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A statistical framework for testing chaotic dynamics via Lyapunov exponents

Ramazan Gençay,

Department of Economics, University of Windsor, 401 Sunset, Windsor, Ont. N9B 3P4, Canada

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Abstract

One common criticism of the existing Lyapunov exponents algorithms is the absence of a distributional theory which provides a framework for the statistical hypothesis testing for the calculated Lyapunov exponents. This paper presents a methodology to calculate the empirical distributions of Lyapunov exponents by using a bootstrapping technique. The methodology of the paper provides a formal test of chaos under the null hypothesis. The numerical examples show that the method works well with small data sets.

1. Introduction

The Lyapunov exponents for a dynamical system, $f : \mathbb{R}^n \to \mathbb{R}^n$, with the trajectory,

$$x_{t+1} = f(x_t) \quad (1)$$

are measures of the average rate of divergence or convergence of a typical trajectory. A positive Lyapunov exponent is a measure of the average exponential divergence of two nearby trajectories. If a discrete nonlinear system is dissipative, a positive Lyapunov exponent is an indication that the system is chaotic.

The last decade led to extensive research in the calculation of Lyapunov exponents of an unknown dynamical system from observations. The first attempt came from Wolf et al. [1] which had very popular applications in a very broad spectrum of disciplines. The introduction of Lyapunov exponents to economics was in Brock [2]. Brock and Sayers [3] note that the Wolf algorithm is sensitive to the number of observations as well as to the degree of measurement or system noise in the observations. Frank et al. [4,5] provide evidence that the Kurths and Herzel [6] algorithm has a positive bias. This observation started a search for new algorithmic designs with improved small sample properties.

The search for an algorithm to calculate Lyapunov exponents with desirable small sample properties has gained momentum in the last couple of years. Three groups of researchers, from three different disciplines, Abarbanel et al. [7–10], McCaffrey et al. [11,12] and Gençay and Dechert [13,14], came up with improved algorithms for the calculation of the Lyapunov exponents from observed data.

The main algorithmic design in [7–10] and [13,14] is to embed the observations in an m-dimensional space, then by theorems of Mañé [15] and Takens [16] the observations are used to reconstruct the dynamics on the attractor. The Jacobian of the reconstructed dynamics as demonstrated in Eckmann et al.

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[17,18] is then used to calculate the Lyapunov exponents of the unknown dynamics.

One common criticism of the existing Lyapunov exponents algorithms is the absence of a distributional theory for testing the statistical significance of the calculated Lyapunov exponents. The problem becomes even more acute if the estimated Lyapunov exponents are positive but very small in magnitude. This is particularly the case for the Mackey-Glass delay equation. For certain parameter values, the largest Lyapunov exponent of this equation is in the of the order of 10^{-2}. In empirical applications the dynamics may not be known to us and it is legitimate to question whether an estimate in the order 10^{-2} reflect truly a chaotic process or it is positively biased due to some type of measurement noise or data snooping problems.

To measure statistical significance of the Lyapunov exponents estimator, one can draw large number of samples from the data generating process and calculate the corresponding Lyapunov exponent estimators. These estimators can then be used to construct the empirical distribution of the estimator under study. Hence, it is possible to test the statistical significance of the estimator at hand with reference to the constructed empirical distribution. The difficulty is that we may only have one set of realizations of the data generating process and no more. The question is then whether one can design a mechanism to construct the empirical distribution of the estimator from the realizations at hand. The theory of bootstrapping deals with this problem and some recent results of this literature have been extended to cover the cases for the weakly dependent stationary sequence of observations.

The objective of this paper is to present a methodology to calculate the empirical distribution of Lyapunov exponents from the observed data as a framework of hypothesis testing for chaotic dynamics. The null hypothesis H_0 for the largest Lyapunov exponent can be formulated [1] as

$$H_0 : \lambda^{\max} = \lambda_1, \quad H_1 : \lambda^{\max} \neq \lambda_1. \tag{2}$$

where $\lambda^{\max}$ is the unknown parameter (largest Lyapunov exponent) which is set to some hypothesized value λ_1, which may be the true Lyapunov exponent if known or some other value.

Section 2 is on the Lyapunov exponents algorithm utilized here. Section 3 is on the description of the bootstrap methodology. The examples which illustrates the procedure and an application to financial time series data are presented in Section 4. Conclusions follow thereafter.

2. The algorithm

In practice one rarely has the advantage of observing the state of the system, x_t, let alone knowing the actual functional form f that generate the dynamics. The model that is widely used is the following: associated with the dynamical system in Eq. (1) there is an observer function $h : \mathbb{R}^n \to \mathbb{R}$ which generates the observations,

$$y_t = h(x_t) + \gamma\epsilon_t, \quad \epsilon_t \sim IID(0, \sigma^2)$$

ϵ_t is an identically and independently distributed measurement error and γ is a constant controlling the degree of noise infiltrated to the system. It is assumed that all that is available to the researcher is the sequence $\{y_t, t = 1, \dots, T\}$. For notational purposes, let

$$y_t^m = (y_t, y_{t+1}, \dots, y_{t+m-1}). \tag{3}$$

When $\gamma = 0$, it is shown in [16] that if the set $\overline{U}$ is compact manifold then for $m \geq 2n + 1$

$$J^m(x) = \left(h(x), h(f(x)), \dots, h(f^{m-1}(x))\right) \tag{4}$$

generically is an embedding. [2] For $m \geq 2n + 1$ there exists a function $g : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$y_{t+1}^m = g(y_t^m)$$

where

[1] The formulation of the null hypothesis is not restricted to the largest Lyapunov exponents. Other lower order Lyapunov exponents can be tested in similar fashion.

[2] By generic is meant that in every neighborhood of f and h there are functions $\tilde{f}$ and $\tilde{h}$ so that the function J^m corresponding to these functions is an embedding of the attractor of $\tilde{f}$ and the image of the image of the attractor under J^m. Here $2n + 1$ is the worst-case upper limit.

$$y_{t+1}^m = (y_{t+1}, y_{t+2}, \ldots, y_{t+m}).$$

But notice that

$$y_{t+1}^m = J^m(x_{t+1}) = J^m(f(x_t)). \quad (5)$$

Hence from Eqs. (3) and (5)

$$J^m(f(x_t)) = g(J^m(x_t)).$$

The function g is topologically conjugate to f. This implies that g inherits the dynamical properties of f.

From Eq. (3) the mapping g which is to be estimated may be taken to be

$$g : \begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+m-1} \end{bmatrix} \rightarrow \begin{bmatrix} y_{t+1} \\ y_{t+2} \\ \vdots \\ v(y_t, y_{t+1}, \ldots, y_{t+m-1}) \end{bmatrix} \quad (6)$$

and this reduces to estimating

$$y_{t+m} = v(y_t, y_{t+1}, \ldots, y_{t+m-1}).$$

Here v is an unknown map and is estimated by feedforward networks.[3]

Linearization of the map g yields

$$\Delta y_{t+1}^m = (Dg)_{y_t^m} \Delta y_t^m.$$

The solution can be written as

$$\Delta y_t^m = (Dg^t)_{y_0^m} \Delta y_0^m$$

where

$$(Dg^t)_{y_0^m} = (Dg)_{y_{t-1}^m} (Dg)_{y_{t-2}^m} \cdots (Dg)_{y_0^m}$$

and

$$(Dg)_{y_t^m} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ v_{1,t} & v_{2,t} & v_{3,t} & \ldots & v_{m-1,t} & v_{m,t} \end{bmatrix}$$

where

$$v_{m,t} = \frac{\partial v}{\partial y_{t+m-1}}, \quad \ldots, \quad v_{1,t} = \frac{\partial v}{\partial y_t}.$$

The Lyapunov exponents can be calculated from the eigenvalues of the matrix $(Dg)_{y_t^m}$ using QR decomposition. This method is discussed in [17,18,23] and a modified version is presented in [10].

3. Empirical distributions of Lyapunov exponents

Let $\lambda^{\max} = \hat{\lambda}$ where $\hat{\lambda}$ is the estimated value of $\lambda^{\max}$. The natural interest is to calculate the sampling distribution of

$$\sqrt{T}(\hat{\lambda}_1 - \lambda_1) \quad (7)$$

to draw inferences for the statistical significance of calculated largest Lyapunov exponent, $\hat{\lambda}_1$. The nature of the Lyapunov exponents poses severe theoretical difficulties in terms of deriving their asymptotic distributions. It is however possible to construct their empirical distributions nonparametrically by a bootstrap methodology.

Bootstrap techniques have been extended to weakly dependent stationary sequence of time series by [19,20]. The results in [19,20] point out that if the time series under study is weakly dependent and has a stationary distribution for all observations then the distribution of the estimator of interest can be constructed consistently by a moving block bootstrap methodology. Some other versions of bootstrap are proposed by [21] for weakly dependent data and [22] for observations satisfying the uniform mixing condition.

The bootstrap technique designed here attempts to provide an empirical distribution to the estimator of $\hat{\lambda}_1$ by utilizing the method of embeddings. Consider a time series of scalar observations $y_t, t = 1, 2, \ldots, T$ which are generated by the observer function $y_t = h(x_t) + \gamma\epsilon_t$. Using these observations create a series of m-histories by $y_t^m = (y_t, y_{t+1}, \ldots, y_{t+m-1})$. This converts T scalars into $T_m = T - m + 1$ vectors with overlapping entries. Under the bootstrap scheme, the resampling is done with replacement from $\{y_1^m, y_2^m, \ldots, y_{T_m}^m\}$ with y_i^m's equally likely to be drawn. Let $k = T/m$ and suppose that k m-histories are drawn and denote the

[3] In [8,7] a truncated Taylor series is used, while in [13] multilayer feedforward networks are used to calculate the function v. These studies use gradient methods (as it is done here) to calculate the Lyapunov exponents, and a comparison of the results can be found in [13]. In [11] the feedforward networks are used to calculate the largest Lyapunov exponent.

resulting sampled m-histories by z_j^m. Note that each z_j^m contains m elements which may be expressed as

$$z_j^m = (y_j, y_{j+1}, \dots, y_{j+m-1}).$$

Let $l = km$ and $\tilde{\lambda}_1$ be the largest Lyapunov exponent computed from the bootstrap sample. In order to estimate the sample distribution of $\sqrt{T}(\hat{\lambda}_1 - \lambda_1)$ under the bootstrap procedure, we may obtain a large number of bootstrap replications $\sqrt{l}(\tilde{\lambda}_1 - \hat{\lambda}_1)$ to form a histogram. This histogram is the proposed bootstrap approximation of the sampling distribution of $\sqrt{T}(\hat{\lambda}_1 - \lambda_1)$.

The bootstrap methodology utilized here requires the following steps:

(i) Using a time series of scalar observations $y_t, t = 1, 2, \dots, T$ create a series of m-histories by $y_t^m = (y_t, y_{t+1}, \dots, y_{t+m-1})$.

(ii) Select k random elements of $\{y_1^m, y_2^m, \dots, y_{T_m}^m\}$ with replacement and denote it by $z_j^m = (y_j, y_{j+1}, \dots, y_{j+m-1})$. All elements in the k sampled m-histories are then regarded as the bootstrap sample. z_j^m is used to estimate the unknown function v and construct the $(Dg)_{z_j^m}$ matrix by

$$(Dg)_{z_j^m} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ v_{1,j} & v_{2,j} & v_{3,j} & \dots & v_{m-1,j} & v_{m,j} \end{bmatrix}$$

(iii) The bootstrapped values of the largest Lyapunov exponent are calculated from the matrix (Dg^k) which is obtained by

$$(Dg^k) = (Dg)_{z_k^m}(Dg)_{z_{k-1}^m} \cdots (Dg)_{z_1^m}.$$

(iv) The final step is to calculate the bootstrapped Lyapunov exponent estimate, $\tilde{\lambda}_1$ from (Dg^k) by using the QR algorithm.

(v) The procedure above is repeated l times to obtain $\tilde{\lambda}_{1,j}, j = 1, 2, \dots, l$. These l bootstrap Lyapunov exponents are then used to construct the empirical distribution of $\sqrt{l}(\tilde{\lambda}_1 - \hat{\lambda}_1)$. In the paper we report the mean, the standard deviation and the confidence intervals for the estimated largest Lyapunov exponent $\hat{\lambda}_1$.

The intuitive explanation for this procedure is that, for a given m, it involves making random searches in the phase space of the dynamical system. For large l, there is enough degrees of freedom to cover all combinations of random samples in the phase space such that the moments of the largest Lyapunov exponent estimator obtained from the bootstrap procedure matches the moments of the distribution of the true largest Lyapunov exponent.

4. Applications

In this section, we present the results on two examples and an application to the Dow Jones Industrial Average Index series.

(i) The Hénon map

$$\begin{aligned} x_{t+1} &= 1 - 1.4x_t^2 + y_t \\ y_{t+1} &= 0.3x_t. \end{aligned} \tag{8}$$

The matrix of derivatives of the Hénon map is

$$\begin{bmatrix} -2.4x_t & 1 \\ 0.3 & 0 \end{bmatrix}. \tag{9}$$

Since the determinant of this matrix is constant, the Lyapunov exponents for this map satisfy

$$\lambda_1 + \lambda_2 = \ln(0.3) \approx -1.2. \tag{10}$$

The two largest Lyapunov exponents of the Hénon map are 0.408 and −1.620. The Hénon map is estimated based on $T = 200$ observations. In the estimation of this map, six hidden units are used in a single layer feedforward network. The degree of the measurement noise γ is set to 0.1, 0.25 and 0.50 and generated from a uniform random number generator.

(ii) A discretized variant of the Mackey-Glass delay equation,

$$x_t = x_{t-1} + \left[\frac{a x_{t-s}}{1 + (x_{t-s})^c} - b x_{t-1}\right] \tag{11}$$

where we used $a = 0.2$, $b = 0.1$, and $c = 10.0$ and $s = 17$. This equation is chosen to show

Table 1
Confidence intervals for the largest Lyapunov exponents estimates

		Hénon map	Mackey-Glass delay equation
$\gamma = 0.05$	estimated largest Lyapunov exponent	0.403	0.0081
	confidence interval	[0.377,0.439]	[0.0026,0.0144]
	mean	0.405	0.0079
	standard deviation	0.062	0.0120
$\gamma = 0.10$	estimated largest Lyapunov exponent	0.381	0.0077
	confidence interval	[0.368,0.448]	[0.0016,0.0159]
	mean	0.403	0.0077
	standard deviation	0.081	0.014
$\gamma = 0.25$	estimated largest Lyapunov exponent	0.375	0.0061
	confidence interval	[0.360,0.457]	[0.0006,0.0168]
	mean	0.401	0.0073
	standard deviation	0.094	0.016
	true largest Lyapunov exponent	0.408	0.0086

the performance of the feedforward network with higher dimensional systems and the resulting Lyapunov exponent estimates. The first two largest Lyapunov exponents of the Mackey-Glass delay equation are 0.0086 and 0.001. In the estimation of this map 16 hidden units and 3500 observations are used in a single hidden layer feedforward network. The degree of the measurement noise γ is chosen to be the same as with the Hénon map.

The results are presented in Table 1. For the Hénon map, we choose the embedding dimension to be $m = 2$ and carry out 100 replications given that $k = (T/m) = 100$. Each cell of the table reports the estimated largest Lyapunov exponent, the 5 percent confidence interval, the mean and the standard deviation of the bootstrapped Lyapunov exponent values. The lower and upper bound of the confidence interval is the 5th lowest and the 5th highest Lyapunov exponent estimates for the 100 bootstrapped samples. If the calculated Lyapunov exponent estimates are within the confidence interval then the null hypothesis is retained. For $\gamma = 0.05$, the estimated largest Lyapunov exponents of the Hénon map is 0.403. Since 0.403 is within [0.377, 0.439], the null hypothesis that 0.403 is equal to the true largest Lyapunov exponent is retained. One important thing to notice is that the mean of the bootstrap samples across the three values of γ are unbiased as it closely approximates the the true Lyapunov exponent value. Also, at higher γ values the standard deviation of the bootstrap samples gets larger which translates into larger confidence intervals. This is expected as the Lyapunov exponent estimators gets less precise at higher noise levels. At $\gamma = 0.25$, the estimated Lyapunov exponent is 0.375 and we retain the null hypothesis $\hat{\lambda}_1 = 0.375$ is not statistically different than the true largest Lyapunov exponent, $\lambda_1 = 0.408$.

Similar findings are obtained for the Mackey-Glass delay equation. The Mackey-Glass delay equation is estimated at $m = 35$ and 100 bootstrap samples are generated. At $\gamma = 0.25$, the confidence interval is [0.0006, 0.0168]. Since the estimated largest Lyapunov exponent is 0.0061, the null hypothesis that $\hat{\lambda}_1 = 0.0086$ is retained.

The procedure discussed above is applied to the daily Dow Jones Industrial Average Index (DJIA) series for the period of January 3, 1984 – June 30, 1988, a total of 1137 observations. Five hundred replications are performed at $m = 2$. The estimated largest Lyapunov exponent and the corresponding confidence interval are -1.2011 and $[-1.3523, -1.1612]$, respectively. The results clearly indicate that the DJIA series does not inherit chaotic dynamics.

5. Conclusions

One common criticism of the existing Lyapunov exponents algorithms is the absence of a distributional theory which provides a framework for the statistical hypothesis testing for the calculated Lyapunov exponents. This paper presents a methodology to construct the empirical distribution of Lyapunov exponents calculated from noisy data. It is shown that the proposed bootstrap methodology can be used to construct empirical confidence intervals for the Lyapunov exponent estimators as a statistical measure of how accurate the estimators are. The numerical examples show that the proposed methodology works well in small samples.

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