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Testing chaotic dynamics via Lyapunov exponents

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Abstract

This paper presents a bootstrap-based test statistic for testing the presence of chaotic dynamics from data by using the Lyapunov exponents. In particular, a one-sided test statistic in Gençay's [Gençay, Physica D 89 (1996) 261–266] framework is designed and its small sample properties are tested on the Hénon map. The numerical examples show that the test statistic has desirable small sample properties. Copyright © 1998 Elsevier Science B.V.

1. Introduction

In a recent paper, Gençay [1] proposed a statistical framework which, under a null hypothesis, tested for chaotic dynamics using a moving blocks bootstrap procedure. The main motivation in Gençay [1] is to design a statistical framework in which a postulated largest Lyapunov exponent of a chaotic system can be tested.

The Lyapunov exponents for a dynamical system

$$S_{t+1} = f(S_t), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and S_t is the state of the system, are measures of the average degree of divergence of nearby orbits in phase-space. If the dynamical system is dissipative, a positive Lyapunov exponent is an indication of chaotic dynamics, i.e. the system has a property of "sensitive dependence on initial conditions": any two solution paths with starting values arbitrarily close together but not equal will diverge at exponential rates. However, globally the dynamics of the system will remain within a bounded set.

Associate with the dynamical system in Eq. (1) an observer function $h: \mathbb{R}^n \to \mathbb{R}$, which generates the observations

$$x_t = h(S_t) + \gamma \varepsilon_t, \quad \varepsilon_t \sim \mathrm{IID}(0, 1),$$

where γ is the noise level and ε_t is the measurement error.

Gençay [1] formulates the null hypothesis, H_0, and the alternative hypothesis, H_1, as

$$\mathrm{H}_0: \lambda^{\max} = \lambda_1 \ \text{ vs. } \ \mathrm{H}_1: \lambda^{\max} \neq \lambda_1,$$

where $\lambda^{\max}$ is the unknown parameter (largest Lyapunov exponent) which is set to some hypothesized value λ_1, which may be the true largest Lyapunov exponent if known or some other value.

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2. Test for chaos

The focus of this paper is to design a one-sided alternative hypothesis such that the one-sided 97.5% confidence interval is constructed by calculating the critical value as $\hat{\lambda}_1 - q(97.5\%)$, following from $E_q\{\Pr((\hat{\lambda}_1 - \lambda_1) \leq q(97.5\%))\} = 0.975$, where $\hat{\lambda}_1$ is an estimate of the largest Lyapunov exponent, and $q(97.5\%)$ is the quantile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$ [2].

The null hypothesis, H_0, and the alternative hypothesis, H_1, in this case are

$$H_0\colon \lambda^{\max} = 0 \quad \text{vs.} \quad H_1\colon \lambda^{\max} > 0.$$

Then, if $\hat{\lambda}_1 - q(97.5\%) > 0$, the null hypothesis is rejected, which means that the dynamics is chaotic. Under the null hypothesis the dynamics is non-chaotic.

3. Numerical examples

The framework is tested on the Hénon map which is described by the following system:

$$x_{t+1} = y_t + 1 - 1.4x_t^2, \quad y_{t+1} = 0.3x_t.$$

Depending on the initial point, (x_0, y_0), the sequence of points obtained by iteration of the mapping either diverges to infinity or tends to a strange attractor. When the initial point is $(x_0, y_0) = (0, 0)$, as in our tests, the sequence of points $\{(x_0, y_0), (x_1, y_1), \ldots\}$ tends to a strange attractor.

To construct a test sample we use the time series $\{x_{1001}, \ldots, x_{1000+N}\}$, given the initial point $(x_0, y_0) = (0, 0)$, and add measurement errors, ε_t, with the noise level, γ, i.e.

$$x_t^{\mathrm{o}} = x_t + \gamma\varepsilon_t, \quad \varepsilon_t \sim \mathrm{NID}(0, 1).$$

This means that we observe the time series $\{x_{1001}^{\mathrm{o}}, \ldots, x_{1000+N}^{\mathrm{o}}\}$. The test results, for significance level 0.025, are found in Tables 1 and 2.

Table 1
The Hénon map when $N = 200$

Block size	$\gamma = 0.05$		$\gamma = 0.25$	
	$\hat{\lambda}_1$	Critical value	$\hat{\lambda}_1$	Critical value
2	0.440	0.385	0.430	0.401
4	0.270	0.145	0.274	0.211
6	0.236	0.176	0.206	0.162
8	0.218	0.198	0.139	0.092
10	0.197	0.199	0.122	0.094

The number of bootstrap values is 400.

Table 2
The Hénon map when $N = 1000$

Block size	$\gamma = 0.05$		$\gamma = 0.25$	
	$\hat{\lambda}_1$	Critical value	$\hat{\lambda}_1$	Critical value
2	0.558	0.514	0.567	0.583
4	0.358	0.238	0.362	0.339
6	0.274	0.188	0.247	0.218
8	0.243	0.198	0.181	0.155
10	0.226	0.218	0.144	0.125

The number of bootstrap values is 400.

In all cases the null hypothesis is rejected. This is in accordance with the true largest Lyapunov exponent which is here $\lambda_1^{\mathrm{true}} = 0.408$.

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