

Figure 5. Unit root process (x_t against x_{t-2})

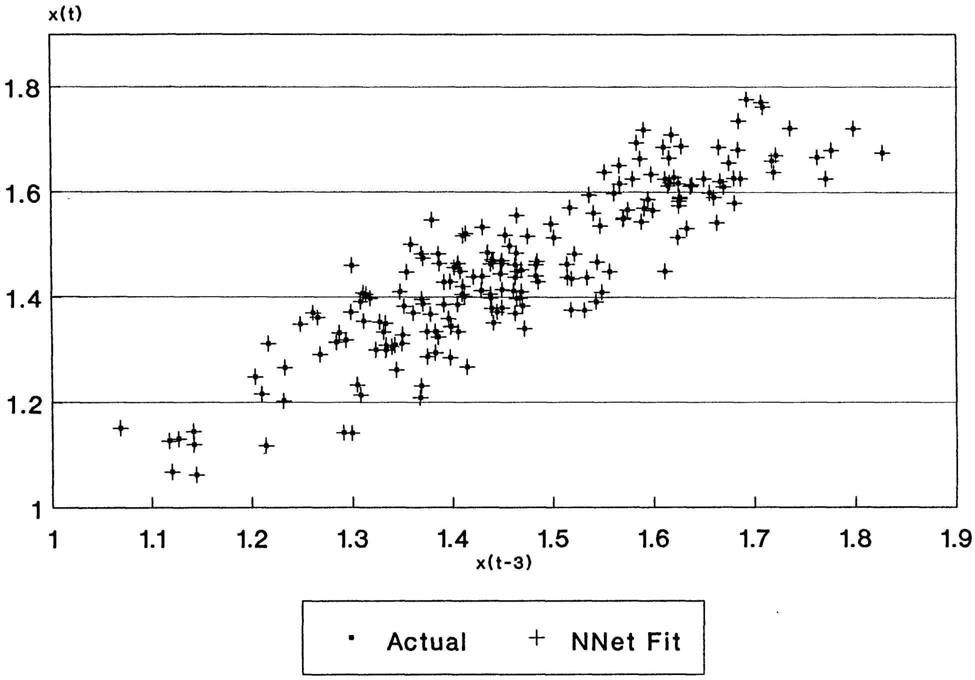

Figure 6. Unit root process (x_t against x_{t-3})

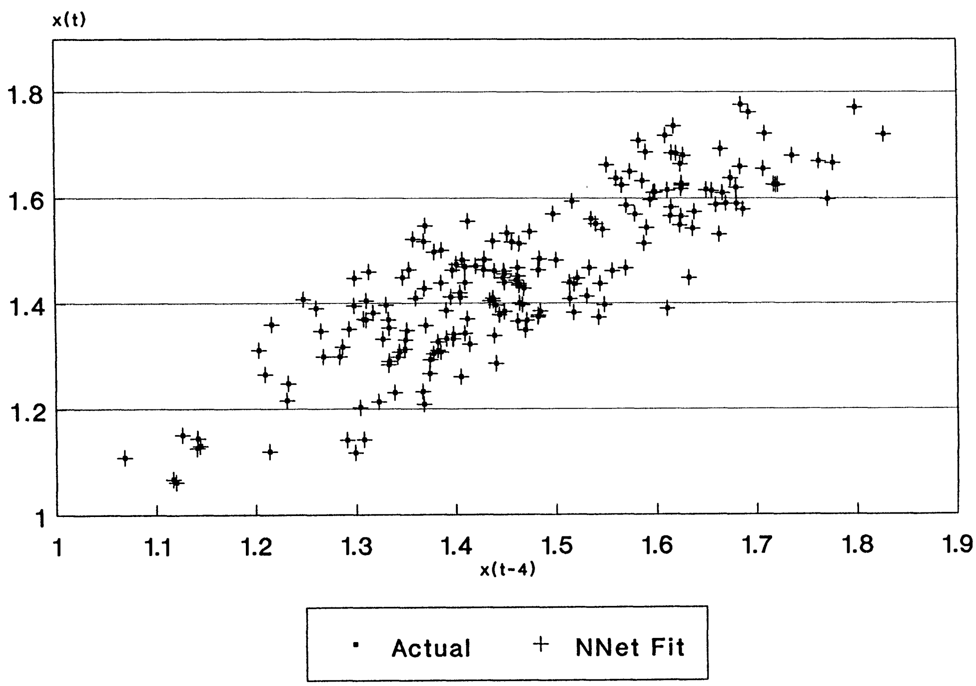

Figure 7. Unit root process (x_t against x_{t-4})

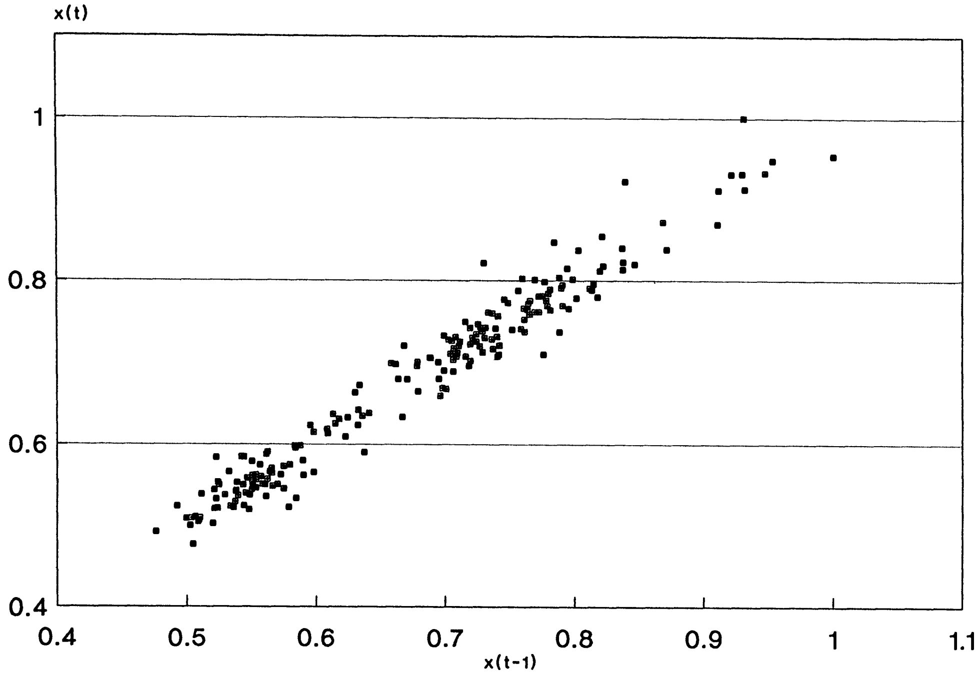

Figure 8. German end-of-month spot exchange rate (x_t against x_{t-1})

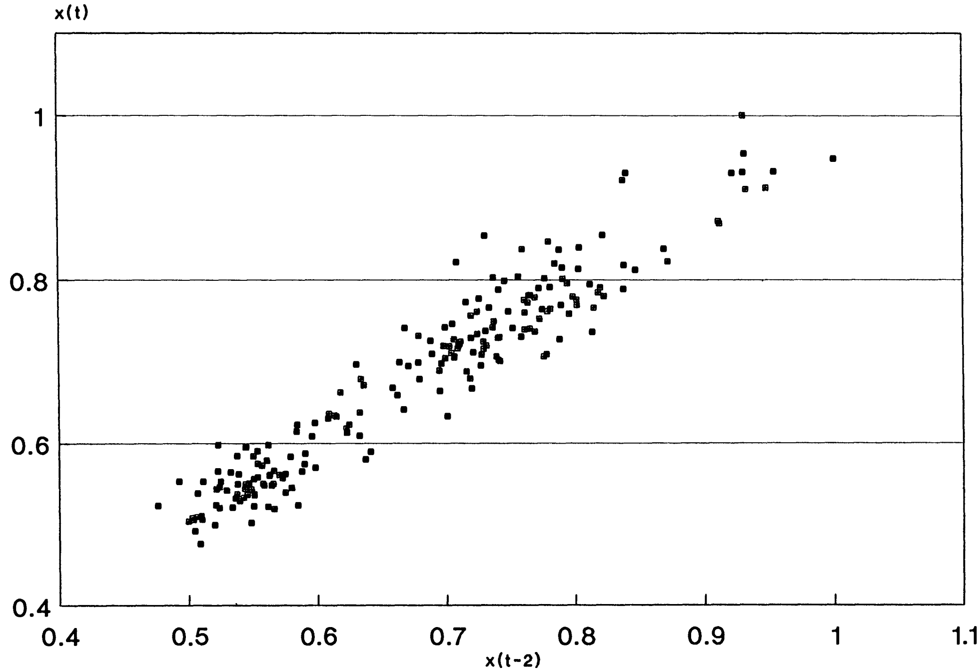

Figure 9. German end-of-month spot exchange rate (x_t against x_{t-2})

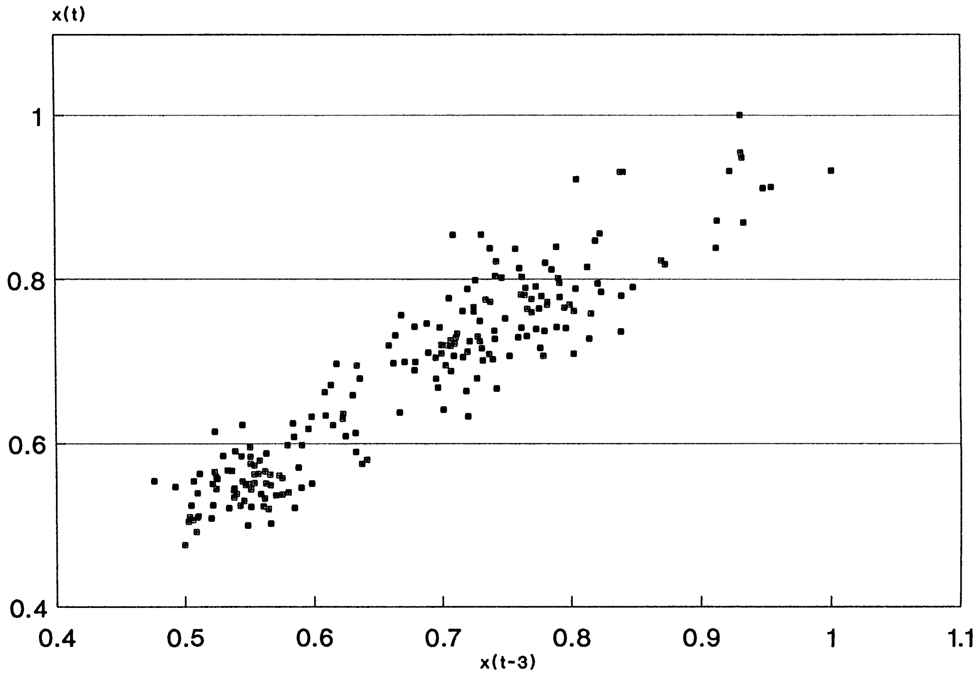

Figure 10. German end-of-month spot exchange rate (x_t against x_{t-3})

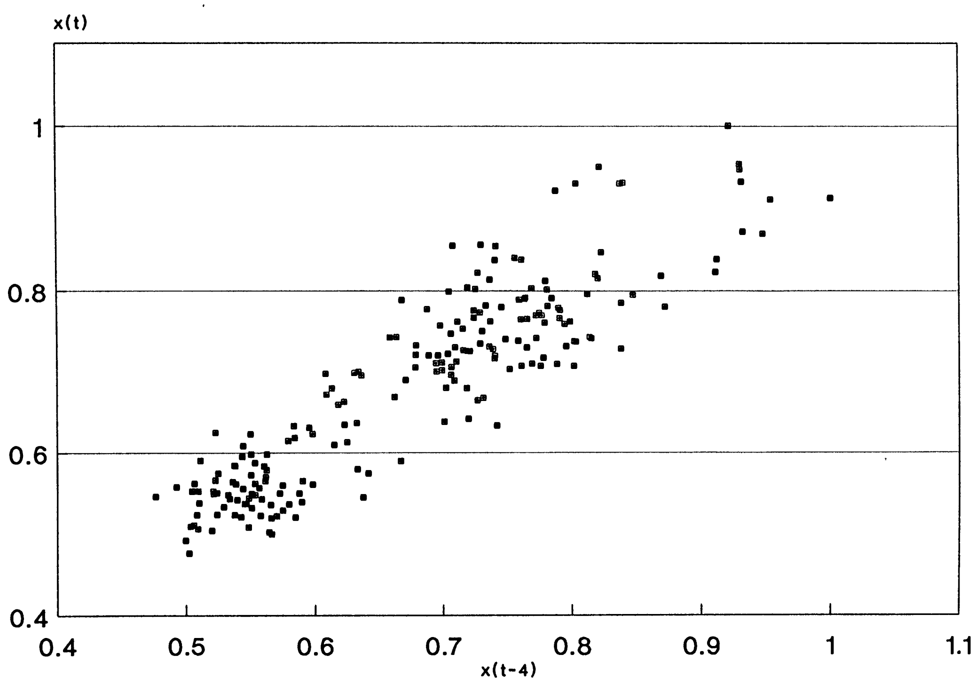

Figure 11. German end-of-month spot exchange rate (x_t against x_{t-4})

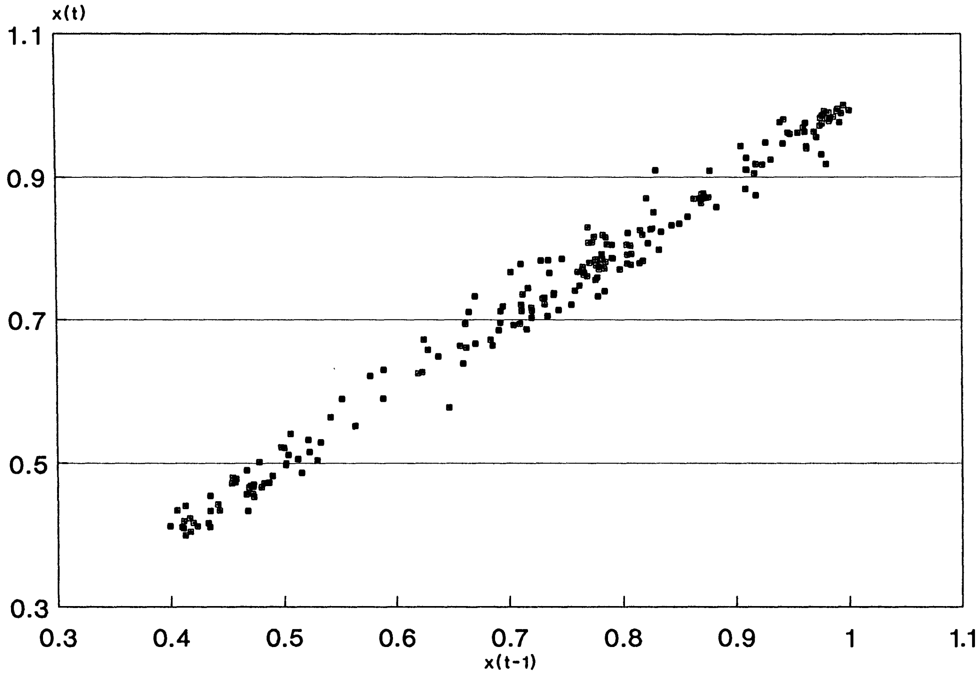

Figure 12. Japanese end-of-month spot exchange rate (x_t against x_{t-1})

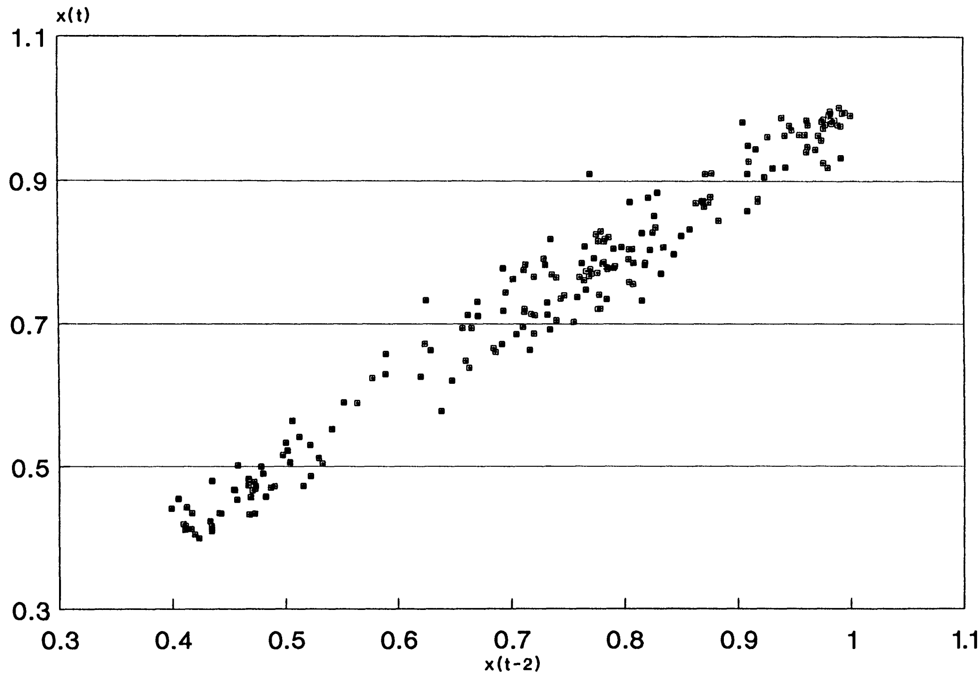

Figure 13. Japanese end-of-month spot exchange rate (x_t against x_{t-2})

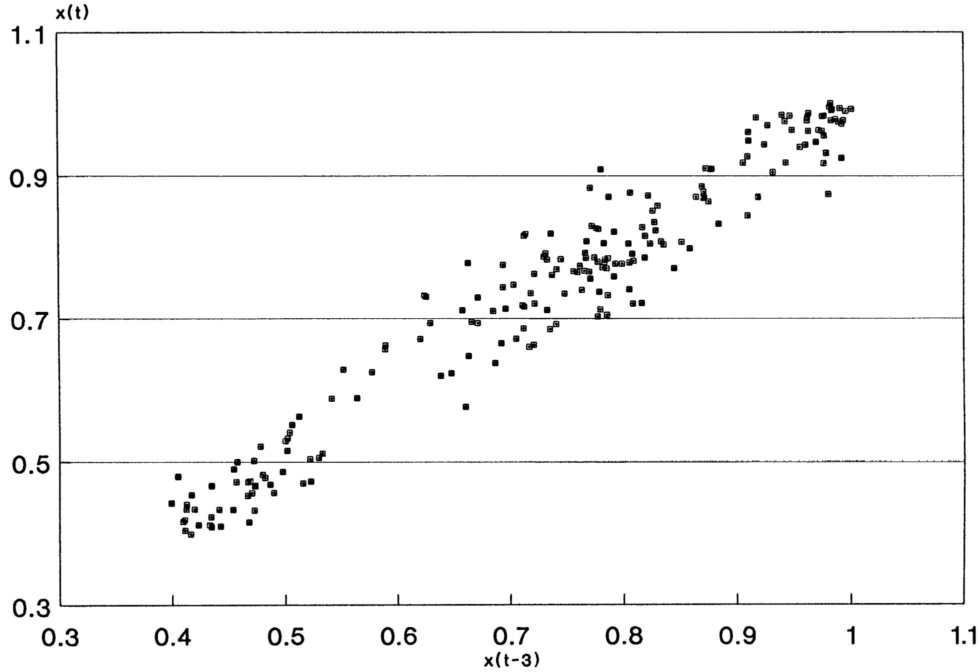

Figure 14. Japanese end-of-month spot exchange rate (x_t against x_{t-3})

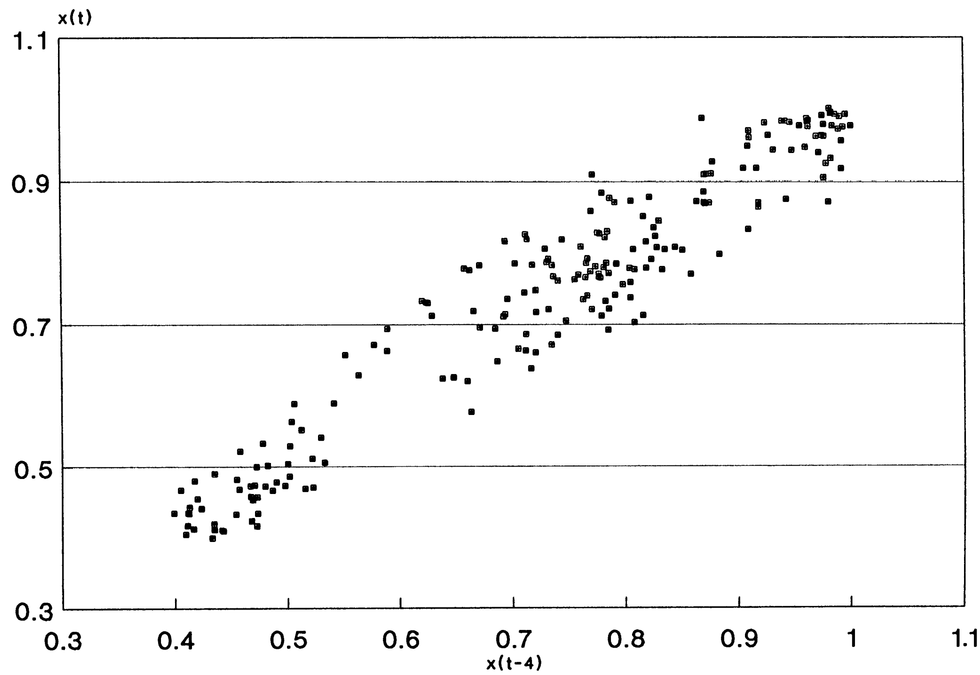

Figure 15. Japanese end-of-month spot exchange rate (x_t against x_{t-4})

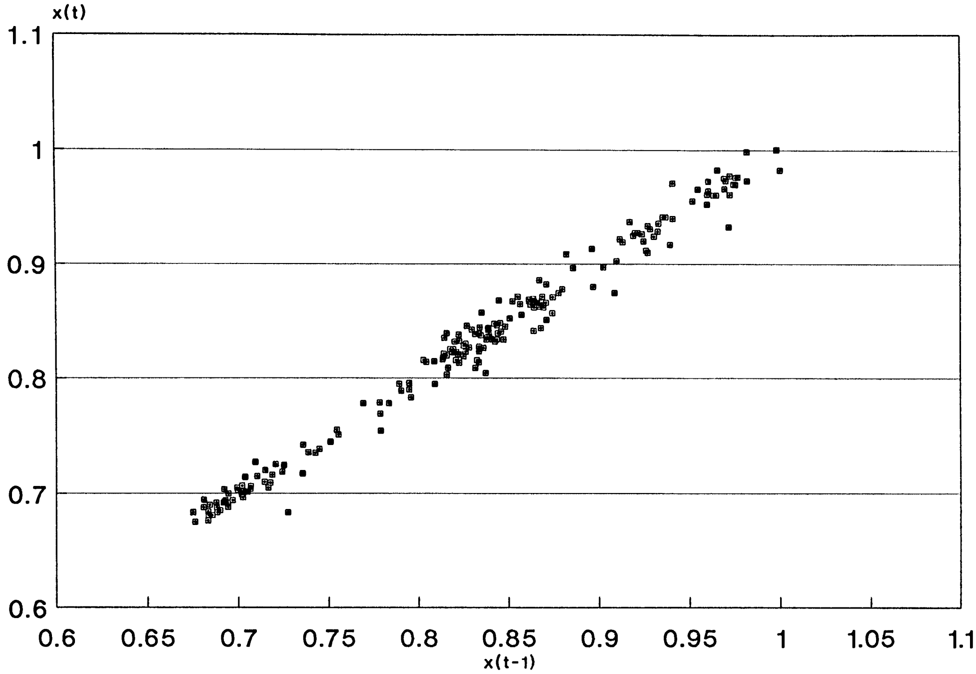

Figure 16. Canadian end-of-month spot exchange rate (x_t against x_{t-1})

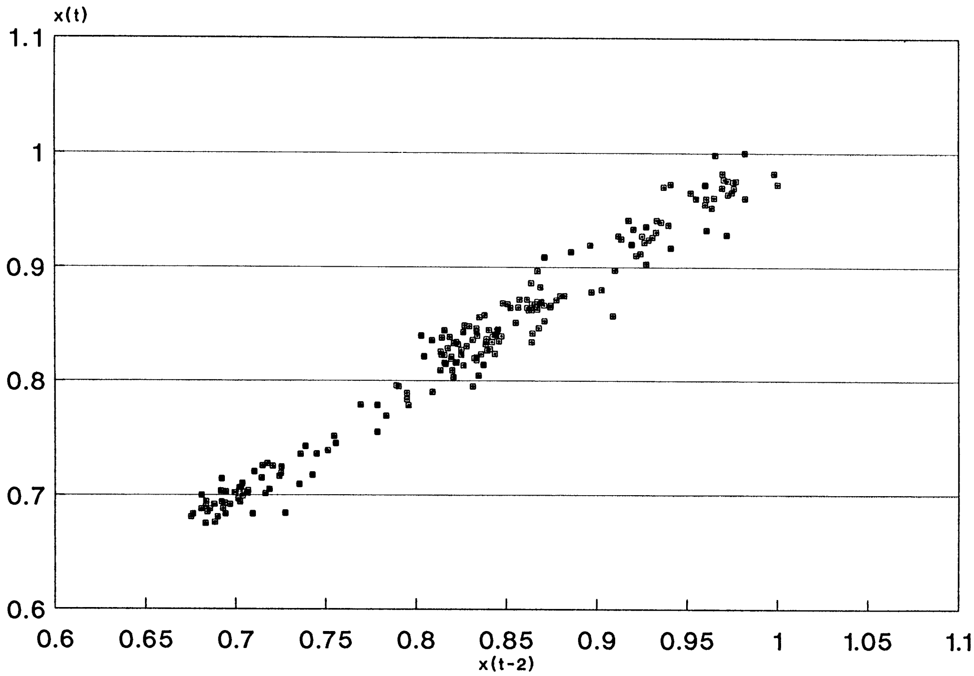

Figure 17. Canadian end-of-month spot exchange rate (x_t against x_{t-2})

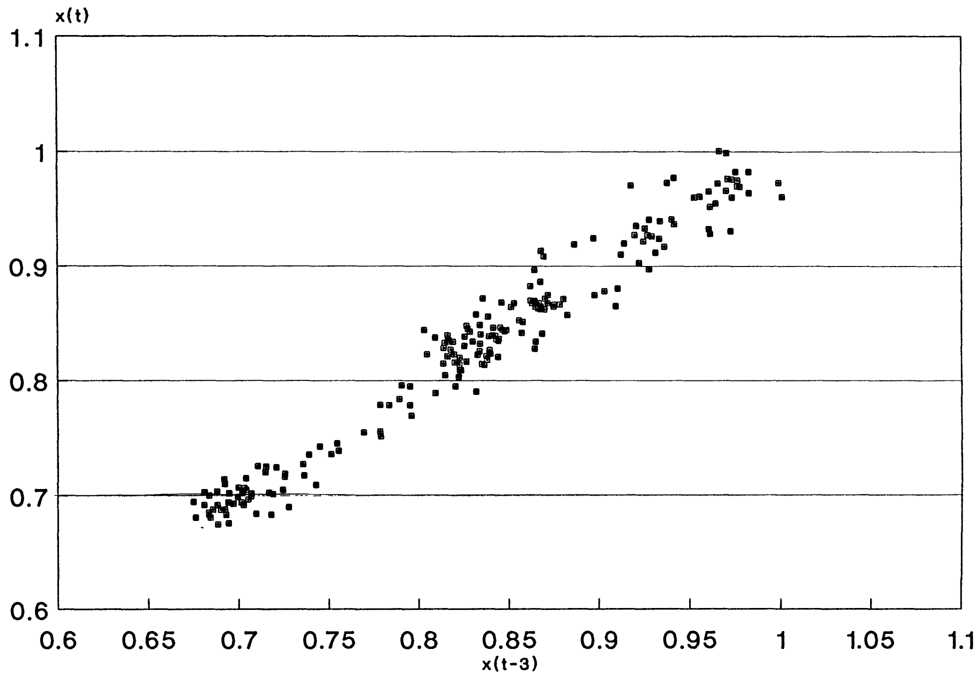

Figure 18. Canadian end-of-month spot exchange rate (x_t against x_{t-3})

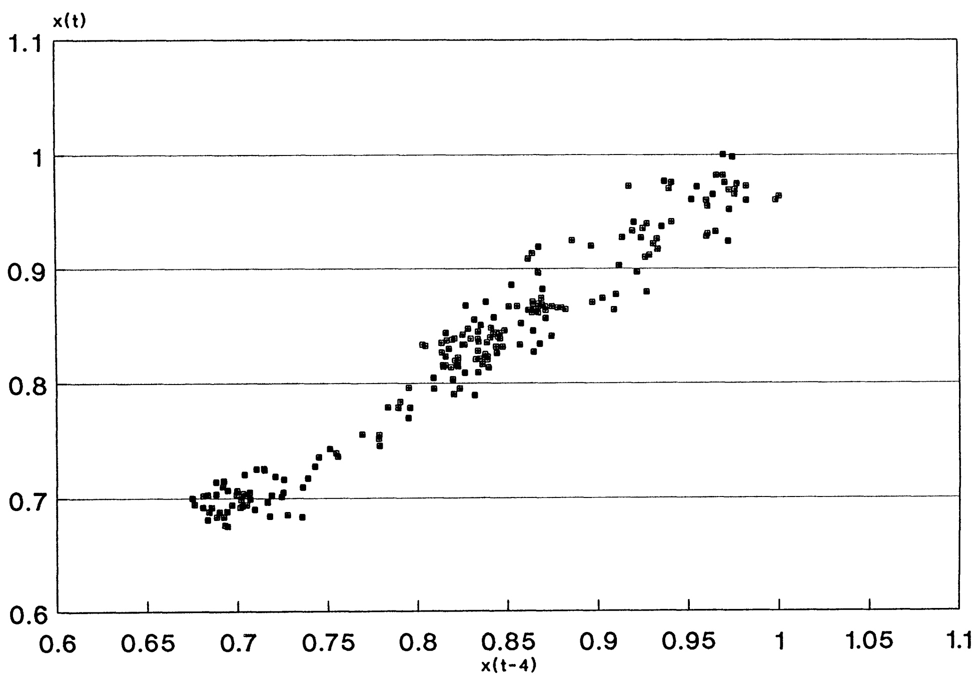

Figure 19. Canadian end-of-month spot exchange rate (x_t against x_{t-4})

Table VI. The Dickey–Fuller unit root tests of the spot exchange-rate data

	Monthly average			
	Fixed and flexible exchange-rate period		Flexible exchange-rate period	
	No constant	With constant	No constant	With constant
Canada	0·0004	−0·0063	0·0007	−0·0004
	(0·9417)	(−1·428)	(0·9952)	(−0·065)
Italy	0·0010	−0·0023	0·0023	−0·0016
	(0·7534)	(−0·7040)	(1·2531)	(−0·2692)
Japan	−0·002*	0·0014	−0·0018	0·0087
	(−2·212)	(0·4344)	(−0·9740)	(1·1180)
Germany	−0·002*	−0·0012	−0·0019	0·0158
	(−2·4629)	(−0·3591)	(−0·8867)	(1·2996)

Notes:
(1) The first entry is the parameter estimate; the second entry in parenthesis is the t-statistic. The critical values of the Dickey–Fuller test statistic at the 5% level are −1·95 with no constant and −2·89 with a constant term. The 10% level critical values with no constant are −1·61 and −2·58 with a constant term.
(2) An asterisk denotes evidence against the null hypothesis of a unit root.

root tests by running Δx_t on x_{t-1} as well as running Δx_t on a constant and x_{t-1} under the assumption that the error term is identically and independently distributed. These statistics are reported in Tables VI and VII. Note that in the first column of Table VI, for Japanese and German average exchange rates, the Dickey–Fuller test rejects the presence of a unit root at the 5 per cent level, although the unit root is clearly there. At the 10 per cent level, the Dickey–Fuller test rejects the presence of unit root for the Italian spot exchange rate in the

Table VII. The Dickey–Fuller unit root tests of the spot exchange-rate data

	End of month			
	Fixed and flexible exchange-rate period		Flexible exchange-rate period	
	No constant	With constant	No constant	With constant
Canada	−0·001	−0·0092	−0·0007	0·0023
	(−1·2171)	(−1·6661)	(−0·7967)	(0·2670)
Italy	−0·002*	−0·0015	0·0024	−0·0005
	(−1·7029)	(−0·5124)	(1·1251)	(−0·0801)
Japan	0·003	0·0034	−0·0017	0·0108
	(1·9974)	(0·6973)	(−0·7881)	(1·2019)
Germany	0·003	0·0004	−0·0016	0·0219
	(1·6881)	(0·0797)	(−0·6539)	(1·5611)

Notes:
(1) The first entry is the parameter estimate; the second entry in parenthesis is the t-statistic. The critical values of the Dickey–Fuller test statistic at the 5% level are −1·95 with no constant and −2·89 with a constant term. The 10% level critical values with no constant are −1·61 and −2·58 with a constant term.
(2) An asterisk denotes evidence against the null hypothesis of a unit root.

first column of Table VII. To check the behaviour of the series in higher dimensions, we embedded each series in embedding dimension two, $m = 2$. These estimates are summarized in the second and fifth columns of Tables IV and V. Note that the largest Lyapunov exponents are preserved close to their $m = 1$ estimates. The second Lyapunov exponent of each series is clearly away from zero and negative. There can be two interpretations of this situation. The first is that each of these series is generated from a two-dimensional system with a second stable root. The second interpretation is that each of these series is generated from a one-dimensional system with a unit root and the smaller Lyapunov exponent is spurious.

One method of distinguishing between these two interpretations is to see whether the first two largest Lyapunov exponents are invariant in their magnitude as the embedding dimension is increased. An example of this situation is given in Table II, where the largest Lyapunov exponents of the Henon map are not changed as the embedding dimension is increased up to four. However, note that the spurious Lyapunov exponents vary in magnitude. An example of this is the unit root process itself. The calculated Lyapunov exponents of the simulated unit root process in Figures 4–7 are $m = 1$, $\lambda_1 = 0\cdot0001$; $m = 2$, $\lambda_1 = 0\cdot0002$, $\lambda_2 = -1\cdot1443$; and $m = 3$, $\lambda_1 = 0\cdot0002$, $\lambda_2 = -2\cdot1401$, $\lambda_3 = -3\cdot1410$.

The Lyapunov exponent estimates of the exchange rate data at $m = 3$ are summarized in third and sixth columns of Table IV and V. The two largest Lyapunov exponents of the German and Japanese end-of-month spot exchange rates for the fixed and flexible exchange-rate period are very close to $m = 2$ and $m = 3$. This gives the indication that there is a second dimensional element for the German and Japanese end-of-month spot exchange rates. For all other series, the dynamics of the data are characterized by a unit root process.

In monthly average as well as in the end-of-month series, the unit root presence in the largest Lyapunov exponent does not show any systematic difference. In higher embeddings, the second and the third Lyapunov exponents have a larger negative value in the flexible exchange-rate period. This gives the indication that the random walk hypothesis is much more apparent in the flexible exchange rate-period.

The Lyapunov exponents are invariant to the form of the observer function as long as the observer function (6) is generic. Therefore, time averaging will have no influence on the deterministic part of the data for a sufficiently large number of observations. However, time averaging may eliminate the fine structure of the stochastic nature of the data if the degree of averaging is high. The comparison of the end-of-month with the monthly average data does not exhibit any systematic variation. An explanation of this is, first, that the number of observations is large enough and second, the degree of averaging is not so severe so that both series provide the same information.

5. CONCLUSION

We have discussed the Lyapunov exponents as a diagnostic of stability analysis with a particular focus on the algorithm proposed by Dechert and Gencay (1990). The implementation of the algorithm is carried out by multilayer feedforward networks. It is shown that it is possible to calculate all Lyapunov exponents of a dynamical system accurately with limited number of observations. The algorithm has good performance with the presence of measurement as well as system noise. This multi-dimensional diagnostic can clearly serve as a useful complementary tool in the stability analysis of time-series data.

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REFERENCES

Brock, W., and C. Sayers (1988), 'Is the business cycle characterized by deterministic chaos?', *Journal of Monetary Economics*, **22**, 71–90.

Brown, R., P. Bryant and D. I. Abarbanel (1991), 'Computing the Lyapunov spectrum of a dynamical system from an observed time series', *Physical Review A*, **43**, 2787–2806.

Cohen, J. E., J. Kesten, and C. M. Newman (eds) (1986), 'Random matrices and their application', *Volume 50 of Contemporary Mathematics*, American Mathematical Society, Providence, RI.

Dechert, W. D., and R. Gencay (1990), 'Estimating Lyapunov exponents with multilayer feedforward network learning', Department of Economics, University of Houston.

Eckmann, J. P., and D. Ruelle (1985), 'Ergodic theory of strange attractors', *Reviews of Modern Physics*, **57**, 617–656.

Ellner, S., A. R. Gallant, D. McCaffrey, and D. Nychka (1991), 'Convergence rates and data requirements for Jacobian-based estimates of Lyapunov exponents from data', *Physics Letters A*, **153**, 357–363.

Frank, M., R. Gencay, and T. Stengos (1988), 'International Chaos?', *European Economic Review*, **32**, 1569–1584.

Frank, M., and T. Stengos (1988), 'Some evidence concerning macroeconomic chaos', *Journal of Monetary Economics*, **22**, 423–438.

Gallant, A. R., and H. White (1992), 'On learning the derivatives of an unknown mapping with multilayer feedforward networks', *Neural Networks*, **5**, 129–138.

Guckenheimer, J., and P. Holmes (1983), *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York.

Hornik, K., M. Stinchcombe and H. White (1990), 'Universal approximation of an unknown mapping and its derivatives using multilayer feedforward networks', *Neural Networks*, **3**, 535–549.

Kurths, J., and Herzel, H. (1987), 'An attractor in a solar time series', *Physica D*, **25**, 165–172.

McCaffrey, D., S. Ellner, A. R. Gallant and D. Nychka (1991), 'Estimating Lyapunov exponents with nonparametric regression', *Journal of the American Statistical Association*, forthcoming.

Nychka, D., S. Ellner, A. R. Gallant, and D. McCaffrey (1992), 'Finding chaos in noisy systems', *Journal of the Royal Statistical Society* **B4**, forthcoming.

Oseledec, V. I. (1968), 'A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical system', *Transactions of the Moscow Mathematical Society*, **19**, 197–221.

Press, W. H., B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling (1986), *Numerical Recipes, The Art of Scientific Computing*, Cambridge University Press, Cambridge.

Raghunathan, M. S. (1979), 'A proof of Oseledec's multiplicative ergodic theorem', *Israel Journal of Mathematics*, **32**, 356–362.

Ruelle, D. (1979), 'Ergodic theory of differentiable dynamical systems', *Publications Mathematiques Institut des Hautes Etudes Scientifiques,* **50**, 27–58.

Takens, F. (1981), 'Detecting strange attractors in turbulence', in D. Rand and I. Young (eds), *Dynamical Systems and Turbulence*, Springer-Verlag, Berlin, pp. 366–381.

Wolf, A., B. Swift, J. Swinney, and J. Vastano (1985), 'Determining Lyapunov exponents from a time series', *Physica D*, **16**, 285–317.