2. Modeling Default : firm value ; first passage ; intensity models

firm value is state var. (asset price) "structural"

- default at maturity — Black-Scholes '73, Merton '74
- at coupon dates — Geske-Johnson '77, '84
- diffusion to exog. barrier — Black-Cox '76; Brennan-Schwartz '80 (convertibles); Longstaff-Schw. '95
  - jump-diffusion — Zhou '97
- diffusion to rational barrier — Fisher-Heinkel-Zech '8; Jones '95; Leland-Toft '96
  - with strategic debt-service [game theoretic] — Anderson-Sund; Theunissen '98

'credit quality' is state var "reduced form"

- fixed Poisson default intensity λ — Hull '89 '95
- deterministic $\lambda(t)$ — Lando '94; Jarrow-Turnbull '95; Duffie-Singleton '97 '95
- switching intensity $\lambda(t) \in$ finite set [credit rating] — Jarrow-Lando-Turnb '9
- diffusion for $\lambda(t)$ — Duffie '98; Duffie-Lando '99 ['Cox process']
- mixed jump-diffusion $\lambda(t)$

4

Probability, Brownian Motion, Ito Processes: jargon

Why continuous time? Allows assumption of infinitesimal price change over infinitesimal time interval so that

- portfolios can be revised before next 'minishock'
- prices can be viewed as locally linear in the things that are varying [permits hedging of risk]

A touch of terminology:

space Ω ; elements $\omega \in \Omega$; subset $A \subset \Omega$; $\emptyset$, A^c

σ-field or σ-algebra $\mathcal{F}$: collection of subsets of Ω such that

(1) $\Omega \in \mathcal{F}$

(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(3) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow A_1 \cup A_2 \cup \ldots \in \mathcal{F}$

measure space $(\Omega, \mathcal{F}, \mu)$: function $\mu : \mathcal{F} \to \mathbb{R}^+$

(1) $\mu(\emptyset) = 0$

(2) $A \in \mathcal{F} \Rightarrow 0 \le \mu(A) \le \infty$

(3) $A_1, A_2 \ldots \in \mathcal{F}$ and $A_i \cap A_k = \emptyset$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

i.e., μ measures 'size'

probability space $(\underbrace{\Omega}_{\text{trial, sample space}}, \underbrace{\mathcal{F}}_{\text{event space ('meaningful events')}}, \overbrace{\mu}^{\text{probability }(A)})$: a measure space with $\mu(\Omega) = 1$ [$P \equiv \mu$]

random variable $X(\omega)$: a real valued function on Ω such that events like $X \in (a, b)$ have meaningful probabilities assigned to them by μ.

["X is a measurable function from $(\Omega, \mathcal{F}, \mu)$ to $(\mathbb{R}, \mathcal{R})$ where $\mathcal{R}$ is the Borel sets")

⑤

almost always, almost everywhere, with prob. 1 : mean statement true $\forall\ \omega \in \Omega$ except on subset N and $P(N)=0$

stochastic process : an indexed set of r.v.'s on the same $(\Omega, \mathcal{F}, P)$

$X(t,\omega)$ or $X_t(\omega)$ t discrete or continuous

ω often suppressed

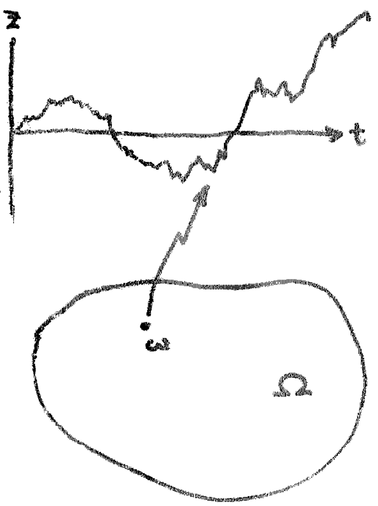
Brownian motion (Weiner process): stochastic process $z(t,\omega)$ on some $(\Omega, \mathcal{F}, P)$ such that

0 start (1) $z(0,\omega) = 0 \quad \forall\, \omega$

Markov (2) $z(t_2) - z(t_1)$ is independent of $z(t_4) - z(t_3)$ $\quad \forall\ t_1 < t_2 < t_3 < t_4$

Normal (3) $z(t_2) - z(t_1) \sim N(0, t_2 - t_1)$

A value for $z(t,\omega)$ is a 'level of the state variable' in finance



A point in Ω is a 'state of the world' in the Arrow-Debreu sense

Remarks: $z(t,\omega)$ is cont. in t for each ω.

: $z(t,\omega)$ is nowhere differentiable in t with prob. 1

(very jiggly!)

martingale : stochastic process such that $E[X(t_2) \mid X(t_1)] = X(t_1)$

i.e., 0 expected change

Remark: though seemingly cumbersome and indirect, the above setup makes it easier to talk about distinct but related stochastic processes on the same underlying probability space Ω

e.g., $X(t,\omega)$ = price of security 1

$Y(t,\omega)$ = " " " 2

$G(t,\omega)$ = amount of X held at time t (controlled)

$V(t,\omega)$ = value of portfolio " " "

④

Itô process: the stochastic process you get when you add 'drift' and nonunitary 'volatility' to modify a Brownian motion

shorthand representation: $ds(t) = \alpha(s,t)\,dt + \sigma(s,t)\,dz(t)$

[stochastic d.e.] or just $ds = \alpha\,dt + \sigma\,dz$

stands for

$$s(t) = s(0) + \underbrace{\int_0^t \alpha(\tau,s)\,d\tau}_{\text{regular integral}} + \underbrace{\int_0^t \sigma(s,\tau)\,dz(\tau)}_{\text{Ito integral}}$$

properties: $\frac{d}{d\tau} E[s(\tau)]\Big|_{\tau=t} = \alpha(s(t),t)$

$$\frac{d}{d\tau} \mathrm{Var}[s(\tau)]\Big|_{\tau=t} = \sigma^2(s(t),t)$$

Itô's lemma: used heavily in cont.^{s} time finance / economics

if $ds = \alpha\,dt + \sigma\,dz$

and $y(t) = f(s,t)$ [and f is 'nice']

then y follows an Ito process

$$dy = \underbrace{\left(f_t + \alpha f_s + \tfrac{1}{2}\sigma^2 f_{ss}\right)}_{\text{drift of } y} dt + \sigma f_s\,dz$$

Example: With constant nominal interest rates r, the nominal value of a bond at time t is $B(t) = e^{rt}$.

Suppose the price level follows a stoch. process

$$dP = \pi P\,dt + \sigma P\,dz$$

The real value of the bond at time t is then

$$b(t) = B(t)/P(t) = e^{rt}/P(t)$$

Applying Ito's lemma, this follows

$$db = (r - \pi + \underline{\underline{\sigma^2}})\,dt + \sigma b\,dz$$

(6)

Basic notions of an Arbitrage argument : road to the fundamental valuation pde

- one 'dimension' of relevant uncertainty for the securities involved (one 'factor' model), reflected in some state variable $s(t)$ following

$$ds = \alpha\, dt + \sigma\, dz$$

- two tradeable securities with prices $A(s,t)$, $B(s,t)$ plus riskless loans at rate r. Ito's lemma $\Rightarrow$

$$dA = \left(\tfrac{1}{2}\sigma^2 A_{ss} + \alpha A_s + A_t\right)dt + \sigma A_s\, dz$$

$$dB = \left(\tfrac{1}{2}\sigma^2 B_{ss} + \alpha B_s + B_t\right)dt + \sigma B_s\, dz$$

- construct a portfolio of 1 unit of A, $-A_s/B_s$ units of B, funded by borrowing [no net investment].
  $P(t)$ denotes the value of the portfolio at time t.
  By construction, $P(0) = 0$. Its value evolves as

$$dP = dA - \frac{A_s}{B_s}\, dB - (\text{borrowings})\, r\, dt$$

$$= \left[\tfrac{1}{2}\sigma^2\left(A_{ss} - \frac{A_s}{B_s}B_{ss}\right) + \alpha\left(A_s - \frac{A_s}{B_s}B_s\right) + \left(A_t - \frac{A_s}{B_s}B_t\right) - \underbrace{\left(A - \frac{A_s}{B_s}B\right)}_{\text{net borrowed}} r\right] dt + \underbrace{0\, dz}_{\therefore\ \text{riskless}}$$

- since 0 was invested, and no risk is born, expression in [...] must be 0 or this strategy (or the reverse) pays for lunch. $\therefore$ [...] $= 0$ if prices are equilibrium.

Rearranging gives

$$\lambda(s,t) \equiv \frac{\frac{1}{2}\sigma^2 A_{ss} + \alpha A_s + A_t - rA}{A_s} = \frac{\overbrace{\frac{1}{2}\sigma^2 B_{ss} + \alpha B_s + B_t}^{E[\text{rate of ret. on } B]} \overbrace{- rB}^{\text{opp. cost}}}{\underbrace{B_s}_{\text{sensitivity to } s\text{-risk}}}$$

market 'price' of s-risk
common to <u>all</u> securities

In terms of λ, we have that $A(s,t)$ satisfies:

$$\frac{1}{2}\sigma^2 A_{ss} + (\alpha - \lambda) A_s + A_t - rA = 0 \qquad *$$

"fundamental valuation pde"

<u>Remarks</u>: (1) doesn't determine $\lambda(s,t)$ [$\lambda > 0$ signifies mkt. aversion to s-risk]

(2) many functions satisfy $*$: added conditions characterize the security and identify particular soln. [boundary conditions]

(3) multi-dimensional analogue if multiple factors on which price depends

(4) obtain same pde if view $\hat{\alpha} \equiv \alpha - \lambda$ as the objective drift, with $\hat{\lambda} \equiv 0$.

[prices same as in 'risk neutral' world with different s-process!]

$\hat{\alpha} \equiv$ 'risk adjusted drift' in s.

(5) if s is price of a traded security (no dividends), then it too must satisfy $*$ trivially. But we know that $A(s,t) \equiv s$ $\therefore A_s = 1$, $A_{ss} = 0 = A_t$

Substituting into $*$ yields $(\alpha - \lambda) - rs = 0$
$\therefore \alpha - \lambda = rs$, which pins down λ, and

$$\frac{1}{2}\sigma^2 A_{ss} + rsA_s + A_t - rA = 0$$ for other 'derivatives'

(1) **Recap:** if state var. $s(t)$ follows $ds = \alpha(\cdot)\,dt + \sigma(\cdot)\,dz$

then value $V(s,t)$ of an asset satisfies

$$\frac{1}{2}\sigma^2 V_{ss} + [\underbrace{\alpha - \lambda}_{\text{risk adjusted drift}}] V_s + \underbrace{c(s,t)}_{\text{'dividend' on } V} + V_t - rV = 0 \qquad \forall s,t$$

Corollary: if $s(t)$ is price of a traded asset itself, can further show

$$\alpha(\cdot) - \lambda(\cdot) = \underbrace{r}_{\text{riskless rate}} s(t) - \underbrace{d(s,t)}_{\text{'dividend' on } s}$$

Examples of 'boundary conditions' for particular assets:

(1) s = stock price, V = call option to purchase at price X at time T

$$V(s,T) = \max\{0, s(T) - X\}$$

(2) $s \equiv r$ = short interest rate, V = bond with mat. value \$1

$$V(r,T) = 1 \qquad \text{CIR model: } \alpha \equiv K(\bar{r} - r),\ \sigma \equiv \omega r^{1/2},\ \lambda \equiv \bar{\lambda} r$$

(3) s = \$US/£, V = exchange rate swap [US flow for £ flow] based on rate $\bar{s}$

i.e., pay $\bar{s}$/yr. dollars, recv. 1 £/yr.

$$V(s,T) = 0$$

$$c(s,t) = (s - \bar{s})$$

(4) s = commodity price, V = call option (perpetual) to purchase at price X

+ policy of exercising option if $s \geq \bar{s}$

$$V(\bar{s},t) = \bar{s} - X$$

$\bar{s}$ optimal $\Longleftrightarrow$ $V_s(\bar{s},t) = \frac{\partial}{\partial s}(s - X) = 1$ [Merton 'high contact' condition]

finding joint soln. for $\bar{s}^*$ and V is 'free boundary' problem.

Interpretations of valuation pde soln $V(s,t)$:

(a) value V would have to trade for if no arbitrage opportunities exist [must hold in any general equilib.

(b) amnt. now that if put into a 'self financing portfolio' would permit the portfolio to replicate all the cash flows and terminal value of V.

(c) Feynman-Kac relation:

$$V(s,t) = \hat{E}\left[b(s(T))\, e^{-\int_t^T r\,d\tau} + \int_t^T c(s(\tau),\tau)\, e^{-\int_t^\tau r\,d\delta}\, d\tau \right]$$

expectation over
$s(t)$ paths where
$ds = (\alpha - \lambda)dt + \sigma dz$
given $s(0)$ start

$b(s)$ gives terminal value

i.e., $\hat{E}$ present value of all cash flows from the asset until it hits a 'stopping boundary'
($\hat{E}$ with respect to 'risk adjusted' process for s)

Et cetera:

— equilibrium futures contract prices $F(s,t)$ satisfy

$$\tfrac{1}{2}\sigma^2 F_{ss} + (\alpha - \lambda)F_s + F_t = 0$$

s.t. $F(s,T)$ = spot price of commodity in state s at T

[CIR, JFE '81]

$\therefore$ $F(s,t) = \hat{E}[\text{spot price at } T \,/\, s(0)]$

— Arrow-Debreu security paying \$1 if $s(T) \in [a,b]$ satisfies pde with $c(\cdot) = 0$ and

$$V(s,T) = \begin{cases} 1 & \text{if } s \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$