Optimal liquidation in dark pools

Peter Kratz & Torsten Schöneborn

To cite this article: Peter Kratz & Torsten Schöneborn (2014) Optimal liquidation in dark pools, Quantitative Finance, 14:9, 1519-1539, DOI: 10.1080/14697688.2014.917434

To link to this article: http://dx.doi.org/10.1080/14697688.2014.917434

Published online: 17 Jun 2014.

Submit your article to this journal

Article views: 201

View related articles

View Crossmark data

Citing articles: 5

Download by: [Simon Fraser University]  Date: 26 January 2016, At: 19:14
Optimal liquidation in dark pools

PETER KRATZ*† and TORSTEN SCHÖNEBORN‡

†Institut de Mathématiques de Marseille, Aix-Marseille Université, 39, rue F. Joliot-Curie, 13453 Marseille Cedex 13, France
‡Deutsche Bank AG, London, UK

(Received 23 January 2014; accepted 11 April 2014)

We consider a large trader liquidating a portfolio using a transparent trading venue with price impact and a dark pool with execution uncertainty. The optimal execution strategy uses both venues continuously, with dark pool orders over-/underrepresenting the portfolio size depending on return correlations; trading at the traditional venue is delayed depending on dark liquidity. Pushing up prices at the traditional venue while selling in the dark pool might generate profits. If future returns depend on historical dark pool liquidity, then sending orders to the dark pool can be worthwhile simply to gather information.

Keywords: Dark pools; Optimal liquidation; Pinging; Market microstructure; Illiquid markets

JEL Classification: C00, C61, G11, G12, G20

1. Introduction

The emergence of so-called dark pools has changed the equity markets in recent years. Dark pools are alternative trading venues which differ in some of their properties (e.g. ownership, crossing procedure and accessibility, see Mittal (2008) or Degryse et al. (2009a) for further details) but share two key features. First, the liquidity available is not displayed openly, resulting in uncertain trade execution but also in no (or less) price impact for dark pool orders.§ Second, dark pools do not determine prices but adopt the prices determined by classical exchanges and settle trades at these prices. In this paper, we analyse trade execution when trading is possible both at the classical exchange as well as in the dark pool. We first propose a model that captures the styled facts above for trading in such a market. Then, we determine the optimal trade execution strategy in a discrete time framework.

We consider a general price impact model for trading at a classical exchange: trade execution can be enforced but this results in higher execution costs due to a stronger price impact. The dark pool in our model provides a limited and previously unknown amount of liquidity that can be used for trade execution without price impact. Trades in the dark pool are executed until the liquidity is exhausted, and there is no way to achieve trade execution in the dark pool for larger orders. Investors hence face the trade-off of execution uncertainty in the dark pool and price impact costs at the classical exchange.

In our model, we prove the existence and uniqueness of optimal execution strategies that trade simultaneously at the primary exchange and in the dark pool. Subsequently, we consider a specific multi-asset market model with linear price impact. The model can be specified by a small set of parameters which can be directly estimated from trade execution data. In this market model, we obtain a linear solution of the optimal trade execution problem which can be computed recursively. This recursive scheme makes the model tractable for practicable applications and allows us to investigate several examples. If a single-asset position is to be liquidated, it is optimal to offer the complete position in the dark pool at all times; in parallel, the position is liquidated at the primary exchange. In comparison to optimal liquidation without dark pool, the optimal trading speed at the exchange is slowed down. Hence, traders need to adjust their trading algorithms fundamentally instead of only adding a component that also places orders in the dark pool.

If an investor needs to liquidate a multi-asset portfolio, it is no longer optimal to place the entire position for all assets in the dark pool. Instead, the optimal dark pool orders depend on the correlation of the assets. For example, it is undesirable to risk the full liquidation of the position in one of the assets if the portfolio is well balanced and exposed to little market risk; hence, only a fraction of the entire portfolio should be placed in the dark pool. This emphasizes again that overly simple modifications of existing trading algorithms can have unfavourable consequences. For dark pool operators, it yields

*Corresponding author. Email: kratz@mathematik.hu-berlin.de
§Conrad et al. (2003) and Fong et al. (2004) provide empirical evidence for lower price impact/transaction costs of dark pools compared to traditional exchanges.
an incentive to offer balanced executions in order to attract more liquidity.

Whenever traders have an influence on market prices, the question of price manipulation arises. The co-existence of two trading venues opens up new opportunities in this regard: a trader could, for example, continuously buy an asset at the exchange, thus pushing its price up, and could thereafter sell the asset at the elevated price in the dark pool. We find that such strategies can indeed be profitable, but can be ruled out in our model if either trades do not have an impact on dark pool transaction prices or if liquidity in the dark pool is limited and exposed to sufficiently high adverse selection.‡

If dark pool liquidity is predictive of future price moves, then knowledge of historical dark pool liquidity is valuable. This leads to ‘pinging’, i.e. the submission of small orders to the dark pool for the purpose of information gathering. Based on the (lack of) execution of these pinging orders, the trader can infer whether liquidity was available in the dark pool or not. We find that anti-gaming measures such as minimum order sizes can increase the cost of information acquisition sufficiently to render pinging unattractive.

Our paper is connected to research on optimal trade execution strategies for a single trader in models where the liquidity effects are given exogenously. For trading at traditional exchanges, this includes the articles by Bertsimas and Lo (1998), Almgren and Chriss (2001), Obizhaeva and Wang (2013), Schied and Schöneborn (2009) and Alfonsi, Fruth, et al. (2010). These articles investigate execution problems when trading is limited to one (traditional) venue. Our discrete time market model for the primary exchange generalizes all of these models. In addition, we introduce the possibility to use a dark pool in parallel. After establishing existence and uniqueness of the optimal trading strategy in this general model, we subsequently follow Almgren and Chriss (2001) and assume a more specific model with linear temporary price impact for trading at the primary venue. This yields a tractable model which allows us to obtain explicit solutions for the optimal trading strategy; nevertheless, it captures price impact effects. Linear price impact models are the basis of many theoretical studies such as Rogers and Sing (2010), Almgren and Lorenz (2007), Carlin et al. (2007) and Schöneborn and Schied (2009). Furthermore, they demonstrated reasonable properties in real world applications and serve as the basis of many optimal execution algorithms run by practitioners (see e.g. Kissell and Glantz (2003), Schack (2004), Abramowitz (2006) and Leinweber (2007)). Our article is most closely connected to Kratz and Schöneborn (2013), which focuses on the special case of independent dark pool liquidity in a continuous time version of the general model presented in section 4 of this article. While keeping the exposition in an accessible discrete time framework, we are able to relax many restrictions of Kratz and Schöneborn (2013) and work in a more general setting which allows us to analyse features such as dark pool pinging (cf. section 6), dependencies of the dark pool liquidity for different assets (cf. section 4.4.2), partial execution of dark pool orders§, permanent price impact of

‡A detailed discussion of adverse selection within our framework can be found in Kratz and Schöneborn (2014) and in Kratz (2014).

§Many dark pool operators do not fully disclose the details of their matching algorithm, however, some of them apply some form of pro rata matching. This matching rule is structurally different to the model

1520

P. Kratz and T. Schöneborn
market, a crossing network and the limit order book. Their findings include that crossing networks provide lower price impact for block trades than the consolidated limit order book. Using the same data-set Næs and Ødegaard (2006) and Næs and Skjeltorp (2003) analyse the effect of adverse selection due to private information on transaction costs. Ready (2010) analyses determinants of volume in dark pools and shows that the volume share of dark pools is negatively correlated to consolidated volume.

In the following, we first introduce the market model (section 2). This consists of a model for the primary trading venue (section 2.1) and for the dark pool (section 2.2). Furthermore, we describe the trader’s objective function (section 2.3). Existence and uniqueness of optimal trade execution strategies are established in section 3. In section 4, we propose a specific tractable model and analyse its properties. We discuss price manipulation in section 5 and conclude with an investigation of dark pool pinging in section 6. We present all proofs in section 7 and conclude in section 8. Appendix A provides a discussion of the risk criterion we consider. Explicit recursions for the optimal trading strategy and the cost functional of section 4 are collected in Appendix B.

2. Model description

The market we consider consists of a risk-free asset and n risky assets. For simplicity of exposition, we assume that the risk-free asset does not generate interest. Large transactions are usually executed within a few hours or at most a few days; the effect of discounting is therefore marginal, and we will not consider it in this paper. We analyse a discrete time model, i.e. we assume that trades can be executed at the (not necessarily equidistant†) time points 0 = t₀ < · · · < tN = T. At each of these time points, we assume that the seller as well as a number of noise traders execute orders. We denote the orders of the seller at time tᵢ at the primary venue by

\[
x(tᵢ) = (x₁(tᵢ),...,xₙ(tᵢ)) \in \mathbb{R}^n
\]  

(1)

and in the dark pool by

\[
y(tᵢ) = (y₁(tᵢ),...,yₙ(tᵢ)) \in \mathbb{R}^n.
\]  

(2)

Positive entries denote sell orders and negative entries denote buy orders. We allow for dynamic liquidation strategies that can react to changes in market prices and to liquidity in the dark pool. To this end, we assume that the orders x and y are adapted stochastic processes relative to a stochastic basis (Ω, ℱ, ℙ = (ℱ(tᵢ))ᵢ∈\{0,...,N\}, ℙ). In the following sections 2.1 and 2.2 we describe the different effects of the orders x(tᵢ) and y(tᵢ). The execution of the order x(tᵢ) at the primary venue is guaranteed but has an adverse effect on the market price. The execution of the order yᵢ in the dark pool is uncertain but has no price impact (irrespective of whether it is executed or not). In section 2.3, we define the trading objective of the investor and specify the set of admissible strategies.

†For example, the distance can be taken in volume time to adjust for the U-shaped intraday pattern of market volatility and liquidity.

2.1. Transaction price and impact of primary venue orders

We assume that the transaction price \( P(tᵢ) \in \mathbb{R}^n \) at the primary venue at time \( tᵢ \) can be decomposed into the price impact of the primary venue trades \( (x(tᵢ))_{i=0,...,N} \) of the large trader and the ‘fundamental’ asset price \( \tilde{P}(tᵢ) \in \mathbb{R}^n \) that would have occurred in the absence of large trades. We model the fundamental asset price \( \tilde{P}(tᵢ) \) as a stochastic process with independent increments \( \epsilon(tᵢ) \in \mathbb{R}^n \) that have zero expectation‡ \( (\mathbb{E}[\epsilon(tᵢ)] = 0) \):

\[
\tilde{P}(tᵢ+1) = \tilde{P}(tᵢ) + \epsilon(tᵢ+1).
\]  

(3)

To avoid technical difficulties, we assume that the underlying probability space \( \Omega \) is finite and all its elements have positive probability.§ We do not make assumptions on the distributions of the \( \epsilon(tᵢ) \). In particular, they can have different distributions. The random price changes \( \epsilon(tᵢ) \) reflect the noise traders’ actions as well as all external events, e.g. news. The assumption of independence of the \( \epsilon(tᵢ) \) implies that the random price changes do not exhibit autocorrelation. Autocorrelation in the market model would shift the focus from optimal liquidation to optimal investment: even without any initial asset position, the mathematical model will recommend high-frequency trading to exploit the autocorrelation. But this effect is not related to the original question of optimal execution. Furthermore, many investors do not have an explicit view on autocorrelation and thus choose an execution algorithm that is optimal under the assumption of independence of price increments. Finally, for realistic parameters, the effect of autocorrelation on the optimal execution strategy and the resulting execution cost is marginal as was demonstrated by Almgren and Chriss (2001). For these reasons, we do not include autocorrelation in our market model.

As the time horizon for portfolio liquidation is usually short, i.e. several hours or a few days, the assumption of zero drift in the fundamental asset price does not constitute a major divergence from reality. Furthermore, any drift in the model would again introduce a trading motivation independent of the original liquidation intention.

It has long been documented that trades have an impact on subsequent market prices, see for example Kraus and Stoll (1972). Several explanations of this effect have been suggested, for example, limited liquidity (see e.g. Grossman and Miller (1988) and Duffie (2010)) and private information (see e.g. Kyle (1985) and Glosten and Milgrom (1985)). We do not seek to explicitly model the underlying mechanisms resulting in price impact, but instead assume an exogenously given relationship. This allows the model to capture a wide range of complex price impact dynamics which are usually not explicable by tractable full equilibrium models. We allow a general form of the impact of the trades\[ x(t₀),...,x(tᵢ) \] on the transaction price \( P(tᵢ) \):

‡The fundamental price process is hence a martingale.

§The results of this paper also hold for infinite Ω if the price increments \( \epsilon(tᵢ) \) satisfy suitable conditions and the set of admissible strategies is chosen appropriately.

In reality, the trade \( x(tᵢ) \) is often not executed as one instantaneous order at time \( tᵢ \), but rather as a sequence of smaller orders over the time interval \( [tᵢ, tᵢ+1) \). For the purposes of this paper, we do not specify when and how exactly the trade \( x(tᵢ) \) is executed but only assume that the transaction price is given by \( P(tᵢ) \) as in equation (4).
\[ P(t_i) = \frac{\hat{P}(t_i)}{\sum_{j=0}^{N} f_j(x(t_0), \ldots, x(t_i)). \uparrow \text{Fundamental’ asset price}} \]

\[ f_i : \mathbb{R}^{n \times (i+1)} \rightarrow \mathbb{R}^n. \] By allowing \( f_i \) to depend on \( x(t_i) \), we allow the order \( x(t_i) \) to influence its own execution price (in the form of a temporary price impact). Adverse selection effects at the primary venue can be captured in the model through this price impact formulation. We define the price impact costs of trading as

\[ \sum_{j=0}^{N} x(t_i)^{T} f_j(x(t_0), \ldots, x(t_i)). \]

**Assumption 2.1** The price impact costs fulfil the following two conditions.

(i) \( \sum_{j=0}^{N} x(t_i)^{T} f_j(x(t_0), \ldots, x(t_i)) \) is strictly convex in \((x(t_0), \ldots, x(N))\).

(ii) \( \sum_{j=0}^{N} x(t_i)^{T} f_j(x(t_0), \ldots, x(t_i)) \) grows super-linearly as \( \| (x(t_0), \ldots, x(N)) \| \rightarrow \infty \).

This framework generalizes many of the existing market impact models of liquidity, e.g. the model suggested by Almgren and Chriss (2001) and Obizhaeva and Wang (2013) (see Kratz (2011) for more details).

### 2.2. Trade execution in the dark pool

Contrary to the primary venue, the dark pool does not guarantee trade execution, since it only provides limited liquidity. We introduce the random variables \( a(t_i), b(t_i) \in [0, \infty]^{n} \) that model the liquidity which can be drawn upon by the trader in the time interval \([t_{i-1}, t_i)\) for buy (ask side) and sell (bid side) orders, respectively. The amount

\[ z(t_i) = (z_1(t_i), \ldots, z_n(t_i))^T \in \mathbb{R}^n \]

which is executed in the dark pool during time \( t_i \) and \( t_{i+1} \) is given by

\[ z_k(t_i) = \begin{cases} \min(y_k(t_i), b_k(t_{i+1})) & \text{if } y_k(t_i) \geq 0 \\ -\min(-y_k(t_i), a_k(t_{i+1})) & \text{if } y_k(t_i) < 0. \end{cases} \]

where \( y_k(t_i) \) is the order \( x_k(t_i) \) in the dark pool for the \( k \)-th asset at time \( t_i \). While matching rules vary significantly between dark pools, our model seeks to capture their joint key characteristics: unknown and (possibly) limited liquidity. Several different matching rules investigated in the existing literature fit into our framework. Hendershott and Mendelson (2000), Daniëls et al. (2013) and Zhu (2014) use a stochastic matching rule, where the priority of orders in the book is determined randomly just prior to matching. For this matching rule, \( a(t_i) \) and \( b(t_i) \) should be interpreted as the liquidity that the trader has access to given her randomly determined queue priority.

Ye (2011) discusses a model where queue priority in the dark pool is determined by differences in technology (with better technology allowing traders to obtain higher queue priority) and by dark pool rules (with more sophisticated/toxic traders being allocated a lower queue priority by the dark pool operator). The outcome of these two factors is a deterministic queue priority for each trader. Such a deterministic priority matching rule can be reflected in our model by again setting \( a(t_i) \) and \( b(t_i) \) as the liquidity that the trader has access to (not necessarily as the liquidity available in the dark pool overall, cf. Footnote 3 in the introduction section of this paper).

We will make the following assumption regarding the dark pool liquidity \((a(t_i), b(t_i))\) available to the trader.

**Assumption 2.2** The dark pool liquidity fulfils the following three conditions.

(i) For \( i = 1, \ldots, N+1, a(t_i) \) and \( b(t_i) \) are independent of previous liquidity in the dark pool \( a(t_1), \ldots, a(t_{i-1}), b(t_1), \ldots, b(t_{i-1}) \) and of previous price moves \( e(t_1), \ldots, e(t_{i-1}). \) Furthermore, subsequent price moves \( e(t_{i+1}), \ldots, e(t_N) \) are independent of \( a(t_i) \) and \( b(t_i) \).

(ii) Dark pool execution is not guaranteed, i.e. for \( i = 1, \ldots, N+1, k = 1, \ldots, n, P[a_k(t_i) = 0] > 0, P[b_k(t_i) = 0] > 0 \).

(iii) For all \( i = 1, \ldots, N+1, k = 1, \ldots, n, \) with \( P[a_k(t_i) = p], P[b_k(t_i) = q] > 0, \) we have

\[ \mathbb{E}[e_k(t_i)|a_k(t_i) = q] \geq \mathbb{E}[e_k(t_i)|a_k(t_i) = p], \]

\[ \mathbb{E}[e_k(t_i)|b_k(t_i) = q] \leq \mathbb{E}[e_k(t_i)|b_k(t_i) = p], \]

\[ \mathbb{E}[e_k(t_i)|a_k(t_i) > 0] \leq \mathbb{E}[e_k(t_i)|b_k(t_i) > 0]. \]

(iv) For all \( i = 1, \ldots, N+1, k = 1, \ldots, n \) and every collection \( a_1, \ldots, a_n, b_1, \ldots, b_n \) with \( a_k, b_k > 0 \), there exists a non-negative real number \( p_{a_k, b_k} \) independent of \( a_k \) and \( b_k \) such that

\[ P[a_k(t_i) = a_k|a_k(t_i) = a_l, b_k(t_i) = b_l, l \neq k] = P[a_k(t_i) = a_k] \cdot p_{a_k, b_k}, \]

\[ P[b_k(t_i) = b_k|a_k(t_i) = a_l, b_k(t_i) = b_l, l \neq k] = P[b_k(t_i) = b_k] \cdot p_{a_k, b_k}. \]

Due to Assumption 2.2 (i) and the independence of price increments \((e(t_j))_{j=1 \ldots N} \), the investor cannot derive any predictions of future dark pool liquidity and price moves from previous observations of the market. In section 6, we investigate the consequences of weakening this assumption. Assumption 2.2 (i) however does allow for a dependence of the liquidity parameters \((a(t_i), b(t_i))\) and the price move \( e(t_i) \) for the same

\[ \text{The large trader can only be either selling or buying in the dark pool during } [t_1, t_{i+1}). \text{ but not both. She can however trade in opposite directions in the dark pool and on the primary venue; see section 5 for a discussion.} \]

\[ \text{Ye (2011) discusses in particular the two extreme cases of the strategic trader either having consistently top or having consistently bottom queue priority.} \]

\[ \text{Deterministic and stochastic priority rules reflect different extremes of the spectrum and can easily be combined. For example, consider the following batched time priority matching mechanism. First, the dark pool operator gathers orders between time } t_i \text{ and } t_{i+1}. \text{ Subsequently, she matches the orders using time priority just prior to } t_{i+1} \text{ and cancels all unmatched orders at the same time. At time } t_{i+1}, \text{ the trader then faces an empty dark pool order book and all market participants rush to enter their orders into the dark pool. Such a batched operation can be approximated by a mixture of the deterministic and stochastic priority rules, where technology and dark pool rules have a broadly deterministic influence on queue priority while stochastic variations in order submission latency provide a random component to queue priority.} \]
trading period $t_i$. This enables us to incorporate the simultaneous occurrence of price jumps and liquidity in the dark pool which can lead to adverse selection.\footnote{See Kratz and Schöneborn (2014) for a discussion.} Assumption 2.2 (ii) is needed in order to ensure uniqueness of the optimal strategy. Economically, it means that price moves in the dark pool are not instantaneous, i.e. the stronger the demand in the dark pool, the stronger the price at the primary venue is expected to move upwards; and the stronger the supply in the dark pool, the stronger the price is expected to move downwards. In other words: a large amount of liquidity in the dark could be a sign for an impending favourable price move. The case of strict inequality in Inequalities (8), (9) or (10) for some $p > q \geq 0$ can lead to adverse selection. Assumption 2.2 (iv) limits the dependence of the dark pools in the various assets. Whether or not there is liquidity for asset $l$ in the dark pool can have an influence on the likelihood of liquidity for asset $k$, but it cannot change the relative likelihood of large liquidity versus small liquidity.

We do not restrict our model to specific joint distributions of price increments $\epsilon$ and dark pool liquidity $(a, b)$. Instead, we will derive the results of this paper directly from sets of assumptions such as Assumption 2.2.

While the dark pool has no impact on prices at the primary venue, it is less clear to which extent the price impact $f_j$ of the primary venue is reflected in the trade price of the dark pool. If, for example, the price impact $f_j$ is realized predominantly in the form of a widening spread, then the impact on dark pools that monitor the mid quote can be much smaller than $f_j$. In sections 3–6, we will make the simplifying assumption that trades in the dark pool are not influenced by the price impact $f_j$ at all, i.e. that they are executed at the fundamental price $P(t_i)$. If trading in the dark pool reflects the price impact $f_j$, then market manipulating strategies can become profitable. We investigate this phenomenon in section 5.

### 2.3. The liquidation problem

For fixed $i = 0, \ldots, N$, we consider an investor who has executed trades $x(t_0), \ldots, x(t_{i-1}) \in \mathbb{R}^n$ at times $t_0, \ldots, t_{i-1}$ and needs to liquidate a portfolio

$$X(t_i) = (X_1(t_i), \ldots, X_n(t_i))^\top \in \mathbb{R}^n$$

of $n$ assets within a finite time horizon $[t_i, T]$. For $X_k(t_i) > 0$, this implies liquidating a long position in asset $k$ (selling), whereas $X_k(t_i) < 0$ implies liquidating a short position in asset $k$ (buying). In both cases, we speak of ‘liquidation’ or ‘sale’. We require that at all times $t_j \geq t_i$ the investor’s orders $x(t_j)$, $y(t_j)$ depend only on past information $(\epsilon(t_1), \ldots, \epsilon(t_j), a(t_1), \ldots, a(t_j))$ and $(b(t_1), \ldots, b(t_j))$ and thus assume that the filtration $\mathbb{F}$ = $\{\mathcal{F}(t_j)\}_{j=1,\ldots,N}$ is given by $\mathcal{F}(t_j) = \sigma(\epsilon(t_1), \ldots, \epsilon(t_j), a(t_1), \ldots, a(t_j), b(t_1), \ldots, b(t_j))$. This implies deterministic (also called static) strategies, i.e. strategies that do not depend on any $\epsilon(t_i)$, $a(t_i)$ or $b(t_i)$.

\footnote{In reality, the investor does not know the exact dark pool liquidity $(a(t_j), b(t_j))$ at time $t_j$ but only the executed fraction $z(t_{j-1})$. Economically, the filtration should therefore be given by $\mathcal{F}(t_j) = \sigma(\epsilon(t_1), \ldots, \epsilon(t_j), z(t_{j-1}), a(t_1), \ldots, a(t_j), b(t_1), \ldots, b(t_j))$. Because of the independence of $(a(t_j), b(t_j))$ this is mathematically irrelevant.}

\footnote{Traders liquidating a portfolio for a client can face trading restrictions in practice, e.g. they might not be allowed to short any of the stocks in the portfolio or to change the trading direction intended by the client. We do not consider such restrictions in this article and only want to remark that the results of section 3 remain valid under such constraints. Additional material on this topic is available from the authors upon request.}

\footnote{The convention for the traded amount $z$ is such that $z(t_i)$ refers to the trade executed at time $t_{i-1}$ resulting from the dark pool order $y(t_i)$ submitted at time $t_i$. It does not refer to the dark pool trade executed at time $t_i$ (which would be $z(t_{i-1})$).}

\begin{definition}
Let $i = 0, \ldots, N$ and $X(t_i) \in \mathbb{R}^n$ be the portfolio position at time $t_i$. We call a sequence of $\mathbb{F}$-adapted orders $(x, y) = (x(t_j), y(t_j))_{j=i,\ldots,N}$ an admissible liquidation strategy if it fulfills the following conditions.

(i) The portfolio is liquidated by time $t_N$: $\sum_{j=i}^{N} x(t_j) + z(t_j) = X(t_i)$ for all $\omega \in \Omega$.

(ii) For $j = i, \ldots, N$, $k = 1, \ldots, n$, $y_k(t_j) \in \{-\max_{\omega \in \Omega} a_k(t_{j+1}, \omega), \max_{\omega \in \Omega} b_k(t_{j+1}, \omega)\}$.

We denote the set of admissible liquidation strategies by $\mathcal{H}(t_i, X(t_i))$.

Let us shortly comment on Definition 2.3. We recursively define for $j \geq i + 1$,

$$X(t_{j+1}) := X(t_j) - x(t_j) - z(t_j).$$

By abuse of notation, the liquidation constraint in (i) is then equivalent to $X(t_{N+1}) = 0$. By Assumption 2.2 (ii), all admissible liquidation strategies must satisfy $x(t_N) = X(t_N)$, $y(t_N) = z(t_N) = 0$. Definition 2.3 (ii) ensures that admissible strategies only place dark pool orders which have a chance of being fully executed. This consideration is required in order to ensure uniqueness of the optimal strategy. Due to order submission fees, the assumption is natural in practice.

Definition 2.3 imposes no constraints on the portfolio evolution from $X(t_0)$ to $X(t_{N+1}) = 0$. In particular, the seller may temporarily increase the position in an asset in the portfolio or may temporarily change the direction of the position from long to short or vice versa. As we will see in section 4.4, such liquidation strategies can be optimal as they can be used in a multi-asset portfolio to decrease market risk at low cost.\footnote{The convention for the traded amount $z$ is such that $z(t_i)$ refers to the trade executed at time $t_{i-1}$ resulting from the dark pool order $y(t_i)$ submitted at time $t_i$. It does not refer to the dark pool trade executed at time $t_i$ (which would be $z(t_{i-1})$).}

Figure 1 illustrates the time line of the liquidation process. At time $t_i$ two new pieces of information become available to the trader. First, the dark pool liquidity $(a(t_i), b(t_i))$ is realized. If the trader found liquidity in the dark pool, then (part of) the previously submitted order $y(t_{i-1})$ is executed at the price $P(t_{i-1})$; the traded amount $z(t_{i-1})$ is observed by the seller at time $t_i$. Second, $e(t_i)$ is observed and the new fundamental price $P(t_i)$ for the next trades is determined. After receiving these two pieces of information, the trader places new orders $x(t_i)$ at the primary venue and $y(t_i)$ in the dark pool. As a last step at time $t_i$, the order at the primary venue is executed in full at the price $P(t_i)$. The dark pool order remains unexecuted until time $t_{i+1}$ when the variables $a(t_{i+1}), b(t_{i+1})$ and hence $z(t_i)$ are realized.

For an admissible strategy $(x, y) \in \mathcal{H}(t_i, X(t_i))$, the trader’s cost of execution is given by the \textit{implementation shortfall} (see Perold (1988)):
\[
\mathcal{R}(t_i) := \mathcal{R}(t_i, x(t_0), \ldots, x(t_{i-1}), X(t_i); (x, y)) := X(t_i)^T \tilde{P}(t_i) - \sum_{j=1}^{N} \left( x(t_j)^T P(t_j) + z(t_j)^T \tilde{P}(t_j) \right)
\]

\[
= \sum_{j=1}^{N} \left( x(t_j)^T \left( \tilde{P}(t_i) - P(t_j) + f_j(x(t_0), \ldots, x(t_j)) \right) + z(t_j)^T \left( \tilde{P}(t_i) - P(t_j) \right) \right).
\]

The trade-off between expected proceeds and risk is an important driver of optimal liquidation and has been the focus of several investigations including Almgren and Chriss (2001), Almgren and Lorenz (2007), Schied and Schöneborn (2009) and Schied et al. (2012). In this paper, we assume that the investor wants to minimize the following function\(^\dagger\) of execution cost:

\[
J(t_i, x(t_0), \ldots, x(t_{i-1}), X(t_i); (x, y)) := \mathbb{E}\left[ \mathcal{R}(t_i, x(t_0), \ldots, x(t_{i-1}), X(t_i); (x, y)) \right] + \alpha \mathbb{E}\left[ \sum_{j=1}^{N} X(t_j)^T \Sigma(t_{j+1}) X(t_j) \right],
\]

where \(\alpha \geq 0\) is the coefficient of risk aversion and \(\Sigma(t_j)\) is the covariance matrix of the increments \(\alpha(t_j)\). The risk costs \(\alpha \sum_{j=1}^{N} X(t_j)^T \Sigma(t_{j+1}) X(t_j)\) reflect the market risk of the portfolio and thus penalize slow execution and poorly diversified portfolios. In the setting of Almgren and Chriss (2001) for optimal liquidation without dark pools, this is equivalent to minimizing the mean-variance functional \(\mathbb{E}[\mathcal{R}(t_i)] + \alpha \text{Var}[\mathcal{R}(t_i)]\) over all deterministic strategies. Schied et al. (2012) show that this in turn is equivalent to maximizing the utility of investors with constant absolute risk aversion.

The value function of the optimization problem is thus given by

\[
v(t_i, x(t_0), \ldots, x(t_{i-1}), X(t_i)) := \inf_{(x,y)\in \mathcal{H}(t_i, X(t_i))} \text{OPT}.
\]

We call a strategy \((x, y) \in \mathcal{H}(t_i, X(t_i))\) optimal if it realizes the minimum in equation (OPT) and denote optimal strategies by \((x^*, y^*)\) for the remainder of the paper. The amount executed in the dark pool in \([t_i, t_{i+1}]\) associated with the optimal order \(y^*(t_i)\) is denoted by \(x^*(t_i)\). Note that our optimization criterion penalizes risk due to market moves \(\epsilon(t_j)\), but not the risk due to execution uncertainty in the dark pool. Since the market risk usually outweighs the liquidity risk, disregarding the latter should not lead to significantly different results while at the same time simplifying the analysis considerably. ‘Selective risk aversion’ focusing only on market risk and disregarding liquidity risk has been applied before by Walia (2006) and Rogers and Singh (2010) in the context of stochastic liquidity and hedging. We discuss the effect of this choice in more detail in Appendix A.

3. Optimal liquidation

The following theorem establishes the existence and uniqueness of an optimal trading strategy that exploits both the trading opportunities at the primary exchange and the dark pool. \(\dagger\)

**Theorem 3.1** Assume that the assumptions of section 2 are satisfied. Let \(i = 0, \ldots, N - 1\), \(x(t_0), \ldots, x(t_{i-1}) \in \mathbb{R}^n\) be the previous trades of the investor and \(X(t_i) \in \mathbb{R}^n\) be the portfolio position at time \(t_i\). Then there exists a unique optimal strategy \((x^*, y^*) \in \mathcal{H}(t_i, X(t_i))\) realizing the minimum in equation (OPT).

4. Linear price impact

In the previous section, we established an existence and uniqueness result for a general market model. In order to obtain additional insight into the structure of the optimal liquidation strategy, we now consider the case of linear temporary price impact, which can be solved in explicit form. In section 4.1, we specify the model in terms of its price impact functions \(f_j\), the fundamental price process \(P\) and the liquidity in the dark pool \(a(t_0), b(t_0), \ldots, a(t_N), b(t_N)\). In section 4.2, the value function \(v\) and the optimal orders \(x^*(t_i), y^*(t_i)\) at times \(t_i\) are proven to be of quadratic respectively linear form and shown to satisfy a backward recursion. In sections 4.3 and 4.4, we study the effects of dark pools for liquidation of a single-asset position and a two asset portfolio, respectively.

\(\dagger\) The results of this section also hold for fundamental price processes \(\tilde{P}\) with drift: the proof of Theorem 3.1 does not use the assumption \(\mathbb{E}[\epsilon(t_i)] = 0\).
4.1. Model specification

In this section, we assume that price impact is linear and purely temporary. We specify the precise form of the impact costs, the distributions of the fundamental asset price $\hat{P}$ and the dark pool liquidity $(a, b)$ in the following way.

**Assumption 4.1**

(i) For $i = 0, \ldots, N$, $f_i(x(t_i)) := f_i(x(t_0), \ldots, x(t_i)) := \lambda x(t_i)$ for a positive definite matrix $\Lambda \in \mathbb{R}^{n \times n}$.

(ii) The covariance matrix of the increments of $\hat{P}$ is constant: $\Sigma(t_i) := \Sigma$ for all $i, j = 0, \ldots, N$.

(iii) Dark pool orders are executed fully or not at all, i.e. for $i = 1, \ldots, N + 1, k = 1, \ldots, n, a_i(t_i), b_k(t_i) \in \{0, \infty\}$. Furthermore, $(a_i(t_i), b_k(t_i))_{i=1}^{N+1}$ is identically distributed and $(a_i(t_i), b_k(t_i))$ and $\epsilon_i(t_i)$ are independent.

(iv) For $i = 1, \ldots, N + 1, k = 1, \ldots, n$,

$$\mathbb{P}[a_i(t_i) = \infty | a_i(t_i), b_k(t_i), l \neq k] = \mathbb{P}[b_k(t_i) = \infty | a_i(t_i), b_k(t_i), l \neq k]. \quad (19)$$

Assumption 4.1 (i) implies convexity and superlinear growth of the price impact costs, so that Assumption 2.1 is satisfied. We call the matrix $\Lambda$ the **price impact matrix** and say that the price impact is **linear** and **temporary** since the function $f_i$ only depends on the trade $x(t_i)$ at time $t_i$ and not on past trades $x(t_0), \ldots, x(t_{i-1})$. As a direct consequence of Assumption 4.1 (i), $v(t_i, x(t_i), \ldots, x(t_{i-1}), X(t_i))$ is independent of $x(t_0), \ldots, x(t_{i-1})$.

The martingale property of $\hat{P}$ together with the independence of future price moves of dark pool liquidity and the liquidation constraint implies $\mathbb{E}\left[\sum_{j=1}^{N} (x(t_j) + z(t_j))^\top (\hat{P}(t_j) - \hat{P}(t_{j-1}))\right] = 0$. The Optimization Problem (OPT) hence becomes

$$v(t_i, X(t_i)) := v(t_i, x(t_i), \ldots, x(t_{i-1}), X(t_i))$$

$$= \inf_{(x,y) \in \mathcal{A}(t_i, X(t_i))} \left\{ \mathbb{E}\left[\sum_{j=1}^{N} x(t_j)^\top \Lambda x(t_j) \right] \right.$$

$$+ \alpha \mathbb{E}\left[\sum_{j=1}^{N} X(t_j)^\top \Sigma X(t_j) \right]\right\}. \quad (OPT-LPI)$$

By Assumption 4.1 (iv), Assumption 2.2 (iv) is satisfied. Additionally, the assumption implies that supply and demand in the dark pool are distributed symmetrically.

4.2. Optimal liquidation

The following theorem establishes that the value function $v(t_i, \cdot)$ of the Optimization Problem (OPT-LPI) is quadratic and that the optimal orders $x^*(t_i)$ placed at the primary venue and $y^*(t_i)$ placed in the dark pool are linear functions of the portfolio $X(t_i)$ at any time $t_i$.

**Theorem 4.2** For $i = 0, \ldots, N$ there exist matrices $A(t_i), B(t_i), C(t_i) \in \mathbb{R}^{n \times n}$ such that for any portfolio $X(t_i) \in \mathbb{R}^n$ the unique optimal strategy $(x^*, y^*) \in \mathcal{A}(t_i, X(t_i))$ and the value function fulfill

$$x^*(t_i) = A(t_i) X(t_i), \quad y^*(t_i) = B(t_i) X(t_i), \quad (20)$$

$$v(t_i, X(t_i)) = X(t_i)^\top C(t_i) X(t_i)$$

with positive definite $C(t_i)$. An explicit recursion for $A(t_i), B(t_i)$ and $C(t_i)$ is given in Appendix B.

4.3. Liquidating a single-asset position

The most transparent case to analyse is the liquidation of a position $X(t_0)$ in a single asset ($n = 1$), for which we derive a closed form solution. Let $p := \mathbb{E}[\alpha(t_i) = \infty] = \mathbb{P}[b(t_i) = \infty]$ be the probability of order execution in the dark pool. For a given $p \in [0, 1)$, we denote the matrices (now real numbers) $A(t_i), B(t_i)$ and $C(t_i)$ introduced in Theorem 4.2 by $A(t_i, p), B(t_i, p)$ and $C(t_i, p)$ in order to highlight their dependence on $p$. For the remainder of the section, we let $X(t_0) > 0$. The results are symmetric in the sign of $X(t_0)$ and can be easily transferred to negative initial asset positions $X(t_0)$. In the following, we provide closed-form solutions for the optimal strategy and the value function which follow directly from Theorem 4.2 by standard methods for the solution of linear difference equations. The result is a generalization of the corresponding result in Almgren and Chriss (2001) for optimal liquidation without dark pools ($p = 0$). Let

$$\kappa(p) := \text{arcosh} \left( \frac{\sqrt{1 - p}}{\sqrt{\frac{\alpha \Sigma}{\Lambda} + 1}} \right). \quad (21)$$

Then the optimal orders are given by $x^*(t_i) = A(t_i, p) X(t_i)$ and $y^*(t_i) = B(t_i, p) X(t_i)$ with

$$A(t_i, p) = 1 - \frac{\sinh(\kappa(p)(N-i))}{\sqrt{1-p} \sinh(\kappa(p)(N+1-i))}, \quad (22)$$

$$B(t_i, p) = 1 - A(t_i, p)$$

$$= \frac{\sinh(\kappa(p)(N-i))}{\sqrt{1-p} \sinh(\kappa(p)(N+1-i))} < 1. \quad (23)$$

In particular, $0 < x^*(t_i), y^*(t_i) < X(t_i)$ and $x^*(t_i) + y^*(t_i) = X(t_i)$ for $i \neq N$. The value function is given by $v(t_i, X(t_i)) = C(t_i, p) X(t_i)^2$ with

---

†See Ye (2011) for an endogenous dark pool trading model with linear price impact at the primary venue.

‡Many market models dissect price impact into a temporary and a permanent component, for example Almgren and Chriss (2001) and Obizhaeva and Wang (2013). In this section, we omit the permanent impact component for two reasons. First, permanent price impact complicates the mathematics (most notably requires to keep track of the already accumulated permanent impact over time) without significantly changing the qualitative features of the optimal strategy. Second, it appears reasonable that permanent price impact also influences trade prices in the dark pool, since otherwise a permanent divergence between trade prices at the primary venue and in the dark pool would be created. Allowing for such a spillover effect on dark pool prices can lead to price manipulation as we will see in section 5.

§The corresponding continuous-time result can be found in Kratz and Schöneborn (2013), section 4.1. It can be shown that the discrete-time solution converges to the continuous-time solution as the number of time steps $N$ approaches infinity, see Kratz (2011).
the expected asset position over all scenarios. Solid thin lines refer to the optimal strategy without dark pool. Of Kratz and Schöneborn (2013).†The same result was derived by Ye (2011) in an endogenous model.

\[ C(t_j, p) = \frac{\Lambda}{1 - p} \left( \frac{\sqrt{1 - p} \sinh(\kappa(p)(N + 1))}{\sinh(\kappa(p)(N + 1 - i))} - 1 \right) \]  

We can also express \( X(t_j), x^*(t_j) \) and \( y^*(t_j) \) as functions of \( X(t_0) \). To this end, we define recursively,

\[ X^{ne}(t_0, p) := X(t_0), \]
\[ X^{ne}(t_j, p) := X^{ne}(t_{j-1}, p) - x^*(t_{j-1}). \]

By a simple forward induction, we obtain

\[ X(t_i) = X^{ne}(t_i, p) \]
\[ = \frac{1}{\sqrt{1 - p}} \sinh(\kappa(p)(N + 1 - i)) X(t_0) \]  

if the dark pool order has not been executed before time \( t_i \). Else, we have \( X(t_i) = x^*(t_i) = y^*(t_i) = 0 \).

Equations (22) and (23) answer the question of how to use a dark pool optimally for \( n = 1 \). It is always optimal to place the remainder of the asset position (minus the optimal order in the primary venue) in the dark pool as the investor pays no price impact there. Consequently, the liquidation task is finished as soon as the dark pool order is executed by Assumption 4.1 (iii).

Using the dark pool also changes optimal trading in the primary exchange. The trader slows down the trading speed in the primary venue as she wants as much as possible to be executed in the dark pool. If the position is not yet executed towards the end, she has to speed up in order to finish the liquidation until time \( T \). It can be shown that the optimal trading trajectory until order execution in the dark pool is strictly increasing in the dark pool liquidity (i.e. \( p \)). Consequently, the relative amount traded in the dark pool is increasing in \( p \).† On the other hand, the expected asset positions at time \( t_i \), \( E[X(t_i)] \), is strictly decreasing in \( p \).‡ Figure 2 illustrates how the dark pool changes the optimal strategy in the primary venue.

The costs of an admissible liquidation strategy \( (x, y) \) are composed of the impact costs of trading at the primary venue \( \Lambda \cdot E[\sum_{t_j} x(t_j)^2] \) and the risk costs \( \alpha \Sigma \cdot E[\sum_{t_j} X(t_j)^2] \). Using a dark pool reduces the overall costs; more generally, it can be shown that \( C \) is decreasing in \( p \) (e.g. by backward induction using

\[ E \left[ \sum_{t_j} X(t_j)^2 \right] \text{ strictly decreasing for } p \in (0, 1). \]

(i) If \( \alpha \Sigma > 0 \), then the risk costs \( \alpha \Sigma \cdot E[\sum_{t_j} X(t_j)^2] \) are strictly increasing for \( p \in (0, \frac{\alpha \Sigma}{\Lambda + \alpha \Sigma}) \) and strictly decreasing for \( p \in (\frac{\alpha \Sigma}{\Lambda + \alpha \Sigma}, 1) \).

We illustrate the dependence of the two components of the costs of the optimal strategy on \( p \) in figure 3. The left graph shows that the impact costs are decreasing in \( p \) on the whole interval \( (0, 1) \) while the risk costs are increasing for small \( p \) (right graph). Overall, the reduction of the impact costs outweighs the increase of the risk costs for small \( p \). Note that in continuous time, the risk costs are always decreasing in the dark pool liquidity (see Kratz and Schöneborn (2013) Proposition 4.1 (vi)). Hence the effect of Proposition 4.3 (ii) disappears as \( N \to \infty \).

The left graph of figure 4 illustrates the costs of different trading strategies dependent on the probability of execution \( p \). The dotted line denotes the costs of not using a dark pool (which is obviously independent of \( p \)). The solid line represents the costs of the optimal strategy and the dashed line represents the costs of the following naïve strategy:

Use the optimal strategy without dark pools for the primary venue and place the remainder of the position in the dark pool!

This strategy is cheaper than not using dark pools, as both impact costs and risk costs are saved if an order in the dark pool is executed before time \( T \). However, it is significantly more expensive than the optimal strategy, which unlocks additional cost saving potential by deferring execution at the primary venue.

For the performance of the optimal trading strategy, it is essential to estimate the parameters \( \Lambda, \Sigma, \) and \( p \) appropriately. Especially for the probability of execution this is difficult: as
orders are not reported openly in dark pools, it is hard to obtain useful data.

Let us assume that we have estimated the average number of executions in \([0, T]\) to be \(N \cdot q\). We have seen already that applying the optimal strategy \((x(t_i, q), y(t_i, q))\), reduces liquidation costs significantly, provided that \(q\) equals the real-world probability of execution \(p\). If we have underestimated \(p\), i.e. \(q < p\), the strategy \((x(t_i, q), y(t_i, q))\) is still cheaper than the optimal strategy without using dark pools. On the other hand, overestimating \(p\) can make the strategy \((x(t_i, q), y(t_i, q))\) more expensive than the optimal strategy without using dark pools. The middle graph of figure 4 illustrates the costs of \((x(t_i, q), y(t_i, q))\), dependent on \(p\).

### 4.4. Liquidating a portfolio of two assets

If a risk averse investor has to liquidate a portfolio of multiple assets \((n \geq 2)\), then correlation between the assets comes into play. It might no longer be optimal to always place the remaining portfolio into the dark pool. For example, a trader liquidating a well diversified portfolio consisting of two assets will most likely not want to risk losing her balanced position by being executed in only one of the two assets. In section 4.3, we introduced the naïve strategy of applying the optimal strategy without dark pools in the primary venue and placing the remainder of the position in the dark pool. Although suboptimal, this strategy performed strictly better than the best strategy without dark pool in the case \(n = 1\). A simple numerical example confirms that this is no longer the case for \(n \geq 2\). This highlights that it is not advisable to apply the naïve strategy for a portfolio of more than one asset. In this section, we will investigate the dependence of the value function and optimal strategy on the model parameters in a two asset setting. Note that explicit solutions for the value function and optimal strategy are unavailable in general in the multi-asset case \((n \geq 2)\).

We will see that dark pool trading is sensitive to the correlation of price increments and to the dependence of dark pool liquidity between the two assets. In order to simplify the exposition, we assume that there is no cross-asset price impact so that we can set:

\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}, \quad \Sigma = \begin{pmatrix}
\sigma_1^2 & \rho\sigma_1\sigma_2 \\
\rho\sigma_1\sigma_2 & \sigma_2^2
\end{pmatrix},
\]

(28)

For the purposes of this section, we assume that the variances \(\sigma_1^2\) and \(\sigma_2^2\) of the two assets as well as the risk aversion parameter \(\alpha\) are strictly positive.

#### 4.4.1. Dependence on the price correlation \(\rho\)

If the correlation of the two assets is positive \((\rho > 0)\), a portfolio consisting of a long position in one asset and a short position in the other one is more desirable than long positions (or short positions) in both assets; in the former case, a part of the risk of each asset is hedged by the other asset. Conversely, if \(\rho < 0\), it is more desirable to have long (or short) positions in both assets. In the following, we hence call a portfolio \(X = (X_1, X_2)\) well diversified if either the signs of the positions are equal \((\text{sgn}(X_1) = \text{sgn}(X_2))\) and \(\rho < 0\) or if the signs of the positions are different and \(\rho > 0\). Otherwise, the portfolio is called poorly diversified.

We proceed by describing the dependence of the liquidation costs and the optimal liquidation strategy on \(\rho\).

We first note that changing the sign of the position of only one asset of a well diversified portfolio (hence rendering the portfolio poorly diversified) increases the liquidation costs. Moreover, the larger the correlation \(|\rho|\) of a well diversified portfolio, the cheaper the liquidation costs. The right graph of figure 4 illustrates the dependence of the value function on the correlation \(\rho\) for a portfolio that is long in both assets. For \(\rho < 0\), this portfolio is well diversified and the value function is increasing in \(\rho\). For positive \(\rho > 0\), the portfolio is poorly diversified. This leads to elevated liquidation costs

\[\text{sgn}(X_1) = \text{sgn}(X_2)\]

Note that these observations are consistent with the corresponding results in continuous time (see Kratz and Schöneborn (2013) Propositions 4.5, 4.6 and 4.8) and can mostly be proven by similar types of reasoning. Detailed proofs are available from the authors upon request.

---

†Consider a risk averse investor (\(\alpha = 1\)) liquidating \(X_0 = (1, -1)^\top\) with \(\Sigma = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}\) at two trading times \((N = 1)\). Assume that the first stock is less liquid than the second:

\[
\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{P}[\text{No d.p. ex.}] = \frac{1}{2},
\]

\[
\mathbb{P}[\text{D.p. ex. of asset 1 only}] = 0, \quad \mathbb{P}[\text{D.p. ex. of asset 2 only}] = \frac{1}{2}.
\]

Using Theorem 4.2, we can compute the costs of the optimal strategy without dark pools and the naïve strategy. The former is less costly than the latter (2.98 versus 3.22).

‡These observations are consistent with the corresponding results in continuous time (see Kratz and Schöneborn (2013) Propositions 4.5, 4.6 and 4.8) and can mostly be proven by similar types of reasoning. Detailed proofs are available from the authors upon request.
We consider the following two portfolios in more depth:

(i) Long positions in both stocks, i.e. a poorly diversified portfolio: \( X(t_0) = (1, 1)^T \).

(ii) A long position in the first and a short position in the second stock, i.e. a well-diversified portfolio: \( X(t_0) = (1, -1)^T \).

Figure 5 shows the evolution of the two portfolios if a risk averse investor \( (\alpha = 4) \) applies the optimal strategy. For the poorly diversified portfolio, the trader tries to improve her risky position by trading out of the second stock. For this stock, trading in the primary venue is less expensive and being executed in the dark pool is more probable. It can easily be computed that this process on average evolves significantly faster if the trader uses the dark pool than without the dark pool. For the well-diversified portfolio, the portfolio position is decreasing almost linearly in time in all cases. We expect to trade only slightly faster if we use the dark pool. Note that this corresponds to the intuition given at the beginning of the section: it is most profitable to trade out of the position almost evenly.

Additionally, orders in the dark pool are very large for the poorly diversified portfolio and comparatively small for the well-diversified portfolio. The reason can be observed in figure 6. As long as the portfolio is poorly diversified, the risk is relatively large and is significantly decreased by a large execution in the dark pool (left picture). However, if it is well diversified as in the right picture, each execution in the dark pool increases the risk. Therefore, the dark pool saves price impact costs but potentially increases risk costs in this case.

In this subsection, we have seen that the optimal execution strategy changes in a systematic way when \( \rho \) is flipped from negative to positive. The dependence of optimal orders \( (x^*, y^*) \) on \( \rho \) within \( \rho \in [0, 1] \) or within \( \rho \in [-1, 0] \) however is not necessarily monotonic, as can be observed in figure 7. If \( \rho = 0 \) then the total order is \( x(t_i) + y(t_i) = X(t_i) \). For small positive \( \rho \), the total order is slightly increased for both assets to achieve a well-diversified portfolio if exactly one asset is executed in the dark pool. For large \( \rho \), the total order for the second asset is significantly larger than \( X_2(t_i) \); this maximizes the benefits of a dark pool execution of only the second asset. For the first asset, the preference for a large total order in the scenario...
of dark pool execution of only the first asset is increasingly
ominated by the preference for a small order in the scenario
of joint execution of both assets (in which case the position in
the second asset already flips sign).

4.4.2. Dependence on the interaction of dark pool liquidi-
ties for the two assets. The dependence of liquidity between
the two assets in the dark pool is governed by the three param-
eters \( p_1 \), \( p_2 \) and \( p \) defined in the previous subsection. The
effect of these parameters on the optimal liquidation strategy
is complex. In this subsection, we therefore restrict ourselves
to two examples that highlight several intuitive relationships
and are connected to results of previous subsections.

As a first example we consider the case
\( p \in [0, 1/1000] \),
\( p_1 = p_2 = 1/1000 - p \), \( \rho_0 = 998/1000 + p \); all other param-
eters are in the related numerical examples of section 4.4.1.
Note that the marginal probabilities of execution are inde-
dent of \( p \), e.g. the probability of dark pool execution for the
first asset is \( p_1 + p = \frac{1}{1000} \). Figure 8 illustrates the desired position
\( X_k(t_0) - x_k^0(t_0) - y_k^0(t_0) \) at time \( t_1 \) dependent on \( p \) for a poorly
diversified portfolio (\( X(t_0) = (1, 1)^\top \), \( \rho = 0.9 \); left picture)
and a well-diversified portfolio (\( X(t_0) = (1, -1)^\top \), \( \rho = 0.9 \);
right picture). For \( p < 1/1000 \) (respectively \( p_1, p_2 > 0 \)), the
position is well diversified if the original position \( X(t_0) \) was
well diversified. For poorly diversified \( X(t_0) \), we observe that
\( X_k(t_0) - x_k^0(t_0) - y_k^0(t_0) \) has a different sign than \( X_k(t_0) \)
for both assets. In both cases, \( X_k(t_0) - x_k^0(t_0) - y_k^0(t_0) \) approaches
zero as \( p \to 1/1000 \) and dark pool liquidity becomes perfectly
synchronized.

As a second example, we consider the case \( p_1 \neq p_2 \). We let
\( p \in [0, 1/1000] \), \( p_1 = \frac{1}{1000} - p \), \( p_2 = \frac{6}{1000} - p \), \( p_0 = \frac{993}{1000} + p \)
and all other parameters as before. We illustrate the desired
position at time \( t_1 \) in figure 9. The behaviour for larger \( p \) is
qualitatively different from the case \( p_1 = p_2 \) of figure 8.
For the poorly diversified portfolio, the desired position for
the second (liquid) asset (dashed line) is negative for all \( p \in
[0, p_1] \) and does not approach zero since even at \( p = p_1 \)
we have \( p_2 > 0 \) and hence a chance that only the order for
the second asset is executed. In contrast to that, the desired
position for the first (illiquid) asset is positive for sufficiently
large \( p \); in this case, execution of the dark pool order for the
first asset indicates that probably the order for the second asset
is also executed. This would turn the positive initial position
in the second asset into a short position, which is best hedged
by a long position in the first asset.† For the well-diversified
portfolio neither of the desired positions approach zero as
\( p \to 1/1000 \); the probability of simultaneous execution is
relatively small and hence the trader does not risk to lose her
well-diversified position.

Figures 8 and 9 also illustrate that the optimal strategy de-
pend strongly on the relative liquidity of the two stocks. Risk
mitigation plays a minor role for the illiquid stock as impact

†The desired position for the first asset is positive because \( p_1 < p_2 \)
and not because it is less liquid (\( \lambda_1 > \lambda_2 \)); if \( p_1 \) and \( p_2 \) are reversed,
the desired position for the second asset is positive for sufficiently
large \( p \).
costs outweigh risk. Therefore, both in the poorly and in the well-diversified case, the orders in the dark pool for the illiquid stock are relatively close to the remainder of the position in the stock, whereas this is not the case for the liquid stock.

5. Trading prices in the dark pool

So far we have assumed that trades in the dark pool are executed at the unaffected price $\tilde{P}$. Within this section, we assume instead that dark pool orders are executed at the exchange quoted price $P$, which includes the temporary market impact of the orders $x(t_i)$. As indicated in section 2.2, this might be a more appropriate assumption for some dark pools. However, this results in profitable market manipulating strategies unless the model parameters are chosen with great care as we shall show in this section. For simplicity, we assume the single-asset model described in section 4.3 and furthermore assume that the investor is risk neutral ($\alpha = 0$) in this section.

Market manipulation is a concern in all market models where a large trader’s orders have a feedback effect on the execution price of her own orders. Huberman and Stanzl (2004) and Gatheral (2010) derive necessary conditions for market models that exclude profitable market manipulation at a primary exchange. Both papers disregard trading opportunities in dark pools. For the primary exchange, the market model introduced in section 4 fulfills the requirements established in these papers, i.e. it is not possible to generate profits from market manipulation by trading only at the primary exchange. However, it might be possible to generate profits from market manipulation if orders are placed cleverly in parallel in the dark pool. It is unclear whether such profitable market manipulation strategies exist in reality; given that such strategies were used and had to be forbidden (see Gatheral (2010) for an exposition), such
opportunities seem to be available at least sometimes. Nevertheless, we agree with Huberman and Stanzl (2004), Gatheral (2010), Alfonsi and Schied (2010) and Alfonsi, Schied, et al. (2012) that an appropriate market model should exclude profitable market manipulation.

For the purposes of this section, we define market manipulation strategies in the following way.

**Definition 5.1** Let \( i = 0, \ldots, N \) and \( X(t_i) \in \mathbb{R} \). We call a strategy \( (x(t_i), y(t_i)) \) a market manipulation strategy if

\[
\text{sgn}(X(t_i)) \neq \text{sgn}(x(t_i)) \quad \text{or} \quad \text{sgn}(X(t_i)) \neq \text{sgn}(y(t_i)).
\]

(32)

As we saw in section 4.4, such orders can be attractive as risk mitigation tools in a multi-asset setting. In the single-asset setting of this section, this justification does not apply, and we saw in section 4.3 that if trades are executed in the dark pool at fundamental prices, then market manipulation as defined here is never optimal.

In the following, we consider in particular a market manipulation strategy similar to the classical ‘pump and dump’ strategy†. In our market model, selling the stock at the primary exchange after artificially elevating its price (‘pumping’) cannot generate profits due to the associated price reaction. A liquidation in the dark pool however does not face such a price penalty. Consider the following strategy:

Assume that the initial asset position is zero and that the number of trading time points \( N + 1 \) is divisible by four. From \( t_0 \) until \( t_{(N+1)/4} \), the investor buys a stock quantity \( X \) at each point in time at the primary exchange. Simultaneously, she seeks to dump shares by placing a sell order for \( \frac{(N+1)X}{4} \) in the dark pool until the order gets executed (if at all). At time \( t_{(N+1)/4} \), the investor either holds a long or short position of \( \frac{(N+1)X}{4} \) in the asset, which she liquidates at a constant rate over the remaining time points \( t_{(N+1)/4}, \ldots, t_N \). The expected trading proceeds are then

\[
\mathbb{E}
\left[
\sum_{i=0}^{N}
(x(t_i) + z(t_i))P(t_i)\right] = \mathbb{E}
\left[
\sum_{i=0}^{N}
(x(t_i) + z(t_i))\tilde{P}(t_i)\right]
\]

\[
= \mathbb{E}
\left[
\sum_{i=0}^{N}
(x(t_i) + z(t_i))\lambda x(t_i)\right]
\]

\[
= \Lambda(N + 1)\left(\frac{1}{6} - \frac{(1 - p)^{(N+1)/4}}{2}\right)X^2.
\]

(33)

(34)

The last expression is positive if the number of trading time points \( N + 1 \) is large enough. Furthermore, the expected proceeds grow in the position sizing factor \( X \): the larger the bets, the larger the expected proceeds. The following proposition summarizes the issues we found.

**Proposition 5.2** Assume that trades in the dark pool are executed at the market price \( P \). If

\[
N + 1 \geq \lfloor 4 \log(1/3)/\log(1 - p) \rfloor + 1,
\]

then profitable market manipulation strategies exist and optimal strategies do not exist.

In section 4.3, we assumed both infinite liquidity in the dark pool if trading is possible \( (a(t_i), b(t_i) \in [0, \infty]) \) and that price moves \( e(t_i+1) \) and dark pool liquidity \( a(t_i), b(t_i) \) are independent. We replace Assumption 4.1 (iii) by the following assumption.

**Assumption 5.3**

(i) Liquidity in the dark pool is bounded: \( a(t_i), b(t_i) \in [0, L] \) \( (i = 1, \ldots, N + 1) \) for some \( L \in (0, \infty) \).

(ii) \( (a(t_i), b(t_i))_{i=1,\ldots,N+1} \) is identically distributed and

\[
\mathbb{E}[e(t_i)]a(t_i) = \infty \quad \text{and} \quad \mathbb{E}[e(t_i)]b(t_i) = \infty.
\]

(36)

for a non-negative constant \( \Gamma \).

Let us comment on Assumption 5.3 (ii). In section 4, we assumed that the price increments \( e(t_i) \) and the liquidity variables \( a(t_i), b(t_i) \) are independent. This assumption is not always satisfied in reality. For example, several large traders might be using the dark pool and the exchange in parallel to execute trades. We can split these traders into a group of buyers and a group of sellers. Traders in the larger of these two groups face increased competition in the dark pool (and hence a reduced probability of execution in the dark pool) as well as adverse price movements at the exchange reflecting the net price impact of the large traders. Those traders in the smaller group find unusually high liquidity in the dark pool (and thus an increased likelihood of execution) while prices move in their favour at the exchange. All that the individual trader observes directly is that price changes \( e(t_i) \) at the primary venue and liquidity \( a(t_i), b(t_i) \) in the dark pool are correlated. Liquidity seeking traders find that their trades in the dark pool are usually executed just before a favourable price move, i.e. exactly when they do not want them to be executed since they miss out on the price improvement. In advance of adverse price movements, they observe that they rarely find liquidity in the dark pool. Traders in the dark pool are hence suffering adverse selection.‡

By limiting dark pool liquidity, market manipulating strategies with very large trades cannot be profitable. On the other hand, adverse selection makes market manipulation by small trades unprofitable. The following proposition shows that if Assumption 4.1 (iii) is replaced by Assumption 5.3 and adverse selection is sufficiently large, then the undesirable properties outlined in Proposition 5.2 disappear.§

**Proposition 5.4** Let \( i = 0, \ldots, N \). Assume that trades in the dark pool are executed at the market price \( P(t_j), j = \)

†’Pump and dump’ schemes, also known as ‘hype and dump manipulation’, involve the touting of a company’s stock [...] After pumping the stock, fraudsters make huge profits by selling their cheap stock into the market.” (From http://www.sec.gov/answers/pumpdump.htm)

‡A detailed discussion of adverse selection within our framework including a review of the relevant literature can be found in Kratz and Schöneborn (2014).

§Ye (2011) proposes a model where transactions in the dark pool are executed at the price \( P \) including the market impact of primary venue orders. In Ye’s model, market manipulation is not optimal. This is qualitatively consistent with Proposition 5.4 since Ye assumes limited dark pool liquidity (by a power-law) as well as a (proportional) commission for dark pool orders which has a similar effect as the adverse selection cost in our model.
We consider the following optimization problem:

\[
\tilde{\varepsilon}(t_i, X(t_i)) = \inf_{(x, y) \in \tilde{A}(t_i, X(t_i))} \mathbb{E}\left[ \sum_{j=1}^{N} (x(t_j) + z(t_j)) \top (\tilde{P}(t_i) - P(t_j)) \right].
\]

If \( \Gamma > \Lambda L \), then there exist optimal strategies realizing the minimum in equation (37) and these are not market manipulating.

The assumptions of Proposition 5.4 are strong; we leave it for future research (such as Klöck et al. (2014)) to determine tighter necessary and sufficient conditions for the exclusion of profitable market manipulation in markets with dark pools. We only want to remark that our assumptions in Proposition 5.2 are not too restrictive for dark pool usage in general: for large initial asset positions \( X(t_0) \), the optimal strategy places orders in the dark pool in a non-market-manipulating fashion.

### 6. Pinging

In the previous section, we assumed that price moves \( \epsilon(t_i) \) and dark pool liquidity \( (a(t_i), b(t_i)) \) can be dependent for the same point in time \( t_i \) (cf. Assumption 5.3 (ii)). In reality, price moves after time \( t_i \) might depend on dark pool liquidity at time \( t_i \). For example, traders looking to buy in the dark pool might continue to buy at the primary venue later on, thus driving up prices in the future. If this is the case, then traders can infer whether prices will rise or fall in the future from historical dark pool liquidity. Market participants, however, do not have full transparency of historical dark pool liquidity \( (a, b) \), but can only infer part of this information from their own execution experience \( z \) in the dark pool. In this section, we investigate this situation and show that ‘pinging’ the dark pool can be profitable, i.e. the submission of orders to the dark pool not for the purpose of portfolio liquidation but instead in order to learn about dark pool liquidity and hence future price moves.

We consider the single asset, linear price impact model of section 4 where dark pool orders are executed either fully or not at all with \( p := \mathbb{P}[a(t_i) = \infty] = \mathbb{P}[b(t_i) = \infty] > 0 \) (in particular, the transaction price in the dark pool is \( \tilde{P} \) in contrast to the model of section 5). We only remove the assumption that future price moves are independent of current dark pool liquidity and replace it in the following way.

**Assumption 6.1** We assume that Assumption 4.1 (i), (ii) and (iv) hold and replace (iii) by the following modification:

- **(iii)** Dark pool orders are executed fully or not at all, i.e. for \( i = 1, \ldots, N + 1, a(t_i), b(t_i) \in [0, \infty] \). Furthermore, \((a(t_i), b(t_i))_{i=1,\ldots,N+1}\) is identically distributed and

\[
\mathbb{E}[\epsilon(t_{i+1})|a(t_i) = \infty] = -\Gamma, \\
\mathbb{E}[\epsilon(t_{i+1})|b(t_i) = \infty] = \Gamma.
\]

for a non-negative constant \( \Gamma \).

Because of the martingale property \( \mathbb{E}[\epsilon(t_{i+1})] = 0 \), a drift for time \( t_{i+1} \) is expected not only if \( a(t_i) = \infty \), but also if

\[
a(t_i) = 0; \tilde{\varepsilon}(t_i, X(t_i)) = \mathbb{E}[\epsilon(t_{i+1})|a(t_i) = 0] = \frac{p}{1-p} \Gamma, \\
\mathbb{E}[\epsilon(t_{i+1})|b(t_i) = 0] = -\frac{p}{1-p} \Gamma.
\]

We assume that there is no further dependence between \((a(t_i), b(t_i))\) and \( \epsilon(t_i) \).

The trader does not observe \( a(t_i) \) or \( b(t_i) \) directly, but observes \( z(t_{i-1}) \) at time \( t_i \). This allows her to infer \( b(t_i) \) if \( y(t_{i-1}) > 0 \); if \( z(t_{i-1}) > 0 \) then dark pool liquidity must have been available, i.e. \( b(t_i) = \infty \). Conversely if \( z(t_{i-1}) = 0 \) then \( b(t_i) = 0 \) must have been the case. Similarly, the trader can infer \( a(t_i) \) if \( y(t_{i-1}) < 0 \). No further information about \( a \) and \( b \) is available to the trader. In particular, she can infer neither \( a(t_i) \) nor \( b(t_i) \) if \( y(t_{i-1}) = 0 \).

Let us consider a risk neutral (\( \alpha = 0 \)) strategic trader with no initial position \((X(t_0) = 0)\) who seeks to maximize trading profits, i.e. to minimize the cost functional in the Optimization Problem (OPT). A pinging strategy refers to the following trading pattern. The trader’s first action in the markets (say at time \( t_i \)) is placing a small buy (or sell) order in the dark pool in order to learn about liquidity available in the dark pool. If the order is executed, a negative (positive) price move is expected for the subsequent time point. To capitalize on this forecast, the trader sells (buys) the asset at the exchange (and possibly also in the dark pool) at \( t_{i+1} \) and buys (sells) it back after the forecast played out, i.e. during \( t_{i+2} \) up until \( t_N \). If the order is not executed, the converse is the case (cf. (40)). The following proposition confirms that such a pinging strategy is indeed optimal.

**Proposition 6.2** Let \( X(t_0) = 0 \). Then there are pinging strategies that generate an expected profit. Furthermore the size of the profit is strictly decreasing with the size of the initial dark pool order. Therefore, no optimal pinging strategy (and no optimal strategy overall) exists.

Figure 10 illustrates a profitable pinging strategy that trades at three time points \( t_0, t_1, t_2 \). In reality, dark pools go to great length to avoid pinging. Minimum order sizes are a frequently used anti-gaming measure. In some dark pools, these are enforced for all orders by the dark pool operator. In other dark pools, traders are allowed to specify a minimum fill size for their dark pool orders, thus preventing matching with only a small pinging order. The following proposition shows that a minimum order size (if large enough) yields pinging unattractive in our model.

**Proposition 6.3** There is a constant \( \tilde{M} \) such that if dark pool orders must satisfy \( |y(t_i)| \geq \tilde{M} \) for all \( i \) then the optimal strategy for a trader with initial position \( X(t_0) = 0 \) is not to trade at all. In particular pinging is not profitable.

\(^{\dagger}\)A possible joint distribution for \((\epsilon(t_{i+1}), a(t_i), b(t_i))\) satisfying these requirements is \( \mathbb{P}[a(t_i) = 0, b(t_i) = 0] = 1 - 2p, \mathbb{P}[a(t_i) = \infty, b(t_i) = \infty] = 0, \mathbb{P}[a(t_i) = \infty, b(t_i) = 0] = \mathbb{P}[a(t_i) = 0, b(t_i) = \infty] = p, \mathbb{E}[\epsilon(t_{i+1})|a(t_i) = 0, b(t_i) = 0] = 0, \mathbb{E}[\epsilon(t_{i+1})|a(t_i) = \infty, b(t_i) = 0] = -\Gamma, \mathbb{E}[\epsilon(t_{i+1})|a(t_i) = 0, b(t_i) = \infty] = \Gamma. \)

\(^{\ddagger}\)This distinction was not necessary in the models of sections 2 to 5, since observing \((a(t_i), b(t_i))\) did not yield any benefits for the trader.
7. Proofs

Proof of Theorem 3.1 Existence of an optimal strategy. Instead of describing a strategy \((x, y) \in \mathcal{H}(t, X(t))\) as a stochastic process, we can alternatively describe it as a vector. Let therefore \(\Omega = \{\omega_1, \ldots, \omega_M\}\). By abuse of notation, we write

\[
w = \begin{pmatrix} x(t_1, \omega_1), \ldots, x(t_1, \omega_M), x(t_1+1, \omega_1), \ldots, x(t_1+1, \omega_M), \ldots, x(t_N, \omega_1), \ldots, x(t_N, \omega_M)\end{pmatrix}^\top \in \mathbb{R}^{n \times M \times (N+1-i)}.
\]

The objective function \(C(w) := J(t_1, x(t_0), \ldots, x(t_i-1), X(t_1); w)\) is continuous in the strategy \(w \in \mathbb{R}^{n \times M \times (N+1-i)}\), and the set of admissible strategies corresponds to a closed subset of \(\mathbb{R}^{n \times M \times (N+1-i)}\). We show that

\[
\lim_{\|w\| \to \infty} C(w) = \infty,
\]

where \(\|\cdot\|\) is the maximum norm on \(\mathbb{R}^{n \times M \times (N+1-i)}\). This allows us to restrict \(w\) to a bounded set, and so the existence of an optimal strategy follows from continuity of \(C\).

It is sufficient to prove equation (42) for \(\alpha = 0\), for which we obtain

\[
C(w) = \mathbb{E}\left[ \sum_{j=1}^{N} x(t_j)^\top f_j(x(t_0), \ldots, x(t_j)) \right] + \mathbb{E}\left[ \sum_{j=1}^{N} x(t_j)^\top (\hat{P}(t_j) - \tilde{P}(t_j)) \right] + \mathbb{E}\left[ \sum_{j=1}^{N} z(t_j)^\top (\hat{P}(t_j) - \tilde{P}(t_j)) \right] =: C_1(w) + C_2(w) + C_3(w).
\]

Note that \(C_1, C_2, C_3\) are not necessarily bounded from below and that \(C_2(w) > 0\) for \(\|w\|\) large enough. Therefore, we have to show that \(C_1\) grows faster in \(w\) than \(C_2\) and \(C_3\).

It follows from Assumption 2.1 (ii) that \(\lim_{\|w\| \to \infty} \frac{C_1(w)}{\|w\|} = \infty\). Since \(\Omega\) is finite, the price process \(\tilde{P}\) is bounded, and thus there exists a constant \(\tilde{C}\) such that for all \(w\), \(C_3(w) \leq \tilde{C}\). If the liquidity in the dark pool is bounded for all assets, then \(\|w\|\) and \(C_3(w)\) are bounded and thus \(\lim_{\|w\| \to \infty} \frac{C_3(w)}{\|w\|} = 0\). If not, we obtain similarly as before for \(w\) such that \(|z| \leq |y|\). \(\lim_{\|w\| \to \infty} \frac{C_3(w)}{\|w\|} \leq \tilde{C}\). Finally, a large order in the dark pool at a given point in time requires large orders in the primary venue with positive probability since by Definition 2.3 (ii), full execution of the dark pool order is possible, while on the other hand future dark pool orders have positive probability of non-execution (cf. Assumption 2.2 (ii)). Thus, there exists a constant \(C\) such that \(\lim_{\|w\| \to \infty} \frac{C_1(w)}{\|w\|} = \infty\).

The inequality follows directly from the convexity of the price impact cost of trading (Assumption 2.1 (i)).

For the induction step we consider two points \((x(t_0), \ldots, x(t_i-1), X(t_1))\) and \((\tilde{x}(t_0), \ldots, \tilde{x}(t_i-1), \tilde{X}(t_1))\). For these points, optimal orders \((x(t_1), y(t_1)) := (x^*(t_1), y^*(t_1))\) respectively \((\tilde{x}(t_1), \tilde{y}(t_1)) := (\tilde{x}^*(t_1), \tilde{y}^*(t_1))\) exist. We define continuous functions \(x(t_1), \ldots, x(t_i), y(t_1), \ldots, y(t_i) : [0, 1] \to \mathbb{R}^n\), \(0 \leq j \leq i\), such that \(x(t_j, 0) = x(t_j), y(t_j, 0) = y(t_j), X(t_j, 0) = X(t_j)\) and for \(0 \leq j \leq i\), \(x(t_j, 1) = \tilde{x}(t_j), x(t_j, 1) = \tilde{y}(t_j), X(t_j, 1) = \tilde{X}(t_j)\). Then by the dynamic programming principle,

\[
v(t_i, x(t_0, s), \ldots, x(t_i-1, s), X(t_i, s)) \leq x(t_i, s)^\top f_i(x(t_0, s), \ldots, x(t_i, s)) + \alpha x(t_i, s)^\top \Sigma(t_i+1) x(t_i, s) + (X(t_i, s) - x(t_i, s))^\top \mathbb{E}[\tilde{P}(t_i) - \tilde{P}(t_i+1)] - \mathbb{E}[z(t_i, s)^\top (\tilde{P}(t_i) - \tilde{P}(t_i+1))] + \mathbb{E}[v(t_{i+1}, x(t_0, s), \ldots, x(t_i, s), X(t_i, s) - x(t_i, s) - z(t_i, s))]
\]

where Inequality (45) is an equality for \(s = 0\) and \(s = 1\). We now assume that the assertion holds for \(i + 1\) and divide the proof of the induction step into three parts.

(i) Let

\[
\tilde{h}_i(s) := h_i(s) + \sum_{j=0}^{i-1} x(t_j, s)^\top f_j(x(t_0, s), \ldots, x(t_j, s)).
\]

We show that \(\tilde{h}_i\) is strictly convex, then \(H_i\) is strictly convex and the optimal strategy at time \(t_i\) is unique.

(ii) We define the functions \(x(t_i, \cdot), y(t_i, \cdot), X(t_i, \cdot)\) in such a way that \(x(t_i, \cdot), \mathbb{E}[\tilde{P}(t_i, \cdot)]\) and \(X(t_i, \cdot)\) are affine linear; here, we use the shorthand notation \(z(t_i, s) := z(t_i, y(t_i, s))\). This is needed to carry out step (iii).

(iii) We show that if \(H_{i+1}\) is strictly convex (induction hypothesis),

\[
x(t_0), \ldots, x(t_i-1), X(t_i), x(t_i), y(t_i)) \neq (\tilde{x}(t_0), \ldots, \tilde{x}(t_i-1), \tilde{X}(t_i), \tilde{x}(t_i), \tilde{y}(t_i))
\]

and \(x(t_j, \cdot), y(t_j, \cdot), X(t_j, \cdot)\) are defined as in (ii), then \(\tilde{h}_i\) is strictly convex in \(s\) on \([0, 1]\). Hence by (i), \(H_i\) is strictly convex.
Figure 10. Trading trajectories of a strategic trader who pings the dark pool with a sell order $y(t_0) = 0.1$. The left picture denotes the scenario where the pinging order is executed, the right picture the one where it is not executed. In both pictures, dashed lines refer to the scenario where the dark pool order at time $t_1$ is executed; solid lines refer to the scenario where it is not executed. $\Lambda = 1$, $\Gamma = 3$, $p = 1/3$.

(i) Let $(x(t_0), \ldots, x(t_{i-1}), X(t_i)) \neq (\tilde{x}(t_0), \ldots, \tilde{x}(t_{i-1}), X(t_i))$ and $s \in (0, 1)$. Then strict convexity of $H_i$ follows as

$$H_i((1 - s)x(t_0) + s\tilde{x}(t_0), \ldots, (1 - s)x(t_{i-1}) + s\tilde{x}(t_{i-1}), (1 - s)X(t_i) + s\tilde{X}(t_i))$$

\[ \overset{(45)}{\approx} \tilde{h}(s) = \tilde{h}_i((1 - s)0 + s\cdot 1) \]

\[ < (1 - s)\tilde{h}_i(0) + s\tilde{h}_i(1) \]

\[ = (1 - s)H_i((x(t_0), \ldots, x(t_{i-1}), X(t_i)) \]

\[ + sH_i((\tilde{x}(t_0), \ldots, \tilde{x}(t_{i-1}), \tilde{X}(t_i)). \]

(ii) We define the functions $x(t_j, \cdot)$ and $X(t_j, \cdot)$ by the convex combinations

\[ x(t_j, s) := (1 - s)x(t_j) + s\tilde{x}(t_j) \]

for all $0 \leq j \leq i$, $0 \leq s \leq 1$, \[(52)\]

\[ X(t_i, s) := (1 - s)X(t_i) + s\tilde{X}(t_i) \]

for all $0 \leq s \leq 1$. \[(53)\]

Note that if we define $y(t_i, s)$ accordingly, the linearity of $y(t_i, \cdot)$ neither carries over to $z(t_i, s)$ nor to $E[z(t_i, s)]$. The key step in the proof is to define $y(t_i, s)$ in such a way that $s \mapsto E[z(t_i, s)]$ is affine linear. We set $y(t_i, s)$ such that $E[z(t_i, s)] = (1 - s)E[z(t_i)] + sE[\tilde{z}(t_i)]$. To this end, we define the function $g(y) := E[z(y)]$. It is injective by Definition 2.3 (ii) and continuous. We define

\[ y(t_i, s) := g^{-1}((1 - s)E[z(t_i)] + sE[\tilde{z}(t_i)]), \]

\[ \overset{(54)}{\in \text{in particular } y(t_i, 0) = y^*(t_i), y(t_i, 1) = \tilde{y}^*(t_i) \text{ and } y(t_i, \cdot) \text{ is continuous on } [0, 1]. \}

Note that $y(t_i, \cdot)$ is piecewise affine linear but not affine linear in general; also, $E[z(t_i, \cdot)]$ is affine linear but pathwise $z(t_i, \cdot)$ is not affine linear.

(iii) If $z(t_i, \cdot)$ is pathwise affine linear on the whole interval $[0, 1]$, then strict convexity of $H_{i+1}$ would directly transfer to strict convexity of $\tilde{h}_i$ by definition of $\tilde{h}_i$ (cf. Inequality (45)). As we only have linearity of $E[z(t_i, \cdot)]$, the line of argument becomes more extensive. The only points where convexity of $\tilde{h}_i$ can break down are the points $s_j$ at which the slope changes for $y_k(t_j, \cdot)$ for some $k = 1, \ldots, n$. We denote the finitely many points at which there is a coordinate $k$ such that

\[ \mathbb{P}[-y_k(t_j, s_j) = a_k(t_{k+1})] = 0 \text{ or } \mathbb{P}[y_k(t_j, s_j) = b_k(t_{k+1})] > 0 \]

\[ \overset{(55)}{\leq} 0 < s_1 < \cdots < s_M < 1. \]

We can assume that at each $s_j$ there is exactly one coordinate $k_j$ such that (55) holds. On $(s_j, s_{j+1})$, the strict convexity of $\tilde{h}_i$ is clear since the map

\[ s \mapsto (x(t_0, s), \ldots, x(t_i, s), z(t_i, s), X(t_i, s)) \]

\[ \overset{(56)}{\text{is pathwise affine linear (note that } y_k(t_j, \cdot) \text{ is affine linear on } (s_j, s_{j+1}), \text{ and thus } \tilde{h}_i \text{ is strictly convex by the strict convexity of } H_{i+1}. \text{ Let therefore } j = 1, \ldots, M' \text{ and } k \in \{1, \ldots, n\} \text{ such that}} \]

\[ \mathbb{P}[y_k(t_j, s_j) = -a_k(t_{k+1})] \cup [y_k(t_j, s_j) = b_k(t_{k+1})] > 0. \]

\[ \overset{(57)}{\leq} \]

We first consider the case $y_k(t_j, s_j) > 0$, i.e.

\[ \mathbb{P}[y_k(t_j, s_j) = b_k(t_{k+1})] > 0. \]

Then $y_k(t_j, s_j) > 0$ for all $s \in (s_{j-1}, s_{j+1})$ by Assumption 2.2 (ii). We assume without loss of generality that $y_k(t_j, s_{j-1}) < y_k(t_j, s_{j+1})$ (the proof for the case $y_k(t_j, s_{j-1}) = y_k(t_j, s_{j+1})$ is straightforward) and define $A := \{y_k(t_j, s_j) \leq b_k(t_{k+1})\} \subseteq \Omega$. Let

\[ \tilde{z}_k(t_j, s) := \left\{ \begin{array}{ll} E[z_k(t_j, s) | A] & \text{on } A \\
E[z_k(t_j, s)] & \text{otherwise,} \end{array} \right. \]

\[ \overset{(58)}{\leq} \tilde{z}_k(t_j, s) := z(t_j, s) \text{ for } l \neq k. \]

Note that for $l \neq k$, $z(t_j, s)$ is pathwise affine linear on the whole interval $(s_{j-1}, s_{j+1})$. On $\Omega \setminus A$, $z_k(t_j, s)$ is independent of $s \in (s_{j-1}, s_{j+1})$. On $A$, we have

\[ E[z_k(t_j, s) | A] = \frac{1}{\mathbb{P}(A)}(E[z_k(t_j, s)] - E[\mathbb{1}_A z_k(t_j, s)]). \]

\[ \overset{(59)}{\leq} \]

$E[z_k(t_j, s)]$ is affine linear by construction and $E[\mathbb{1}_A z_k(t_j, s)]$ is independent of $s$. Therefore, $\tilde{z}(t_j, \cdot)$

\[ \overset{(i)}{\leq} \text{In the case that there are multiple such } k's, \text{ an arbitrary small perturbation of } a(t_{k+1}) \text{ and } b(t_{k+1}) \text{ removes multiplicity, and a simple approximation argument using the fact that } z(t_j) \text{ is continuous in } a(t_{k+1}) \text{ and } b(t_{k+1}) \text{ extends the desired result to full generality.} \]
is pathwise affine linear on the whole interval \((s_{j-1}, s_{j+1})\). We obtain that
\[
\begin{align*}
\sum_{j=0}^{i-1} x(t_j, s)^T f_j(x(t_0, s), \ldots, x(t_j, s)) + \tilde{h}_i(s),
\end{align*}
\]
where
\[
\begin{align*}
\tilde{h}_i(s) := x(t_i, s)^T f_i(x(t_0, s), \ldots, x(t_i, s)) + \alpha \mathbf{X}(t_i, s)^T \mathbf{\Sigma}(t_i+1)X(t_i, s) \\
\quad + (X(t_i, s) - x(t_i, s))^T \mathbb{E}[\mathbf{\tilde{P}}(t_i) - \mathbf{\tilde{P}}(t_{i+1})] \\
\quad - \mathbb{E}[\mathbf{z}(t_i, s)^T (\mathbf{\tilde{P}}(t_i) - \mathbf{\tilde{P}}(t_{i+1}))] \\
\quad + \mathbb{E}[v(t_{i+1}, x(t_0, s), \ldots, x(t_i, s), X(t_i, s) - x(t_i, s) - \mathbf{z}(t_i, s))]
\end{align*}
\]
is strictly convex in \(s\) on \((s_{j-1}, s_{j+1})\) as before by the strict convexity of \(H_{i+1}\). It is clear that \(\tilde{h}_i(s) = h_i(s)\) for \(s \leq s_j\). If \(\tilde{h}_i(s) \leq h_i(s)\) for all \(s \in (s_{j-1}, s_{j+1})\), convexity of \(\tilde{h}_i\) follows at the point \(s_j\).

To this end, we observe that for \(s > s_j\),
\[
\begin{align*}
h_i(s) - \tilde{h}_i(s) &= \mathbb{E}[\mathbb{1}_A (\mathbf{z}_k(t_k, s) - \mathbf{z}_k(t_k, s)) \mathbb{1}_k(t+1)] \\
&\quad + \mathbb{E}[\mathbb{1}_A v(t_{i+1}, x(t_0, s), \ldots, x(t_i, s), X(t_i, s) - x(t_i, s) - \mathbf{z}(t_i, s)) \\
&\quad - \mathbb{E}[\mathbb{1}_A v(t_{i+1}, x(t_0, s), \ldots, x(t_i, s), X(t_i, s) - x(t_i, s) - \mathbf{z}(t_i, s))] \\
&=: C_1 + C_2 - C_3.
\end{align*}
\]

By induction hypothesis, \(v(t_{i+1}, x(t_0, s), \ldots, x(t_i, s), \cdot)\) is strictly convex. Thus, by a standard argument using Jensen's inequality, the disintegration theorem, the definition of \(\mathbf{z}_k\) on \(A\) and Assumption 2.2 (iv), we have \(C_2 \geq C_3\).

The last step is to show that \(C_1 \geq 0\). We let \(m_1\) be the slope of \(y_k(t_i, s)\) for \(s < s_j\) and \(m_2\) be the slope of \(y_k(t_i, s)\) for \(s > s_j\). Note that \(0 < m_1 < m_2\). For \(s < s_j\), we have \(y_k(t_i, s) = \mathbf{z}_k(t_i, s)\) on \(A\). Thus as \(\mathbf{z}_k(t_i, \cdot)\) is affine linear, \(\mathbf{z}_k(t_i, \cdot)\) has slope \(m_1\) on the whole interval \((s_{j-1}, s_{j+1})\). Let now \(s > s_j\). For \(A_1 := \{y_k(t_i, s) = b(t_{i+1})\}\), \(A_2 := \{y_k(t_i, s) < b(t_{i+1})\}\), we have on \(A = A_1 \cup A_2\),
\[
\begin{align*}
\mathbf{z}_k(t_i, s) - \mathbf{z}_k(t_i, s) &= -m_1 (s - s_j) \quad \text{on } A_1 < 0, \\
\mathbf{z}_k(t_i, s) - \mathbf{z}_k(t_i, s) &= (m_2 - m_1)(s - s_j) \quad \text{on } A_2 > 0.
\end{align*}
\]
Furthermore, \(\mathbb{E}[A_1] = (m_2 - m_1)/m_2 \mathbb{E}[A], \mathbb{E}[A_2] = m_1/m_2 \mathbb{E}[A]\). Combining this with equations (64), (65) and Assumption 2.2 (iii), yields \(C_1 \geq 0\).

The cases \(y(t_i, s_j) < 0\) and \(y(t_i, s_j) = 0\) follow similarly with straightforward modifications. Combining these observations, we have strict convexity of \(\tilde{h}_i\) at all points \(s_j\) and on all intervals \((s_j, s_{j+1})\), i.e. on the whole interval \([0, 1]\), completing the proof of (iii).

\textbf{Proof of Theorem 4.2} We prove the theorem by backward induction. The assertions are clear for \(i = N\) as \(\Lambda\) is positive definite. Let now \(i < N\). Due to the linearity of the price impact function, the martingale property and the independence of dark pool liquidity of future price moves, we obtain the Bellman equation
\[
v(t_i, X(t_i)) = \inf_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ x^T \mathbf{A} x + \alpha \cdot X(t_i)^T \mathbf{X} X(t_i) \\
+ \sum_i p_i v(t_{i+1}, X(t_i) - x - Z_i y) \right\};
\]
recalling that the optimal strategy must fulfill \(y_k(t_i) = 0\) for \(k > k_0\) almost surely by Assumption 2.3 (ii). By abuse of notation, \(y\) denotes simultaneously \(y \in \mathbb{R}^{k_0}\) and \(y \in \mathbb{R}^n\) with \(y_k = 0\) for \(k > k_0\), where either way is clear from the context. Function \(v(t_i, X(t_i), \cdot) : \mathbb{R}^n \times \mathbb{R}^{k_0} \rightarrow \mathbb{R}\) given by
\[
\begin{align*}
v(t_i, X(t_i), x, y) := x^T \mathbf{A} x + \alpha \cdot X(t_i)^T \mathbf{X} X(t_i) \\
+ \sum_i p_i (X(t_i) - x - Z_i y)^T C(t_{i+1})(X(t_i) - x - Z_i y);
\end{align*}
\]
\(v(t_i, X(t_i), \cdot)\) is a strictly convex linear-quadratic functional by the induction hypothesis and \(\Lambda\) is positive definite. Therefore, the unique minimum \((x^*, y^*)\) of \(v(t_i, X(t_i), \cdot)\) is given by the solution of
\[
\begin{align*}
\nabla_x v(t_i, X(t_i), x, y) &= 0, \\
\nabla_y v(t_i, X(t_i), x, y) &= 0.
\end{align*}
\]
Solving System (68) for \((x, y)\) by means of elementary calculus, yields equation (B2). Plugging this into \(v(t_i, X(t_i), \cdot)\), we obtain
\[
\begin{align*}
v(t_i, X(t_i)) &= X(t_i)^T \left( \mathbf{A}(t_i)^T \mathbf{A}(t_i) + \alpha \mathbf{\Sigma} + \sum_i p_i (I - A(t_i - Z_i B(t_i))^T \\
&\quad \left( C \cdot C(t_{i+1})(I - A(t_i) - Z_i B(t_i)) \right) X(t_i) \right) X(t_i) \\
&=: C(t_i)
\end{align*}
\]
By the induction hypothesis, \(C(t_i)\) is non-negative definite. To see that \(C(t_i)\) is indeed positive definite, let \(x \in \mathbb{R}^n\), \(x_k \neq 0\). If \(A(t_i)x = 0\), then \(x^T A(t_i)^T \mathbf{A}(t_i) x > 0\). In any case, there exists an \(l\) such that the \(l\)-th diagonal element of \(Z_i\) is 0 and \(p_i > 0\) (cf. Assumption 2.2 (ii)). If \(A(t_i)x = 0\), then \(y := (I - A(t_i) - Z_i B(t_i)) x \neq 0\) and
\[
p_i x^T (I - A(t_i) - Z_i B(t_i))^T C(t_{i+1})(I - A(t_i) - Z_i B(t_i)) x
\]
by equation (26). We note that \(\kappa'(p) < 0\) on \((0, \alpha \Sigma / (\alpha \Sigma + \Lambda), 1)\).
Combining this with Lemma 6.1 of Kratz and Schöneborn (2013), we obtain that the term in equation (71) is strictly increasing for $p \in (0, \alpha \Sigma / (\Lambda + \alpha \Sigma))$ and strictly decreasing for $p \in (\alpha \Sigma / (\Lambda + \alpha \Sigma), 1)$. 

(i) Assertion (ii) and the fact that the overall costs are decreasing in $p$ imply that the impact costs are strictly decreasing for $p \in (0, \alpha \Sigma / (\Lambda + \alpha \Sigma))$. We can thus limit our attention to $p \in (\alpha \Sigma / (\Lambda + \alpha \Sigma), 1)$, in particular $\kappa'(p) > 0$. We obtain (cf. equations (22) and (26))

$$
\mathbb{E}[x(t_i)^2] = (1-p)^{1-p} m(t_i, p)^2 \quad (72)
$$

\[\begin{align*}
\mathbb{E}[x(t_i)^2] &= \frac{1}{1-p} \left( \frac{\sqrt{1-p} \sinh(\kappa(p)(N+1-i)) - \sinh(\kappa(p)(N-i))}{\sinh(\kappa(p)(N+1))} \right)^2 \\
&\quad \times X(t_0)^2.
\end{align*}\]

For $i = 0, \ldots, N$, we set $a = N + 1 - i$, $b = N + 1$ and define the function

$$
g_i(p) := \frac{\sin((a + 1)\kappa(p)) - \frac{1}{\sqrt{1-p}} \sinh(ak(p))}{\sinh(bk(p))}.
$$

Using the addition formulae for hyperbolic functions and the definition of $\kappa(p)$, we show that

$$
\sin((a + 1)\kappa(p)) - \frac{1}{\sqrt{1-p}} \sinh(ak(p)) \\
\geq \frac{p}{2\sqrt{1-p}} (\cosh(ak(p)) - \sinh(ak(p))) > 0.
$$

To finish the proof, it is thus sufficient to show that $\frac{d}{d\kappa} g_i(p) \geq 0$ for all $i$ and $\frac{d}{d\kappa} g_0(p) > 0$ for at least one $i_0$ which follows from elementary yet tedious calculus.

**Proof of Theorem 5.2** Direct consequence of the preceding example of a market manipulating strategy.

**Proof of Proposition 5.4** The same line of argument as in the proof of Theorem 3.1 establishes the existence of optimal strategies since dark pool liquidity is limited.

Consider the optimal strategy for an initial asset position of $X(t_0)$. Assume that at any time $t_i$, an asset position of $X(t_i)$ is being held and orders of $x(t_i)$ and $y(t_i)$ are optimal. By our requirements for admissible strategies, we know that $|y(t_i)| \leq L$ (cf. Definition 2.3 (iii)). We first assume that $\text{sgn}(x(t_i)) \neq \text{sgn}(y(t_i))$. The expected cost of trading at time $t_i$ and thereafter are

$$
\Delta x(t_i)^2 + (1-p) \tilde{v}(t_{i+1}, X(t_i) - x(t_i)) + p \tilde{v}(t_{i+1}, X(t_i) - x(t_i) - y(t_i) + \Delta x(t_i) + \Gamma \text{sgn}(y(t_i)) y(t_i)).
$$

A direct calculation shows that if

$$
\tilde{v}(t_{i+1}, X(t_i) - x(t_i)) < \tilde{v}(t_{i+1}, X(t_i) - x(t_i) - y(t_i) + \Delta x(t_i) + \Gamma \text{sgn}(y(t_i)) y(t_i))
$$

then orders of $\tilde{x}(t_i) = x(t_i)$ and $\tilde{y}(t_i) = 0$ result in lower costs; otherwise, $\tilde{x}(t_i) = x(t_i) + y(t_i)$ and $\tilde{y}(t_i) = 0$ result in lower costs due to Condition (38). In both cases, a contradiction is established. Hence the optimal strategy satisfies $\text{sgn}(x(t_i)) = \text{sgn}(y(t_i))$ at all times $t_i$.

Given that $\text{sgn}(x(t_i)) = \text{sgn}(y(t_i))$, it is obvious that an optimal strategy cannot have

$$
\text{sgn}(x(t_i)) = \text{sgn}(y(t_i)) \neq \text{sgn}(X(t_i)),
$$

i.e. cannot be market manipulating.

**Proof of Proposition 6.2** Construction of a profitable pinging strategy that trades only during the last three time points $t_{N_2}, t_{N_1}$ and $t_0$ is straightforward. Among pinging strategies, the benefit of the initial trade (information acquisition) is independent of the order size, while the cost of liquidating the potentially created undesired position grows with the size of the order. Therefore, no optimal pinging strategy exists since any pinging strategy can be improved by reducing the size of the initial dark pool order (as long as the order still has non-zero size). For a general trading strategy, it is easy to see that the first order of any optimal strategy has to be a dark pool order, since trading at the primary exchange before generation of a price forecast increases costs without benefit.

**Proof of proposition 6.3** From the proof of Proposition 6.2, we know that it is sufficient to investigate strategies that do not trade at the primary venue before placing an initial dark pool order. A backward induction reveals that the potential profits from information about dark pool liquidity are finite, while the cost of liquidating a very large position (independent of its origin) is unbounded in the size of the position. Increasing the size of the initial dark pool order will hence not only reduce the profitability of the strategy but will even turn it into a loss-making strategy if the order is too large.

8. Conclusion

In this article, we establish a model for optimal trade execution in dark pools which trades off price impact costs at the traditional exchange against uncertain execution in the dark pool. By design, our model assumes price impact and dark pool liquidity to be given exogenously; we establish existence and uniqueness of optimal liquidation strategies in this general model.

In order to obtain explicit solutions for the optimal liquidation strategy, we analyse a more specific version of the general model. We assume that the price impact at the primary venue is temporary and linear in the order size and that orders in the dark pool are executed fully or not at all. We obtain explicit recursions and numerical solutions to the resulting quadratic optimization problem. In the single-asset setting, it is always optimal to place the entire asset position in the dark pool and slow down trading at the primary venue. This is no longer true if the trader aims to liquidate a portfolio.

In our model, trades in the dark pool occur at the fundamental price of the asset which is not influenced by the trades of the large investor. If we relax this assumption and allow trading in the primary venue to have an impact on the transaction price in the dark pool, then market manipulation strategies can potentially generate infinite gains and optimal strategies might not exist. These undesirable effects can be avoided if liquidity...
in the dark pool is bounded and adverse selection is sufficiently strong.

If dark pool liquidity is not only related to price moves at the same point in time, but also to price moves at future points in time, then information about historical dark pool liquidity becomes valuable. This can tempt traders to 'light up' the dark pool by placing small 'pinging' orders in the pool for information gathering purposes.

Acknowledgements

We wish to thank Swagato Acharjee, Christoph Baumgarten, Ulrich Horst, Sebastian Jaimungal, Werner Kritz, Daniel Nehren and Mark van Achter for useful comments and discussions. We are also grateful to seminar participants at the University of Bonn, Oxford University, ETH Zürich, the EFA Meeting 2009, the DGF Meeting 2009, the Princeton-Humboldt Workshop 2009, the CFF Research Conference 2010, the GOCPS 2010, the SIAM Conference on Financial Mathematics & Engineering 2010 and the LSE Risk and Stochastics Conference 2013. This research was supported by Deutsche Bank through Quantitative Products Laboratory, German Academic Exchange Service (DAAD) and Institut de Mathématiques de Marseille of Aix-Marseille Université.

References

Glosten, L.R. and Milgrom, P.R., Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. J. Financ. Econ., 1985, 14, 71–100.
Sofianos, G. and Jeria, D., Quantifying the Sigma X crossing benefit. Equity execution strategies: Street smart 31, 2008a, Goldman Sachs.
Appendix A. Market and liquidity risk

As noted in section 2.3, our model set-up penalizes the variance introduced by market risk, but does not consider any uncertainty of liquidation proceeds introduced by liquidity risk. In this appendix, we investigate the impact of this model choice by analysing the single-asset case with linear price impact (see section 4.2). Figure A1 illustrates the distribution of the implementation shortfall \( \mathcal{R} \) realized by the optimal strategy for two different parameter choices of the model that we propose in section 2.2.

If the price impact is small compared to the exogenous volatility of the asset price, then \( \mathcal{R} \) follows a unimodal distribution that is driven by market risk (left hand side of figure A1). If the price impact is large compared to the volatility, then \( \mathcal{R} \) follows a bimodal distribution where the two modes correspond to the cases of presence and absence of liquidity in the dark pool (right hand side of figure A1).

The distribution of \( \mathcal{R} \) is a Gaussian mixture distribution. Liquidation in the dark pool is found at time \( i = 0, \ldots, N - 1 \) for the first time with probability \( p_i := p(1 - p)^i \). In this event, the trades \((x(t_j))_{0 \leq j \leq i}\) and remaining position \((X(t_j))_{0 \leq j \leq i}\) until time \( i \) are equal to \((x^\text{b}(t_j))_{0 \leq j \leq i}\) and \((X^\text{b}(t_j))_{0 \leq j \leq i}\) and are zero thereafter. The implementation shortfall is hence normally distributed with mean \( \mu_i := \sum_{j=0}^{\infty} \Lambda(X^\text{b}) J^2 \) and variance \( \sigma_i^2 := \sum_{j=0}^{\infty} \Sigma(X^\text{b}) J^2 \). Mixing these normal distributions with weights \( w_i \) gives the distribution of \( \mathcal{R} \).

In figure A1, the exact distribution (thick solid line) is compared to two reference distributions. The first one (illustrated by the thin solid line in figure A1) is another mixture distribution of \( N + 1 \) Gaussian distributions, with the same weights \( w_i \) and the same variances \( \sigma_i^2 \) as in the exact distribution of \( \mathcal{R} \). The means of the normal distributions however are changed from \( \mu_i \) to \( \hat{\mu}_i := \sum_{j=0}^{\infty} w_i \hat{\mu}_j \), i.e. the same mean is applied to all underlying Gaussian distributions. The resulting mixture distribution captures the market risk component that is being penalized in our model but does not include the liquidity risk component. As shown in figure A1, the thick solid line (exact distribution) and the thin solid line are similar if the temporary price impact is small compared to the variance, but they are notably different if the temporary price impact is large. In the first case, the variance of the exact distribution is only marginally larger than the market risk component, but in the latter case it is several orders of magnitude larger. In theory, our model might hence be missing an important component of overall variance. In reality, however, price impact is almost always significantly smaller than market volatility. This is reflected, for example, in the difficulty of measuring market impact and evaluating the performance of trade execution algorithms; see for example Almgren et al. (2005), Soﬁanos and Jeria (2008a) and Soﬁanos and Jeria (2008b). Under such circumstances our model appears to capture the primary risk component.

The exact distribution of \( \mathcal{R} \) is a Gaussian mixture distribution and hence not normal. The deviation from normality is obvious in the case of the bimodal distribution for large price impact. However also in the case of the unimodal distribution for small price impact, the distribution is significantly non-normal. For comparison, figure A1 displays a normal distribution with the same mean and variance as the exact distribution (dashed line). It should be noted that this feature is created by the existence of the dark pool; in the absence of a dark pool, i.e. in the model of Almgren and Chriss (2001), the distribution of implementation shortfall is normal. Investors might be concerned about risk features beyond the variance of the distribution. While all of these yield the same set of optimal strategies in Almgren and Chriss (2001), this is not necessarily the case any more in our model. In particular, tail risk aversion might lead to different optimal trading strategies.

Appendix B. Recursions for section 4

Before we state the recursions, we need to introduce the following notation: In each time interval \( [t_i, t_{i+1}) \), there are \( 2^k \) possible 'combinations' or 'scenarios' with respect to joint execution and non-execution of the order \( y(t_j) \in \mathbb{R}^n \) in the dark pool (note that due to Assumption 4.1 (iv), we can assume that execution of an order \( y(t_j) \) is 'independent' of the sign of \( y(t_j) \)). Each of these scenarios occurs with a fixed probability, which we denote by \( p_i \) for the \( i^{th} \) scenario, determined by the distributions of the random variables \( a(t_{i+1}) \) and \( b(t_{i+1}) \). We denote the amount executed in the dark pool at time \( i \) in scenario \( l \) by \( z_l(t_i) \). There exists a diagonal matrix \( Z_l \in \mathbb{R}^{n \times n} \) (with 1's and 0's on the diagonal) such that \( z_l(t_j) = Z_l y(t_j) \). We define the diagonal matrix \( P = (\hat{p}_{l,k})_{k,m=1,\ldots,n} \) by \( P := \sum_l p_l Z_l \), i.e. \( \hat{p}_{l,k} \) is the probability that an order for the \( k^{th} \) asset is executed in the dark pool in \( [t_i, t_{i+1}) \). We re-order the assets in such a way that for \( k_0 \in \{0, \ldots, n\} \), \( \hat{p}_{l,k} = 0 \) if and only if \( k > k_0 \), i.e. \( k_0 = 0 \) refers to the case where the dark pool is not used at all and \( k_0 = n \) to the case where there is liquidity with positive probability for all assets in the dark pool.

\[ \hat{p}_{l,k} = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{otherwise} \end{cases} \]
Finally, for a positive definite matrix $M \in \mathbb{R}^{n \times n}$, we define
\[
\hat{M} = (\hat{m}_{i,j})_{i,j=1,...,n} := \sum_l p_l Z_l M Z_l.
\] (B1)

Note that both $\hat{M} = (\hat{m}_{k,m})_{k,m=1,...,n}$ and $\hat{P}$ are positive definite for $k_0 = n$ but not for $k_0 < n$. In the latter case, $\hat{m}_{k,m} = 0$ for $k > k_0$ or $m > k_0$ and $\hat{p}_{k,k} = 0$ for $k > k_0$. However, the matrices $\hat{M}' := (\hat{m}_{k,m})_{k,m=1,...,k_0} \in \mathbb{R}^{k_0 \times k_0}$ and $\hat{P}' := (\hat{p}_{k,m})_{k,m=1,...,k_0} \in \mathbb{R}^{k_0 \times k_0}$ are positive definite. We therefore use generalized inverses of matrices. We denote the Moore-Penrose Inverse of a matrix $M$ by $M^\dagger$. For regular $M$, we have $M^{-1} = M^\dagger$ (see e.g. the book by Ben-Israel and Greville (2003)). The matrices $A(t)$, $B(t)$ and $C(t)$ in Theorem 4.2 are given recursively by $A_N = I$, $B_N = 0$, $C_N = \Lambda + \alpha \Sigma$ and $A(t_N) = I$, $B(t_N) = 0$, $C(t_N) = \Lambda + \alpha \Sigma$ and
\[
\begin{align*}
A(t) &= (\Lambda + D(t+1))\dagger D(t+1), \\
B(t) &= \hat{C}(t+1)^\dagger \hat{P} C(t+1)(I - A(t)), \\
C(t) &= \alpha \Sigma + D(t+1) - D(t+1)(\Lambda + D(t+1))\dagger D(t+1),
\end{align*}
\] (B2)
\[
\begin{align*}
A(t) &= (\Lambda + D(t+1))\dagger D(t+1), \\
B(t) &= \hat{C}(t+1)^\dagger \hat{P} C(t+1)(I - A(t)), \\
C(t) &= \alpha \Sigma + D(t+1) - D(t+1)(\Lambda + D(t+1))\dagger D(t+1),
\end{align*}
\] (B3)
where $D(t+1) := C(t+1) - C(t+1)^\dagger \hat{P} \hat{C}(t+1)^\dagger \hat{P} C(t+1)$. (B4)