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# The topological invariance of Lyapunov exponents in embedded dynamics

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## Abstract

There are two contributions of this paper. One is to provide a theoretical ground for the Jacobian methods by showing that  $n$  of the Lyapunov exponents of the estimated Jacobian are the Lyapunov exponents of an  $n$  dimensional unknown dynamical system. The second is to show that it is necessary that the Lyapunov exponents be calculated in the tangent space of the attractor in the embedding.

The method of reconstruction is to form vectors of  $m$  consecutive observations, which for  $m > 2n$  is generically an embedding. This is Takens' result. A function of  $m$  variables is then fit to the data and the Jacobian matrix is constructed at each point in the orbit of the data. When embedding occurs at dimension  $m = n$ , then the Lyapunov exponents of the reconstructed dynamics are the Lyapunov exponents of the original dynamics. This is the case for the Hénon, the Lorenz and the Mackey–Glass systems with a first component observer. However, if embedding only occurs for an  $m > n$ , then the Jacobian method yields  $m$  Lyapunov exponents, only  $n$  of which are the Lyapunov exponents of the original system. The problem is that as currently used, the Jacobian method is applied to the full  $m$ -dimensional space of the reconstruction, and not just to the  $n$ -dimensional manifold that is the image of the embedding map. Our examples show that it is possible to get 'spurious' Lyapunov exponents that are even larger than the largest Lyapunov exponent of the original system.

## 1. Introduction

Lyapunov exponents measure the average exponential divergence or convergence of nearby initial points in the phase space of a dynamical system. A positive Lyapunov exponent is a measure of the average exponential divergence of two nearby trajectories whereas a negative Lyapunov exponent is a measure of the average exponential convergence of two nearby trajectories. If a discrete nonlinear system is dissipative, a positive Lyapunov exponent is an indication that the system is chaotic.

Last decade led to extensive research in the calculation of Lyapunov exponents of an unknown dynamical system from observations. The first came from Wolf et al. [30] which was widely applied in a very broad range of disciplines. Later, Briggs [4] suggested using Jacobian methods to calculate Lyapunov exponents by least-squares polynomial fitting. Zeng et al. [31] emphasized the computation of Lyapunov exponents from limited experimental data by designing a new procedure by which one can evaluate the Lyapunov exponent spectrum from relatively small data sets of low precision. An excellent review of various

methods for computing the Lyapunov exponents from time series is reviewed by Zeng et al. [32].

The introduction of Lyapunov exponents to economics was in Brock [5]. Brock and Sayers [6] note that the Wolf algorithm is sensitive to the number of observations as well as the degree of measurement or system noise in the observations. Frank et al. [15,16] provide evidence that the Kurths and Herzel [20] algorithm has a positive bias. This observation started a search for new algorithmic designs with improved finite sample properties. The search for an algorithm to calculate Lyapunov exponents with desirable finite sample properties has gained momentum in the last couple of years. Three groups of researchers, Abarbanel et al. [7,2,1,3], McCaffrey et al. [22,14] and Gençay and Dechert [18,10], researchers from three different disciplines, came up with improved algorithms for the calculation of the Lyapunov exponents from observed data. Gençay [17] worked on the calculation of the Lyapunov exponents with noisy data when feedforward networks are used as the estimation technique.

The main algorithmic design in [7,2,1,3] and [18,10] is to embed the observations in an  $m$ -dimensional space, then by theorems of Mañé [21] and Takens [29] the observations are used to reconstruct the dynamics on the attractor. The Jacobian of the reconstructed dynamics as demonstrated in Eckmann et al. [12,13] is then used to calculate the Lyapunov exponents of the unknown dynamics.

There are two contributions of this paper. One is to provide a theoretical ground for the Jacobian methods of [12,13] by showing that  $n$  of the Lyapunov exponents of the estimated Jacobian are the Lyapunov exponents of the unknown dynamical system. The second is to show that it is necessary that the Lyapunov exponents be calculated in the tangent space of the attractor in the embedding.

As we show below, the reconstructed Jacobian is known, except for its first row. It is therefore possible to calculate all  $n$  Lyapunov exponents of an unknown dynamical system with a degree of accuracy determined by the degree of accuracy of the estimates of the derivatives in the first row. Indeed [18] shows that all Lyapunov exponents of the Hénon map, Lorenz

system and the Mackey–Glass delay equation can be estimated from data with small error and with reasonably small data sets. This is the case when the observer function is a projection onto the first coordinate of these systems.

The method of reconstruction is to form vectors of  $m$  consecutive observations, which for  $m > 2n$  is generically an embedding. This is Takens' result. A function of  $m$  variables is then fit to the data and the Jacobian matrix is constructed at each point in the orbit of the data. When embedding occurs at dimension  $m = n$ , then the Lyapunov exponents of the reconstructed dynamics are the Lyapunov exponents of the original dynamics. This is the case for the Hénon, the Lorenz and the Mackey–Glass systems with a first component observer. However, if embedding only occurs for an  $m > n$ , then the Jacobian method yields  $m$  Lyapunov exponents, only  $n$  of which are the Lyapunov exponents of the original system. The problem is that as currently used, the Jacobian method is applied to the full  $m$ -dimensional space of the reconstruction, and not just to the  $n$ -dimensional manifold that is the image of the embedding map. Our examples show that it is possible to get 'spurious' Lyapunov exponents that are even larger than the largest Lyapunov exponent of the original system.

The existence of the spurious Lyapunov exponents of a calculated attractor is also studied by Čenys [8]. Čenys [8] indicates the existence of two types of Lyapunov exponents which are labelled as the internal and the external Lyapunov exponents. The internal exponents do not depend on the map-generating attractor and are completely defined by the trajectories on the attractor. On the other hand, external Lyapunov exponents depend on the approximating map than of the attractor and can take any value. Parlitz [24] focuses on the identification of the spurious Lyapunov exponents by presenting a method for experimental data. This method is based on the observation that the true Lyapunov exponents change their signs upon time reversal whereas the spurious exponents do not. Parlitz's [24] method can be a useful tool for identification purposes, especially for continuous time systems. For discrete chaotic systems it is not possible in general to run time backwards, since the dynamics are not one-

to-one.

In Section 2 we briefly give the definition of Lyapunov exponents which we utilize here. Section 3 is on the main theorem of the paper. Section 4 discusses the identification issues of spurious Lyapunov exponents. The numerical results are presented in Section 5. We conclude thereafter.

## 2. Lyapunov exponents

The Lyapunov exponents for a dynamical system,  $f : R^n \rightarrow R^n$ , with the trajectory,

$$x_{t+1} = f(x_t), \quad t = 0, 1, 2, \dots, \tag{2.1}$$

are measures of the average rate of divergence or convergence of a typical trajectory.<sup>1</sup> For an  $n$ -dimensional system as above, there are  $n$  exponents which are customarily ranked from largest to smallest:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

*Definition 2.1.* Let  $f : R^n \rightarrow R^n$  define a discrete dynamical system and select a point  $x \in R^n$ . Let  $(Df)_x$  be the matrix of partial derivatives of  $f$  evaluated at the point  $x$ . Suppose that there are subspaces  $R^n = V_1^1 \supset V_1^2 \dots \supset V_1^{n+1} = \{0\}$  in the tangent space of  $R^n$  at  $f^t(x)$  and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  such that

- (a)  $(Df^t)_x(V_1^j) \subseteq V_1^{j-1}$
- (b)  $\dim V_1^j = n + 1 - j$
- (c)  $\lambda_j = \lim_{t \rightarrow \infty} t^{-1} \ln \|(Df^t)_x v\|$  for all  $v \in V_0^j \setminus V_0^{j+1}$

then the  $\lambda_j$  are called the Lyapunov exponents of  $f$ .

The subspace  $V_0^1 \setminus V_0^2$  consists of those vectors that grow at the fastest average rate.  $V_0^2 \setminus V_0^3$  consists of those vectors that grow at the next most rapid rate, etc.

It is a consequence of Oseledec's Theorem [23], that the Lyapunov exponents exist for a broad class

of functions.<sup>2</sup> The additional properties of Lyapunov exponents and a formal definition<sup>3</sup> are given in [19]. Notice that for  $j \geq 2$  the subspaces  $V^j$  are sets of Lebesgue measure zero, and so for almost all  $v \in R^n$  the limit in part (c) of Definition 2.1 equals  $\lambda_1$ . This is the basis for the computational algorithm of [30] which is a method for calculating the largest Lyapunov exponent.

### An example

If  $x$  is a fixed point, then the subspaces  $V_t^j = V^j$  do not depend upon  $t$ . Let us consider the mapping  $f(x)$  at the fixed point  $x$ . Choose  $V^1 = R^2$ ,  $V^2 = \text{span}\{(0, 1)\}$  and  $V^3 = \{0\}$ . For  $|\mu_1| > |\mu_2|$  consider<sup>4</sup>

$$Df(x) = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}. \tag{2.2}$$

This will satisfy parts (1) and (2) of Definition 2.1 and we will have

$$\lambda_1 = \lim_{t \rightarrow \infty} t^{-1} \ln(|\mu_1^t v_1 + \mu_2^t v_2|) = \ln |\mu_1|$$

$$\text{for } v \in V^1 \setminus V^2$$

$$\lambda_2 = \lim_{t \rightarrow \infty} t^{-1} \ln(|\mu_1^t v_1 + \mu_2^t v_2|) = \ln |\mu_2|$$

$$\text{for } v \in V^2 \setminus V^3.$$

This definition mainly generalizes the idea of eigenvalues to give average linearized contraction and expansion rates on a trajectory.

An attractor is a set of points towards which the trajectories of  $f$  converge. More precisely,  $A$  is an attractor if there is an open set  $U \subset R^n$  with  $A \subset U$ ,  $f(\bar{U}) \subset U$  and

$$A = \bigcap_{t \geq 0} f^t(U)$$

<sup>2</sup> Also see [25,26,9] for precise conditions and proofs of the theorem.

<sup>3</sup> Definition 2.1 differs slightly from [19] in that we use set containment rather than set equality in part (a) of the definition. When equality holds in part (a) we will say that the dynamics are of full dimension.

<sup>4</sup> This example is from [19, p. 284].

<sup>1</sup> The trajectory is also written in terms of the iterates of  $f$ . With the convention that  $f^0$  is the identity map, and  $f^{t+1} = f \circ f^t$ , then we also write,  $x_t = f^t(x_0)$ . A trajectory is also called an orbit in the dynamical system literature.

where  $\bar{U}$  is the closure of  $U$ .<sup>5</sup> The attractor  $A$  is said to be indecomposable if there is no proper subset of  $A$  which is also an attractor. An attractor can be chaotic or ordinary (or non-chaotic). There is more than one definition of a chaotic attractor in the literature. In practice the presence of a positive Lyapunov exponent is taken as a signal that the attractor is chaotic.

### 3. Theoretical results

In practice one rarely has the advantage of observing the state of the system,  $x_t$ , let alone knowing the actual functional form  $f$  that generate the dynamics. The model that is widely used is the following: associated with the dynamical system in Eq. (2.1) there is an observer function  $h : R^n \rightarrow R$  which generates the observations,

$$y_t = h(x_t).$$

It is assumed that all that is available to the researcher is the sequence  $\{y_t\}$ . For notational purposes, let

$$y_t^m = (y_t, y_{t+1}, \dots, y_{t+m-1}). \quad (3.1)$$

If the set  $\bar{U}$  is a compact manifold then for  $m \geq 2n + 1$

$$J^m(x) = (h(x), h(f(x)), \dots, h(f^{m-1}(x))) \quad (3.2)$$

generally is an embedding.<sup>6</sup> For  $m \geq 2n + 1$  there exists a function  $g : R^m \rightarrow R^m$  such that

$$y_{t+1}^m = g(y_t^m)$$

where

$$y_{t+1}^m = (y_{t+1}, y_{t+2}, \dots, y_{t+m}).$$

But notice that

<sup>5</sup> More generally, the set  $R^n$  is replaced with a topological space  $X$ . This is necessary, for example, in the case of the Logistic map,  $f(x) = 4x(1-x)$ , in order for the interval  $[0, 1]$  to be an attractor. See [11, Definition 5.3, p. 203].

<sup>6</sup> By generic is meant that in every neighborhood of  $f$  and  $h$  there are functions  $\tilde{f}$  and  $\tilde{h}$  so that the function  $J^m$  corresponding to these functions is an embedding of the attractor of  $\tilde{f}$  and the image of the image of the attractor under  $J^m$ . Here  $2n + 1$  is the worst-case upper limit.

$$y_{t+1}^m = J^m(x_{t+1}) = J^m(f(x_t)). \quad (3.3)$$

Hence from Eqs. (3.1) and (3.3)

$$J^m(f(x_t)) = g(J^m(x_t)).$$

The function  $g$  is topologically conjugate to  $f$ . This implies that  $g$  inherits the dynamical properties of  $f$ .

*Theorem 3.1.* Assume that  $M$  is a smooth manifold dimension  $n$ ,  $f : M \rightarrow M$  and  $h : M \rightarrow R$  are (at least)  $C^2$ . Define  $J^m : M \rightarrow R^m$  by  $J^m(x) = (h(x), h(f(x)), \dots, h(f^{m-1}(x)))$ . Let  $\mu_1(x) \geq \mu_2(x) \geq \dots \geq \mu_n(x)$  be the eigenvalues of the symmetric matrix  $(DJ^m)'_x(DJ^m)_x$ , and suppose that

$$\inf_{x \in M} \mu_n(x) > 0, \\ \sup_{x \in M} \mu_1(x) < \infty.$$

Let  $\lambda_1^f \geq \lambda_2^f \geq \dots \geq \lambda_n^f$  be the Lyapunov exponents of  $f$  and  $\lambda_1^g \geq \lambda_2^g \geq \dots \geq \lambda_m^g$  be the Lyapunov exponents of  $g$ , where  $g : J^m(M) \rightarrow J^m(M)$  and  $J^m(f(x)) = g(J^m(x))$  on  $M$ . Then generically  $\{\lambda_i^f\} \subset \{\lambda_i^g\}$ .<sup>7</sup>

*Proof.* (Notation: for any differentiable function,  $\phi : R^k \rightarrow R^l$ ,  $(D\phi)_x$  denotes the  $l \times k$  matrix of partial derivatives of  $\phi$ .) Let  $x_0 \in M$ ,  $y_0 = J^m(x_0)$ , and let

$$A_t = (Df)_{x_{t-1}}(Df)_{x_{t-2}} \dots (Df)_{x_0} = (Df^t)_{x_0} \\ B_t = (Dg)_{y_{t-1}}(Dg)_{y_{t-2}} \dots (Dg)_{y_0} = (Dg^t)_{y_0}.$$

For ease of notation let

$$F_t = (Df)_{x_t} \\ G_t = (Dg)_{y_t} \\ J_t = (DJ^m)_{x_t}.$$

Then since  $J^m \circ f(x_t) = g \circ J^m(x_t)$ ,

$$J_{t+1}F_t = G_tJ_t \quad t = 0, 1, \dots \quad (3.4)$$

<sup>7</sup> The results of this paper are for the case that the attractor is a smooth manifold. Sauer et al [28] show how to extend the Takens Theorem [29] to attractors with fractal dimension. Note that in Section 5 we show that our theoretical results hold for the Hénon map.

Also, since  $\inf \mu_n(x) > 0$ ,  $J_t J_t$  is invertible for all  $t$ . Multiply equation (3.4) by the projection

$$P_{t+1} = J_{t+1}(J'_{t+1}J_{t+1})^{-1}J'_{t+1}$$

to get

$$J_{t+1}F_t = P_{t+1}G_tJ_t,$$

so that

$$P_{t+1}G_tJ_t = G_tJ_t.$$

Therefore for all  $v \in R^n$ ,

$$G_tJ_tv \in P_{t+1}(R^m).$$

If we multiply Eq. (3.4) by

$$(J'_{t+1}J_{t+1})^{-1}J'_{t+1}$$

we get

$$F_t = (J'_{t+1}J_{t+1})^{-1}J'_{t+1}G_tJ_t$$

and

$$\begin{aligned} F_{t-1}F_{t-2}\dots F_0 \\ = (J'_tJ_t)^{-1}J'_tG_{t-1}P_{t-1}G_{t-2}P_{t-2}\dots P_1G_0J_0 \end{aligned}$$

and hence

$$J_tF_{t-1}F_{t-2}\dots F_0 = P_tG_{t-1}P_{t-1}G_{t-2}\dots P_1G_0J_0.$$

But

$$P_sG_{s-1}J_{s-1} = G_{s-1}J_{s-1} \text{ for } s = t, t-1, \dots, 1$$

and so

$$J_tF_{t-1}F_{t-2}\dots F_0 = G_{t-1}G_{t-2}\dots G_0J_0$$

or

$$J_tA_t = B_tJ_0.$$

Next, let

$$R^n = V_0^1 \supset V_0^2 \supset \dots \supset V_0^n \supset V_0^{n+1} = \{0\}$$

be the subspaces in the tangent space at  $x_0$  such that

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{-1} \ln |A_tv| = \lambda_j^f, \quad v \in V_0^j \setminus V_0^{j+1}, \\ \text{for } j = 1, \dots, n. \end{aligned}$$

Let  $v \in V_0^j \setminus V_0^{j+1}$  and  $w = J_0v$ . Then

$$J_tA_tv = B_tJ_0v = B_tw$$

and so

$$\lim_{t \rightarrow \infty} t^{-1} \ln |J_tA_tv| = \lim_{t \rightarrow \infty} t^{-1} \ln |B_tw| \quad (3.5)$$

if these limits exist. Now,

$$\sqrt{\mu_n(x_t)}|A_tv| \leq |J_tA_tv| \leq \sqrt{\mu_1(x_t)}|A_tv|$$

and so

$$\begin{aligned} \frac{\ln \mu_n(x_t)}{2} + \ln |A_tv| &\leq \ln |J_tA_tv| \\ &\leq \frac{\ln \mu_1(x_t)}{2} + \ln |A_tv|. \end{aligned}$$

Since by assumption both  $t^{-1} \ln \mu_n(x_t) \rightarrow 0$  and  $t^{-1} \ln \mu_1(x_t) \rightarrow 0$ ,

$$\lim_{t \rightarrow \infty} t^{-1} \ln |J_tA_tv| = \lambda_j^f.$$

Therefore, by Eq. (3.5)

$$\lim_{t \rightarrow \infty} t^{-1} \ln |B_tw| = \lambda_j^f$$

and hence  $\lambda_j^f$  is one of the eigenvalues of  $g$ .<sup>8</sup>  $\square$

By Theorem 3.1,  $n$  of the Lyapunov exponents of  $g$  are the Lyapunov exponents of  $f$ . Our approach is to estimate the function  $g$  based on the data sequence  $\{J^m(x_t)\}$ , and to calculate the Lyapunov exponents of  $g$ . The identification of the  $n$  Lyapunov exponents of  $f$  from the  $m$  Lyapunov exponents of  $g$  is discussed in Section 4.

From Eq. (3.1) the mapping  $g$  which is to be estimated may be taken<sup>9</sup> to be

<sup>8</sup>Note that the Euclidean norm on  $R^n$  has been used in the proof. However, since the Lyapunov exponents are independent of (equivalent) norm, the result holds in any norm which is equivalent to the Euclidean norm.

<sup>9</sup>Here, the time step is assumed to be equal to the delay time.

$$g : \begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+m-1} \end{bmatrix} \rightarrow \begin{bmatrix} y_{t+1} \\ y_{t+2} \\ \vdots \\ v(y_t, y_{t+1}, \dots, y_{t+m-1}) \end{bmatrix} \quad (3.6)$$

and this reduces to estimating

$$y_{t+m} = v(y_t, y_{t+1}, \dots, y_{t+m-1}).$$

Here  $v$  is an unknown map. We (and others) use a specification-free estimation technique.<sup>10</sup>

Linearization of the map  $g$  yields

$$\Delta y_{t+1}^m = (Dg)_{y_t^m} \Delta y_t^m.$$

The solution can be written as

$$\Delta y_t^m = (Dg^t)_{y_0^m} \Delta y_0^m$$

where

$$(Dg^t)_{y_0^m} = (Dg)_{y_{t-1}^m} (Dg)_{y_{t-2}^m} \cdots (Dg)_{y_0^m}$$

and

$$(Dg)_{y_t^m} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ v_1 & v_2 & v_3 & \dots & v_{m-1} & v_m \end{bmatrix}$$

where

$$v_m = \frac{\partial v}{\partial y_{t+m-1}}, \dots, v_1 = \frac{\partial v}{\partial y_t}.$$

The Lyapunov exponents can be calculated from the eigenvalues of the matrix  $Dg^m$  using  $QR$  decomposition. This method is discussed in [12,13,27] and a modified version is presented in [3].

### Continuation of the example in Section 2

As in Section 3 consider a map with a fixed point. Suppose that the observations come from the following:

$$y = h(x) = x_1 + x_2 \quad (3.7)$$

where  $h : R^2 \rightarrow R$ . Let us consider a 3-embedding history generated from  $h(x)$  so that,

$$J^3(x) = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \end{bmatrix} x \quad (3.8)$$

and

$$J^3 \circ f(x) = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \\ \mu_1^3 & \mu_2^3 \end{bmatrix} x. \quad (3.9)$$

Let

$$g(y) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\mu_1 \mu_2 & \mu_1 + \mu_2 \end{bmatrix} y$$

for  $y \in R^3$ . Then,

$$g \circ J^3(x) = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \\ \mu_1^3 & \mu_2^3 \end{bmatrix} x = J^3 \circ f(x).$$

Therefore, the condition for conjugacy is satisfied. Also,

$$(Dg)_y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\mu_1 \mu_2 & \mu_1 + \mu_2 \end{bmatrix}. \quad (3.10)$$

Let  $W^1 = R^3$ ,  $W^2 = \text{span}\{(1, 0, 0), (1, \mu_2, \mu_2^2)\}$ ,  $W^3 = \text{span}\{(1, 0, 0)\}$  and  $W^4 = \{0\}$ . Then,

$$(Dg)_y(W^1) = \text{span}\{(1, \mu_1, \mu_1^2), (1, \mu_2, \mu_2^2)\} \subset W^1,$$

$$(Dg)_y(W^2) = \text{span}\{(1, \mu_2, \mu_2^2)\} \subset W^2$$

and

$$(Dg)_y(W^3) = \{0\} \subset W^3.$$

(Notice that the sets  $(Dg)_y W^j$  can be proper subsets of  $W^j$ . See footnote 3. In this example, this comes

<sup>10</sup> In [2,7] a truncated Taylor series is used, while in [18] multilayer feedforward networks are used to calculate the function  $v$ . These studies use gradient methods (as we do here) to calculate the Lyapunov exponents, and a comparison of the results can be found in [18]. In [22] the feedforward networks are used to calculate the largest Lyapunov exponent.

about since the dynamics of  $g$  are not of full dimension, which is immediately apparent from Eq. (3.10.) If  $v \in V^1 \setminus V^2$  then

$$v = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \alpha \neq 0,$$

and

$$(DJ^3)v = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}.$$

Here,  $\alpha \neq 0$  implies that  $(DJ^3)v \in W^1 \setminus W^2$ . If  $v \in V^2 \setminus V^3$  then

$$v = \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \beta \neq 0,$$

and

$$(DJ^3)v = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}.$$

Also  $\beta \neq 0$  implies that  $(DJ^3)v \in W^2 \setminus W^3$ . If  $w \in W^1 \setminus W^2$  then

$$w = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \alpha \neq 0,$$

and

$$|(Dg^t)'_y w| = \left| \alpha \mu_1^t \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} \right|.$$

Hence  $\lim_{t \rightarrow \infty} t^{-1} \ln |(Dg^t)'_y w| = \ln |\mu_1|$ .

If  $w \in W^2 \setminus W^3$  then

$$w = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \beta \neq 0,$$

and

$$|(Dg^t)'_y w| = \left| \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} \right|.$$

Hence  $\lim_{t \rightarrow \infty} t^{-1} \ln |(Dg^t)'_y w| = \ln |\mu_2|$ .

If  $w \in W^3 \setminus W^4$  then

$$w = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma \neq 0,$$

and  $|(Dg^t)'_y w| = 0$ . Therefore  $\lim_{t \rightarrow \infty} t^{-1} \times \ln |(Dg^t)'_y w| = -\infty$ . This example shows Theorem 3.1 at work. The two largest Lyapunov exponents of  $g$  are the Lyapunov exponents of  $f$ , and in this example the 'spurious' third exponent of  $g$  is  $-\infty$ .

#### 4. Spurious Lyapunov exponents

In [10,18] the numerical studies showed the  $n$  Lyapunov exponents of  $f$  turned out to be the largest  $n$  Lyapunov exponents of  $g$ .

These results were obtained using an observation function of the form:

$$h(x_1, x_2, \dots, x_n) = x_1 \quad (4.1)$$

which has been widely used in simulation studies of nonlinear dynamical systems. As Čenys pointed out, all Lyapunov exponents of the Hénon map, Lorenz and Rossler equations are internal whereas the distinction between the internal and external exponents arise for the Mackey–Glass delay equation. It is therefore possible to get spurious Lyapunov exponents which are larger than the true Lyapunov exponents even with the observer function in (4.1).

We will comment more on the lack of uniqueness of the function  $g$  below and the consequences of choosing different observation functions of the same attractor in Section 5.

Consider the following variation on the example of Section 2. The dynamics are the same linear dynamics of Eq. (2.2) and the observation function is the same as equation (3.7). From this we obtain the same embedding equations as (3.8) and (3.9). Now however, consider the following function  $g$ : for any  $a \in \mathbb{R}$ , let

$$g(y) = \begin{bmatrix} a - a(\mu_1^{-1} + \mu_2^{-1}) & a\mu_1^{-1}\mu_2^{-1} \\ 0 & 0 & 1 \\ 0 & -\mu_1\mu_2 & \mu_1 + \mu_2 \end{bmatrix} y \quad (4.2)$$

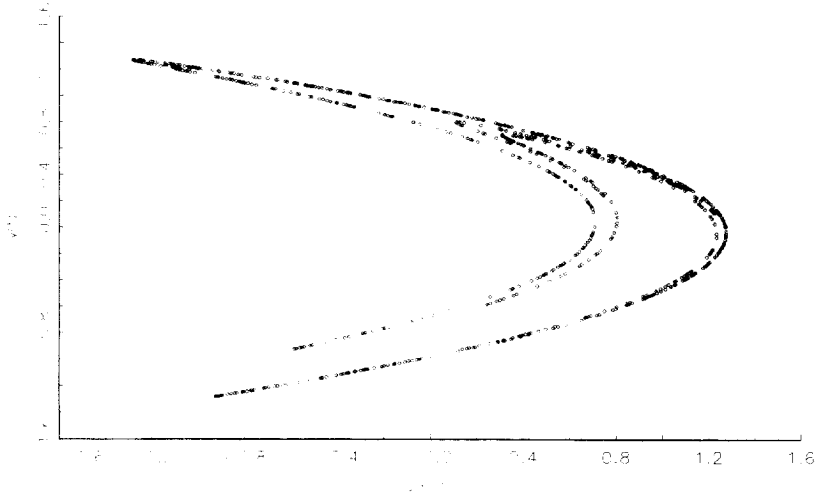


Fig. 1. Hénon map. Observer:  $y(t) = x(t) + \alpha x(t - 1)$ ,  $\alpha = 0.0$ .

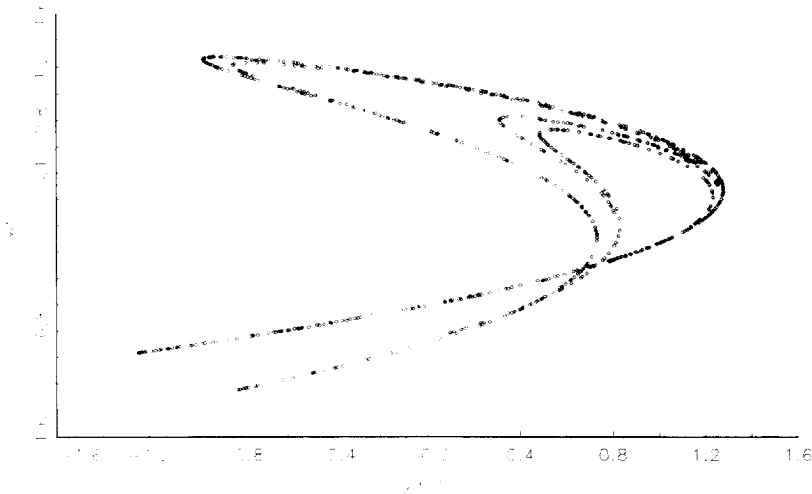


Fig. 2. Hénon map. Observer:  $y(t) = x(t) + \alpha x(t - 1)$ ,  $\alpha = 0.25$ .

for  $y \in \mathbb{R}^3$ . Notice that this is not in the form of Eq. (3.6), however it does satisfy

$$g \circ J^3(x) = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \\ \mu_1^3 & \mu_2^3 \end{bmatrix} x = J^3 \circ f(x).$$

and therefore the condition for conjugacy is satisfied.<sup>11</sup> Also,

<sup>11</sup> This shows that there can be many functions that can generate the same dynamics. In our case we are interested in the impact that the observer function has on this multiplicity of representations,

$$(Dg)_y = \begin{bmatrix} a & 1 - a(\mu_1^{-1} + \mu_2^{-1}) & a\mu_1^{-1}\mu_2^{-1} \\ 0 & 0 & 1 \\ 0 & -\mu_1\mu_2 & \mu_1 + \mu_2 \end{bmatrix}.$$

If  $|\mu_2| > |a|$ , let  $W^1 = \mathbb{R}^3$ ,  $W^2 = \text{span}\{(1, 0, 0), (1, \mu_2, \mu_2^2)\}$ ,  $W^3 = \text{span}\{(1, 0, 0)\}$  and  $W^4 = \{0\}$ . Then if  $a = 0$ ,

$$(Dg)_y(W^1) = \text{span}\{(1, \mu_1, \mu_1^2), (1, \mu_2, \mu_2^2)\} \subset W^1,$$

g. Compare this to Čenys [8] who also shows that there can be many functions that generate the same data.

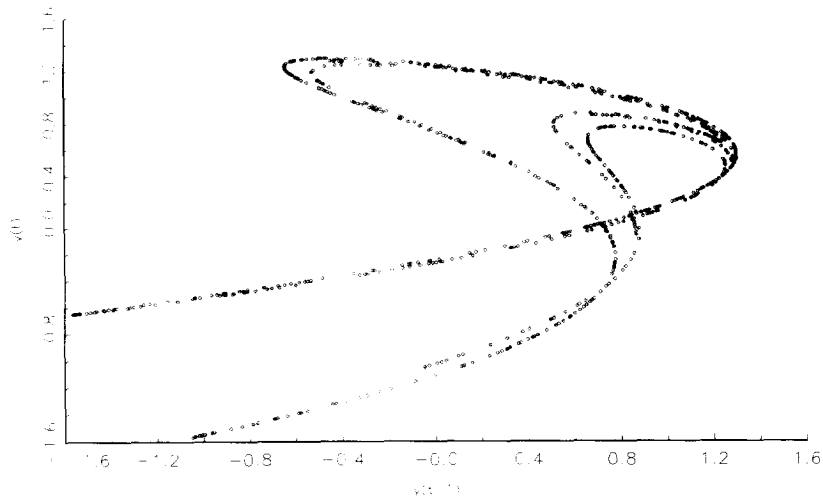


Fig. 3. Hénon map. Observer:  $y(t) = x(t) + \alpha x(t - 1)$ ,  $\alpha = 0.50$ .

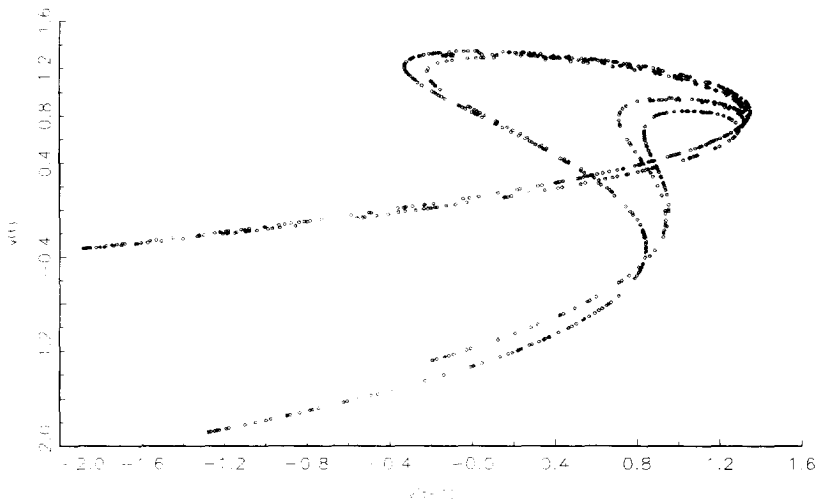


Fig. 4. Hénon map. Observer:  $y(t) = x(t) + \alpha x(t - 1)$ ,  $\alpha = 0.75$ .

$$(Dg)_y(W^2) = \text{span}\{(1, \mu_2, \mu_2^2)\} \subset W^2$$

and

$$(Dg)_y(W^3) = \{0\} \subset W^3.$$

(Notice that the sets  $(Dg)_y W^j$  can be proper subsets of  $W^j$ . See footnote 3.) If  $a \neq 0$  then

$$(Dg)_y(W^1) = W^1,$$

$$(Dg)_y(W^2) = W^2$$

and

$$(Dg)_y(W^3) = W^3.$$

If  $v \in V^1 \setminus V^2$  then

$$v = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \alpha \neq 0,$$

and

$$(DJ^3)v = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}.$$

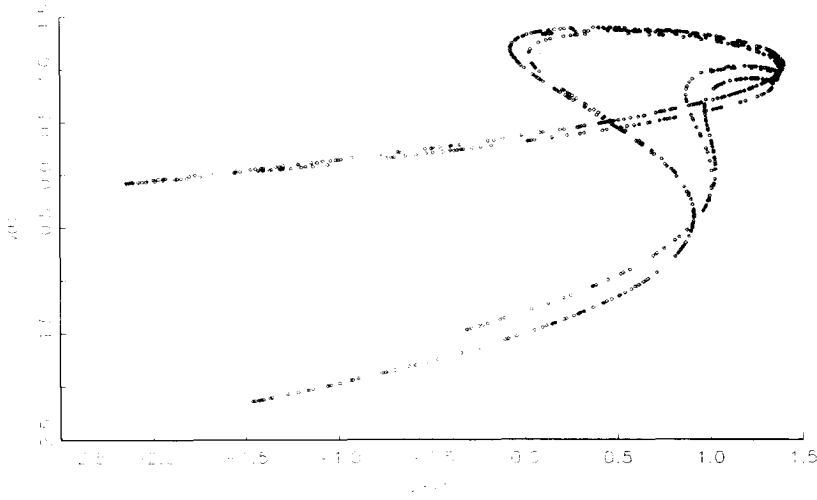


Fig. 5. Hénon map. Observer:  $y(t) = x(t) + \alpha x(t - 1)$ ,  $\alpha = 0.95$ .

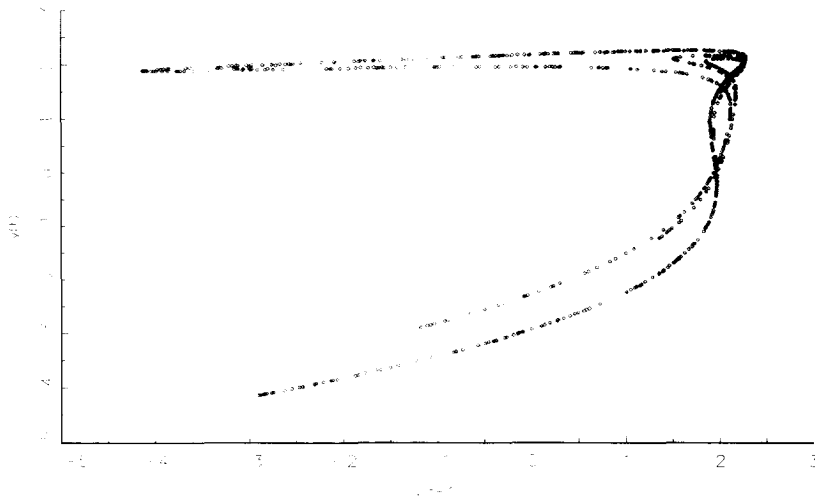


Fig. 6. Hénon map. Observer:  $y(t) = x(t) + \alpha x(t - 1)$ ,  $\alpha = 2.50$ .

Here,  $\alpha \neq 0$  implies that  $(DJ^3)v \in W^1 \setminus W^2$ . If  $v \in V^2 \setminus V^3$  then

$$v = \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \beta \neq 0,$$

and

$$(DJ^3)v = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}.$$

Also  $\beta \neq 0$  implies that  $(DJ^3)v \in W^2 \setminus W^3$ . If  $w \in$

$W^1 \setminus W^2$  then

$$w = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \alpha \neq 0,$$

and

$$|(Dg^t)_{y,w}| = \left| \alpha \mu_1^t \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma a^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right|.$$

Hence  $\lim_{t \rightarrow \infty} t^{-1} \ln |(Dg^t)_{y,w}| = \ln |\mu_1|$ .

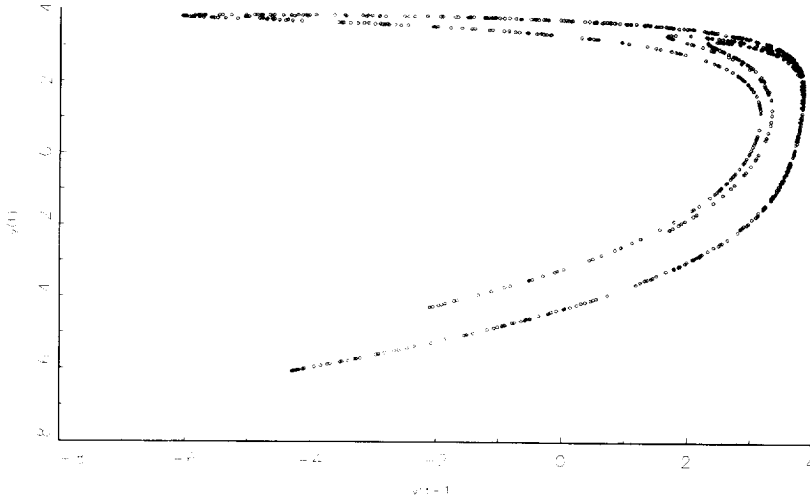


Fig. 7. Hénon map. Observer:  $y(t) = x(t) + \alpha x(t - 1), \alpha = 4.00$ .

If  $w \in W^2 \setminus W^3$  then

$$w = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \beta \neq 0,$$

and

$$|(Dg^t)_{y,w}| = \left| \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma a^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right|.$$

Hence  $\lim_{t \rightarrow \infty} t^{-1} \ln |(Dg^t)_{y,w}| = \ln |\mu_2|$ .

If  $w \in W^3 \setminus W^4$  then

$$w = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma \neq 0,$$

and  $|(Dg^t)_{y,w}| = |\gamma||a|^t$ . Therefore  $\lim_{t \rightarrow \infty} t^{-1} \times \ln |(Dg^t)_{y,w}| = \ln |a|$ . Note that if  $a = 0$  then this third ‘spurious’ Lyapunov exponent is  $-\infty$ .

If  $|\mu_1| > |a| > |\mu_2|$  then the subspace  $W^3$  above needs to be changed so that  $W^3 = \text{span}\{(1, \mu_2, \mu_2^2)\}$ . Then

$$(Dg)_y(W^1) = W^1,$$

$$(Dg)_y(W^2) = W^2$$

and

$$(Dg)_y(W^3) = W^3.$$

The three Lyapunov exponents are:  $\ln |\mu_1|, \ln |a|, \ln |\mu_2|$ . If  $|a| > |\mu_1|$  then change the subspaces so that  $W^2 = \text{span}\{(1, \mu_1, \mu_1^2), (1, \mu_2, \mu_2^2)\}$ ,  $W^3 = \text{span}\{(1, \mu_2, \mu_2^2)\}$  and again

$$(Dg)_y(W^1) = W^1,$$

$$(Dg)_y(W^2) = W^2$$

and

$$(Dg)_y(W^3) = W^3$$

will hold. The three Lyapunov exponents are:  $\ln |a|, \ln |\mu_1|, \ln |\mu_2|$ .

Notice that in all cases the two Lyapunov exponents of  $f$  are two of the Lyapunov exponents of  $g$ . The third Lyapunov exponent of  $g$  can be of any magnitude. The problem comes from the fact that the partial derivatives of  $g$  do not necessarily lie in the tangent space of the image of the attractor under the Takens embedding (3.2). It raises the question of how to identify the  $n$  ‘true’ Lyapunov exponents of  $f$  from the  $m - n$  ‘spurious’ Lyapunov exponents that make up the Lyapunov exponents of  $g$ .

From a theoretical point of view, the answer lies in the fact that the derivatives (and therefore the Lyapunov exponents) of  $g$  are only defined on the tangent space of the image of the manifold under the Takens

Table 1  
Lyapunov exponent estimates of the Hénon map at  $m = 5$

$\alpha$	Number of neurons	LEs	MSE
0.00	7	$\lambda_1 = 0.4401$ $\lambda_2 = -1.6461$ $\lambda_3 = -2.7052$ $\lambda_4 = -2.9970$ $\lambda_5 = -3.2390$	0.21E – 08
0.25	14	$\lambda_1 = 0.4474$ $\lambda_2 = -0.4341$ $\lambda_3 = -1.3021$ $\lambda_4 = -1.6241$ $\lambda_5 = -3.1542$	0.42E – 08
0.50	13	$\lambda_1 = 0.4472$ $\lambda_2 = -0.4123$ $\lambda_3 = -1.1652$ $\lambda_4 = -1.3422$ $\lambda_5 = -2.7612$	0.43E – 08
0.75	15	$\lambda_1 = 0.4469$ $\lambda_2 = 0.2584$ $\lambda_3 = -0.7101$ $\lambda_4 = -1.5794$ $\lambda_5 = -3.4291$	0.44E – 08
1.00	15	$\lambda_1 = 0.4470$ $\lambda_2 = 0.3055$ $\lambda_3 = -1.3810$ $\lambda_4 = -1.6771$ $\lambda_5 = -2.949$	0.11E – 07
2.00	12	$\lambda_1 = 0.5151$ $\lambda_2 = 0.2371$ $\lambda_3 = -0.4854$ $\lambda_4 = -0.9971$ $\lambda_5 = -1.7621$	0.15E – 06
4.00	14	$\lambda_1 = 0.4319$ $\lambda_2 = 0.2006$ $\lambda_3 = -0.2921$ $\lambda_4 = -1.2443$ $\lambda_5 = -2.5381$	0.13E – 06

Number of observations: 200.

Observer:  $h(x) = x_t + \alpha x_{t-1}$ .

embedding  $J^m$ . The dimension of this tangent space is the same as the dimension of the manifold, and so there are precisely  $n$  derivatives of  $g$  in the tangent space of the image of the manifold. However, when an  $m$ -dimensional function is fitted to the data, no matter how good the fit is on the data, the fitted function will have  $m$  derivatives, and hence  $m$  Lyapunov exponents. It is an open question in theory how to identify the Lyapunov exponents of an unknown system.

In practice, it is necessary to identify the tangent

spaces to the attractor in the embedding space at each observation. One suggestion is given in [13]. In order to identify the largest Lyapunov exponent, estimate the degree of spreading of nearby vectors. In order to identify the sum of the two largest Lyapunov exponents, estimate the degree of spreading of nearby plane areas (as determined by pairs of vectors). This process can be repeated to identify all of the Lyapunov exponents.

It may be difficult experimentally to identify all of the Lyapunov exponents, especially the negative ones. This is due to the fact that in a direction in which the dynamics are strongly contracting the data lie close to each other and the rate of contraction is difficult to measure. This phenomenon is exacerbated when there is noise present<sup>12</sup>.

### 5. Numerical results

In this section we use the Hénon map to illustrate the calculation of the Lyapunov exponents with various observers. The Hénon map

$$\begin{aligned} x_{t+1} &= 1.0 - 1.4x_t^2 + z_t \\ z_{t+1} &= 0.3x_t \end{aligned} \tag{5.1}$$

is a widely used example and it can be considered as a benchmark to test our results against the results of the others. We use two different observers to study the Hénon map. The first observer is

$$y_t = h(x_t) = x_t, \tag{5.2}$$

and the second one is

$$y_t = h(x_t) = x_t + \alpha x_{t-1} \tag{5.3}$$

where  $\alpha$  is a real valued constant. The observer in equation (5.2) is extensively studied whereas the realizations of the observer in Eq. (5.3) is not been studied previously.

#### 5.1. Lyapunov exponents of observer $y_t = h(x_t) = x_t$

This case is studied extensively in the previous literature. Lyapunov exponents calculations in Abarbanel

<sup>12</sup> We thank an anonymous referee for pointing this out to us.

et al. [1,2], Ellner et al. [14], Gençay and Dechert [18] and Wolf et al [30], all confirm that two dimensions is sufficient for an embedding to extract the true Lyapunov exponents of the Hénon map and there is no need to go to embedding dimension five ( $m = 5$ ) which is the worst-case upper limit. Accordingly, all but the first two Lyapunov exponents are equal to  $-\infty$ . This is due to the fact that the partial derivatives of the third, fourth and fifth lags of an embedding at embedding dimension five are zero. In simulation experiments, these zero derivatives are very small but not exactly zero (e.g.  $1.0E - 04$ ) so that the calculated Lyapunov exponents are negative but far from being  $-\infty$ . This raises the question of how to identify the spurious Lyapunov exponents from the true ones. Gençay and Dechert [18] showed that the true Lyapunov exponents are numerically stable to increments in the embedding dimension whereas the spurious ones fluctuate in value but keep their signs. Therefore, this approach can be used in applied work in the detection of the spurious Lyapunov exponents from data.

### 5.2. Lyapunov exponents of observer

$$y_t = h(x_t) = x_t + \alpha x_{t-1}$$

This observer function is one of the simplest ones that one can construct. Although our focus will be this additive linear observer one can also generate observers such that it can take the product or the ratio of the states of the data generating process.

In our experiments, we let  $\alpha$  to vary between  $[0, 4]$ . The increments are 0.25 for  $\alpha \in [0, 1]$  and 1.00 for  $\alpha \in [1, 4]$ . The plot of  $y_t$  versus  $y_{t-1}$  for each  $\alpha$  are presented in Figs. 1–7. These figures indicate that function  $g$  may take a much more complicated form depending upon the complexity of the observer function. As the value of  $\alpha$  is increased, the system folds itself in various directions such that this folding effect completely changes the phase portrait of the Hénon map. Our conjecture is that the minimum embedding dimension necessary to get an embedding is a function of the severity of the folding effect.

In Table 1, the results with the observer  $y_t = h(x_t) = x_t + \alpha x_{t-1}$  are presented for 200 observations. As an estimation method, the single layer feedforward net-

Table 2  
Lyapunov exponent estimates of the Hénon map at  $m = 5$

$\alpha$	Number of neurons	LEs	MSE
0.00	8	$\lambda_1 = 0.4301$ $\lambda_2 = -1.6212$ $\lambda_3 = -2.7103$ $\lambda_4 = -2.9871$ $\lambda_5 = -3.2107$	0.24E – 08
0.25	12	$\lambda_1 = 0.4311$ $\lambda_2 = -0.4216$ $\lambda_3 = -0.9611$ $\lambda_4 = -2.3112$ $\lambda_5 = -3.1615$	0.43E – 08
0.50	11	$\lambda_1 = 0.4112$ $\lambda_2 = -0.4211$ $\lambda_3 = -1.1216$ $\lambda_4 = -1.7611$ $\lambda_5 = -1.8112$	0.35E – 08
0.75	13	$\lambda_1 = 0.4211$ $\lambda_2 = 0.2312$ $\lambda_3 = -0.6312$ $\lambda_4 = -1.5512$ $\lambda_5 = -2.9412$	0.43E – 08
1.00	12	$\lambda_1 = 0.4313$ $\lambda_2 = 0.3314$ $\lambda_3 = -1.2516$ $\lambda_4 = -1.7013$ $\lambda_5 = -2.9813$	0.14E – 07
2.00	11	$\lambda_1 = 0.4913$ $\lambda_2 = 0.3012$ $\lambda_3 = -0.7012$ $\lambda_4 = -1.5312$ $\lambda_5 = -2.0012$	0.11E – 06
4.00	13	$\lambda_1 = 0.4612$ $\lambda_2 = 0.2513$ $\lambda_3 = -0.3512$ $\lambda_4 = -1.3519$ $\lambda_5 = -2.6121$	0.12E – 06

Number of observations: 500.

Observer:  $h(x) = x_t + \alpha x_{t-1}$ .

work regression is used. For each  $\alpha$  value, the network is estimated for hidden units from 1 to 15. We use the cross-validated mean square error (MSE) as the model selection criteria and report the MSE of the approximation for each  $\alpha$  value. As seen in Table 1, the quality of approximation is high as all of the MSE calculations are less than  $1.0E - 05$ .

For  $\alpha = 0$ , we are in the observer (5.4) case and embedding dimension two is sufficient to extract the true Lyapunov exponents. The three smallest Lyapunov ex-

Table 3  
Lyapunov exponent estimates of the Hénon map at  $m = 5$

$\alpha$	Number of neurons	LEs	MSE
0.00	10	$\lambda_1 = 0.4177$ $\lambda_2 = -1.5831$ $\lambda_3 = -1.7522$ $\lambda_4 = -2.0953$ $\lambda_5 = -4.8935$	$0.37E - 08$
0.25	11	$\lambda_1 = 0.4176$ $\lambda_2 = -1.5717$ $\lambda_3 = -1.7864$ $\lambda_4 = -2.5527$ $\lambda_5 = -2.9300$	$0.48E - 08$
0.50	10	$\lambda_1 = 0.4166$ $\lambda_2 = -0.7729$ $\lambda_3 = -1.1390$ $\lambda_4 = -1.4201$ $\lambda_5 = -1.8621$	$0.215E - 06$
0.75	9	$\lambda_1 = 0.4162$ $\lambda_2 = 0.2119$ $\lambda_3 = -0.1584$ $\lambda_4 = -1.5940$ $\lambda_5 = -3.5525$	$0.168E - 06$
1.00	10	$\lambda_1 = 0.4613$ $\lambda_2 = 0.3815$ $\lambda_3 = -0.5591$ $\lambda_4 = -1.8065$ $\lambda_5 = -4.6835$	$0.880E - 07$
2.00	11	$\lambda_1 = 0.5754$ $\lambda_2 = 0.4265$ $\lambda_3 = -0.9673$ $\lambda_4 = -1.5627$	$0.6690E - 04$
4.00	13	$\lambda_1 = 0.4575$ $\lambda_2 = 0.3979$ $\lambda_3 = -0.6820$ $\lambda_4 = -1.5732$ $\lambda_5 = -2.7260$	$0.101E - 05$

Number of observations: 2000.

Observer:  $h(x) = x_t + \alpha x_{t-1}$ .

ponents of an embedding at embedding dimension five are large negative numbers. At  $\alpha = 0.75$ , the second largest Lyapunov exponent becomes positive and stays positive for all values of  $\alpha$  up to  $\alpha = 4$ . This example shows that it is possible to get a *spurious* Lyapunov exponent that is a positive large number.

The same experiment is also repeated with different data lengths. The results with two additional data sets which contain 500 and 2000 observations are reported in Tables 2 and 3. The results in these two tables support the ones in Table 1. Some analysis of noisy data

Table 4  
Lyapunov exponent estimates of the Hénon map at  $m = 5$

$\alpha$	Number of neurons	LEs	MSE
0.00	7	$\lambda_1 = 0.3765$ $\lambda_2 = -1.7623$ $\lambda_3 = -2.7989$ $\lambda_4 = -4.1231$ $\lambda_5 = -4.4312$	$0.16E - 05$
0.25	12	$\lambda_1 = 0.3876$ $\lambda_2 = -1.0661$ $\lambda_3 = -1.0945$ $\lambda_4 = -1.1243$ $\lambda_5 = -2.9834$	$0.16E - 05$
0.50	12	$\lambda_1 = 0.3912$ $\lambda_2 = -0.7651$ $\lambda_3 = -0.9834$ $\lambda_4 = -1.2712$ $\lambda_5 = -2.2341$	$0.15E - 05$
0.75	14	$\lambda_1 = 0.3932$ $\lambda_2 = 0.2341$ $\lambda_3 = -0.6832$ $\lambda_4 = -1.8721$ $\lambda_5 = -2.4512$	$0.15E - 05$
1.00	13	$\lambda_1 = 0.4001$ $\lambda_2 = 0.2873$ $\lambda_3 = -0.7623$ $\lambda_4 = -1.6652$ $\lambda_5 = -2.8931$	$0.11E - 05$
2.00	11	$\lambda_1 = 0.4032$ $\lambda_2 = 0.2643$ $\lambda_3 = -0.4923$ $\lambda_4 = -0.7437$ $\lambda_5 = -1.8732$	$0.14E - 06$
4.00	13	$\lambda_1 = 0.4021$ $\lambda_2 = 0.2652$ $\lambda_3 = -0.3012$ $\lambda_4 = -0.6512$ $\lambda_5 = -2.0912$	$0.13E - 06$

Number of observations: 200.

Measurement noise: 0.05 percent.

Observer:  $h(x) = x_t + \alpha x_{t-1}$ .

is also added to the numerical computations. The results with the noisy data set are done with 200 observations and reported in Table 4. Although the presence of noise deteriorates the results a little, the qualitative findings stays the same.

## 6. Conclusions

There are two contributions of this paper. One is to provide a theoretical ground for the Jacobian methods of [12,13] by showing that  $n$  of the Lyapunov exponents of the estimated Jacobian are the Lyapunov exponents of the unknown dynamical system. The second is to show that it is necessary that the Lyapunov exponents be calculated in the tangent space of the attractor in the embedding. We show that when embedding occurs at dimension  $m = n$ , then the Lyapunov exponents of the reconstructed dynamics are the Lyapunov exponents of the original dynamics. This is the case for the Hénon, the Lorenz and the Mackey–Glass systems with a first component observer. However, if embedding only occurs for an  $m > n$ , then the Jacobian method yields  $m$  Lyapunov exponents, only  $n$  of which are the Lyapunov exponents of the original system. The problem is that as currently used, the Jacobian method is applied to the full  $m$ -dimensional space of the reconstruction, and not just to the  $n$ -dimensional manifold that is the image of the embedding map. Our examples show that it is possible to get ‘spurious’ Lyapunov exponents that are even larger than the largest Lyapunov exponent of the original system.

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