Credit Derivatives and Structured Credit Products

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Introduction to Credit Risk

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Risks of Holding an Option

The risks an option holder is exposed to include the following:

- Changes in the market value or volatility of the underlying
- Changes in risk-free interest rates
- Lack of liquidity if sale is intended
- Catastrophic events like those on September 11, 2001
- Fluctuation of the credit quality of the option seller, or worse, failure to honor obligation at option maturity (applies only if option is in the money)

Zoology of Financial Risks

- *Market risk:* possibility of unexpected changes in market prices and rates
- *Operational risk:* possibility of mistake or breakdown in trading/risk management operation
  - mis-pricing of instruments
  - mis-understanding of involved risks
  - fraud ('rough trader')
  - systems failure
  - legal exposure due to inappropriate services

Zoology of Financial Risks (2)

- *Liquidity risk:* possibility of increased costs/inability to adjust position(s)
  - bid-ask spreads widen dramatically over a short period of time
  - access to credit deteriorates
- *Credit risk:* possibility of losses due to unexpected changes in the credit quality of a counterparty or issuer
- *Systemic risk:* market-wide liquidity breakdowns or domino-style correlated defaults

Some Sources of Credit Risk Exposure

- Exposure to the credit risk of an underlying, for example
  - bank loan
  - corporate or sovereign bond
- Exposure to the credit risk of the (OTC) counterparty: vulnerable claims
- Combinations of both, for example with options on corporate bonds

Market vs. Credit Risk

Basically, credit risk is part of market risk. But there are illiquid contracts for which market prices are not available, for example loans. Other factors calling for a distinction are as follows.

	Market Risk	Credit Risk
Time horizon	short (days)	long (years)
Portfolio	static	dynamic
Hedging	standardized	often customized (improving)
Information	market related	contract specific
Data	abound	sparse

Credit Ratings

Ratings describe the credit worthiness of bonds; they are issued by (private) rating agencies.

- S&P: classes AAA, AA, A, BBB, BB, B, CCC, with AAA the best
- Moody's: classes Aaa, Aa, A, Baa, Ba, B, Caa, with Aaa the best
- Bonds with ratings of BBB/Baa and above are considered 'investment grade'; those below are considered non-investment grade

Average Cumulative Default Rates

Complete rating universe of S&P, 2001

Years	1	2	3	4	5	10
AAA	0.00	0.00	0.07	0.15	0.24	1.40
AA	0.00	0.02	0.12	0.25	0.43	1.29
A	0.06	0.16	0.27	0.44	0.67	2.17
BBB	0.18	0.44	0.72	1.27	1.78	4.34
BB	1.06	3.48	6.12	8.68	10.97	17.73
B	5.20	11.00	15.95	19.40	21.88	29.02
CCC	19.79	26.92	31.63	35.97	40.15	45.10

Credit Events

Credit events indicate/document a credit quality change of an issuer.

- Credit rating change
- Restructuring
- Failure to pay
- Repudiation
- Bankruptcy

In case of the last three events we also speak of a 'default'.

Why Measure and Manage Credit Risk?

- In perfect capital markets, adding/subtracting financial risk has no impact on the market value of a firm (Modigliani-Miller)
- In imperfect capital markets, measuring and managing financial risk has significant benefits
- Due to informational asymmetries, credit markets are imperfect

Adverse Selection and Moral Hazard

How can a bank respond to credit market imperfections?

- Increase average interest rates (take a 'lemon's premium')
  - adverse selection: bad risks remain at the bank
  - moral hazard: incentive for borrower to gamble, particularly for large loan sizes
- Reduce exposure, but this may lead to credit rationing and reduces potential profits

In practice, banks would set both interest rates and exposure limits according to the measured credit risk of the borrower.

Treasury vs. Corporate Bond Yields

Maturity (years)	Treasury yield	Corporate bond yield	Yield spread
1	5%	5.25%	0.25%
2	5%	5.50%	0.50%
3	5%	5.70%	0.70%
4	5%	5.85%	0.85%
5	5%	5.95%	0.95%

The *credit yield spread* is the difference in the yield between a defaultable and a non-defaultable (credit risk-free) bond. The spread is the excess yield which compensates the bond holder for bearing the credit risk of the bond issuer.

Credit Spreads

- Two types of bonds: default free and defaultable
- $\bar{B}_t^T$ price at time t of a default free zero-coupon bond paying 1 at T
- Bond yield $\bar{y}(t,T)$ satisfies $\bar{B}_t^T = e^{-\bar{y}(t,T)(T-t)}$
- B_t^T price at time t of a defaultable zero-coupon bond paying 1 at T
- Bond yield $y(t,T)$ satisfies $B_t^T = e^{-y(t,T)(T-t)}$
- Credit yield spread $S(t,T)$ is then

$$S(t,T) = y(t,T) - \bar{y}(t,T) = -\frac{1}{T-t} \ln \frac{B_t^T}{\bar{B}_t^T}$$

- The term structure of credit spreads is the schedule of $S(t,T)$ against T

Default Modeling Paradigms

How can we model the default event, default probabilities, and bond prices?

1. Structural approach: economic arguments about why a firm defaults
   (a) Classic option-theoretic model of Merton (1974)
   (b) First-passage model
2. Reduced form/intensity based approach: tractable ad-hoc financial engineering type approach
3. Hybrid approach: unifies economic and financial engineering type arguments

Goals of This Course

- Provide an understanding of the basic principles of credit risk
- Learn how to model default events, default probabilities, and bond prices
- Learn how to calibrate and apply these models in practice
- Provide an understanding of the structure and rationale of popular derivative products for credit risk insurance
- Learn how to analyze the risk and value of these transactions

Background Reading

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Structural Modeling of Credit Risk

1. Credit spreads and default modeling paradigms
2. Structural modeling: classic approach
3. Structural modeling: first-passage approach
4. Modeling default correlation in the structural approach
5. Implementation in practice

Classic Model of Merton (1974)

- A firm is financed by equity and a single issue of zero-coupon debt with face value F maturing at T
- Markets are frictionless (no taxes, transaction costs...)
- Continuous trading
- The total market (or asset) value of the firm V is modeled by a geometric Brownian motion with drift μ and volatility σ, i.e.

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \quad V_0 > 0$$

  where W is a std. Brownian motion. Thus $V_t = V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$
- V is invariant to the firm's capital structure
- Interest rates are a constant $r > 0$; thus $\bar{B}_t^T = e^{-r(T-t)}$

Default Time

The firm defaults if at the debt maturity T the assets are not sufficient to fully pay off the bond holders (absolute priority). In this case the equity investors surrender the firm to the bond investors which then make use of the remaining assets.

Letting τ denote the default time, we have

$$\tau = \begin{cases} T & : \quad V_T < F \\ \infty & : \quad \text{else} \end{cases}$$

We define the default indicator function

$$1_{\{\tau=T\}} = \begin{cases} 1 & : \quad V_T < F \text{ (default)} \\ 0 & : \quad V_T \geq F \text{ (survival)} \end{cases}$$

Payoffs at Maturity

- With absolute priority, we have the following payoffs at maturity T:

	Bonds	Equity
$V_T \geq F$	F	$V_T - F$
$V_T < F$	V_T	0

- The value of the bonds is $B_T^T = \min(F, V_T) = F - \max(0, F - V_T)$
- And the value of the equity is given by $E_T = \max(0, V_T - F)$

Equity as a Call Option

The payoff $\max(0, V_T - F)$ to equity at T is that of a call option on the assets of the firm V with strike given by the bonds' face value F and maturity T. The value E_0 of equity at time zero is therefore given by the Black-Scholes (1973) call option formula:

$$E_0 = BS_C(\sigma, T, F, r, V_0) = V_0\Phi(d_1) - Fe^{-rT}\Phi(d_2)$$

where Φ is the standard normal distribution function and

$$d_1 = \frac{\ln(\frac{V_0}{F}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Valuing the Bonds

The payoff $\min(F, V_T) = F - \max(0, F - V_T)$ to bond holders at T is that of a riskless loan F and a short put option on the assets of the firm V with strike given by the bonds' face value F and maturity T. The value of the bonds B_0^T at time zero is therefore given by

$$B_0^T = Fe^{-rT} - BS_P(\sigma, T, F, r, V_0) = Fe^{-rT}\Phi(d_2) + V_0\Phi(-d_1)$$

where BS_P is the Black-Scholes put option formula. Equivalently, we can find B_0^T using our previous result as the difference between asset and equity value, i.e. $B_0^T = V_0 - E_0$.

Risk-Neutral Valuation: Idea

- Black and Scholes: prices of derivatives do not depend on agents' preferences toward risk
- Hence, prices of derivatives are the same in a world where investors are risk-neutral (RN)
- But in the RN-world, if there are no arbitrage opportunities, derivatives prices are just given by the expected discounted payoffs.
- Suppose that some contract pays X at time T. Then today's $(t = 0)$ value of this contract is $\tilde{E}[e^{-rT} X]$

Risk-Neutral Valuation

1. There exists a probability $\tilde{P}$ such that all discounted price processes $(X_t)_{t\geq 0}$ are martingales wrt $\tilde{P}$ if and only if there are no arbitrage opportunities in the financial market. The martingale property means that $\tilde{E}[e^{-rT}X_T] = X_0$.

2. $\tilde{P}$ is unique if and only if markets are complete, i.e. every contingent claim can be replicated.

Two probabilities are thus to be distinguished:

1. Real-world (or objective) probability P, e.g. historical default rates

2. Risk-neutral (or martingale or subjective) probability $\tilde{P}$: if we talk about prices of securities

Another Look at Bond Values

In the risk-neutral valuation framework, the prices of securities are given by their expected discounted payoffs under the risk-neutral probability:

$$
\begin{aligned}
B_0^T &= \tilde{E}[e^{-rT}(F - \max(0, F - V_T))] \\
&= e^{-rT}F - e^{-rT}\tilde{E}[(F - V_T)1_{\{\tau=T\}}]
\end{aligned}
$$

which is the present value of a riskless loan with face F less the value of the risk-neutral expected default loss ($1_{\{\tau=T\}} = 1_{\{F>V_T\}}$ is the default indicator function).

Estimating F, r, V_0, and σ

- F can be estimated from balance sheet data.
- r can be estimated from prices of default-free (Treasury) bonds.
- In order to estimate V_0 and σ indirectly, we first observe the equity value E_0 and its volatility σ_E directly from the stock market. Using these quantities, we then solve a system of two equations for V_0 and σ. The first is provided by the equity pricing formula, relating assets, asset volatility and equity:

$$E_0 = BS_C(\sigma, T, F, r, V_0)$$

The second can be obtained via Ito's formula applied to the equity value,

$$\sigma_E E_0 = \Phi(d_1)\sigma V_0$$

Default Probability

We have for the default probability

$$P[\tau = T] = P[V_T < F] = P[V_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} < F]$$
$$= P[W_T < \frac{\ln(\frac{F}{V_0}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma}]$$
$$= \Phi(\frac{\ln(\frac{F}{V_0}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})$$

since W_T is normally distributed with mean zero and variance T. Setting $\mu = r$, we find the risk-neutral default probability

$$\tilde{P}[\tau = T] = \Phi(-d_2) = 1 - \Phi(d_2)$$

Credit Spreads in the Classic Model

From our general spread formula we find

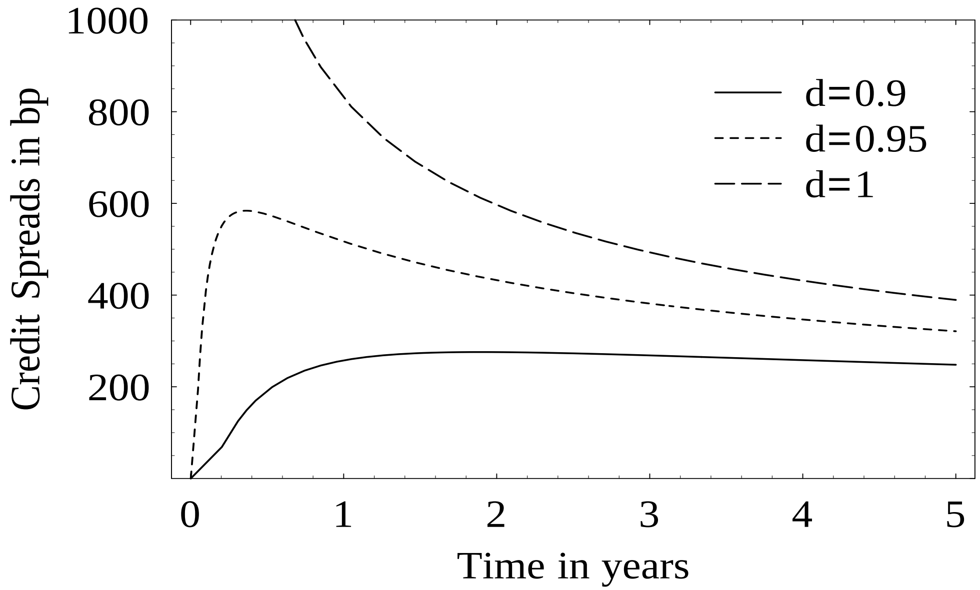
$$S(0,T) = -\frac{1}{T}\ln\left(\Phi(d_2) + \frac{1}{d}\Phi(-d_1)\right)$$

where $d = \frac{F}{V_0}e^{-rT}$ is the discounted debt-to-asset value ratio.

We see that the spread is a function of

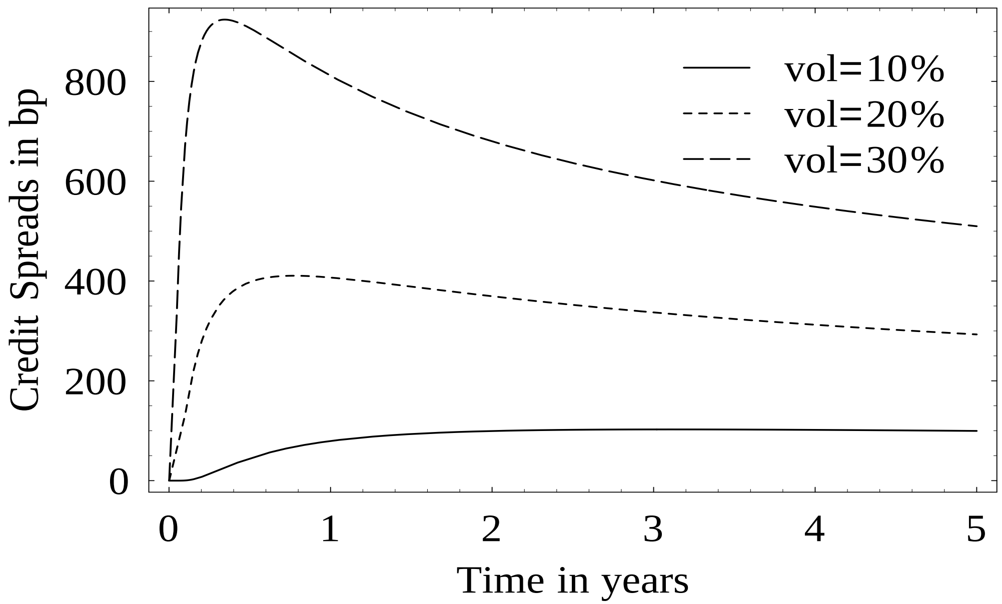
- maturity T
- asset volatility σ (a proxy for the firm's business risk)
- leverage d (a measure of the firm's leverage)

The Term Structure of Credit Spreads



We set the parameters $r = 6\%$ per year, $\sigma = 20\%$ per year. In the figure we plot credit spreads $S(0,T)$ as a function of maturity T for varying degrees of firm leverage d. For $d < 1$ the term structure is hump-shaped, for $d \geq 1$ it is decreasing.

The Term Structure of Credit Spreads (2)



We set the parameters $r = 6\%$ per year, $d = 0.9$. In the figure we plot credit spreads $S(0, T)$ as a function of maturity T for varying degrees of asset volatility σ.

Short Credit Spreads

The short spread is the credit spread for for maturities going to zero. It is the premium bond investors demand as compensation for bearing the issuer's default risk over an infinitesimal period of time.

It can be easily checked that in the classic model the value of the short spread depends on the leverage ratio d:

- $d < 1$: $\lim_{T \downarrow 0} S(0,T) = 0$
- $d \geq 1$: $\lim_{T \downarrow 0} S(0,T) = \infty$

What Can be Criticized?

- Simple capital structure
- Costless bankruptcy
- Perfect capital markets
- Risk-free interest rates constant
- Only applicable to publicly traded firms
- Empirically not plausible
- Default only possible at the maturity of the bonds

First-Passage Default Model

As Black-Cox (1976), we recognize that the firm may default well before T. Let us suppose that a default takes place if the assets fall to some threshold level $D < V_0$ for the first time:

$$\tau = \min\{t > 0 : V_t \leq D\}$$

Note that

$$\{\tau \leq t\} = \{\min_{s \leq t} V_s \leq D\} = \{\min_{s \leq T}(V_0 e^{(\mu - \frac{1}{2}\sigma^2)s + \sigma W_s}) \leq D\}$$

Default Probability

Setting $m = \mu - \frac{1}{2}\sigma^2$, we have for the default probability

$$P[\tau \leq T] = P[\min_{s \leq t}(ms + \sigma W_s) \leq \ln(D/V_0)]$$

That is, we are looking for the distribution function of the running minimum

$$M_t = \min_{s \leq t}(ms + \sigma W_s)$$

of a Brownian motion with drift m and volatility σ. This distribution is known to be inverse Gaussian and we have, setting $x = \ln(D/V_0)$,

$$P[\tau \leq T] = \Phi\left(\frac{x - mT}{\sigma\sqrt{T}}\right) + e^{\frac{2mx}{\sigma^2}}\Phi\left(\frac{x + mT}{\sigma\sqrt{T}}\right)$$

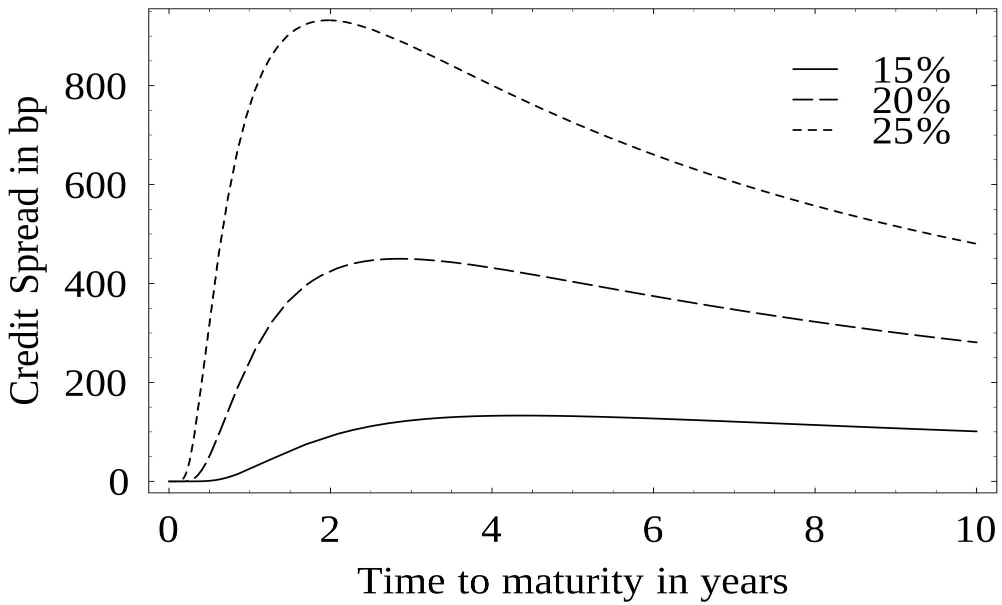
A Quick Look at Bond Prices

Consider a zero coupon bond which has zero recovery–in the event of a default a bond investor loses all of his/her initial investment. If arbitrage opportunities are ruled out, then the price B_0^T of this bond is given by its expected discounted payoff under the risk-neutral probability:

$$B_0^T = \tilde{E}[e^{-rT} 1_{\{\tau > T\}}] = e^{-rT}(1 - \tilde{P}[\tau \leq T])$$

where $\tilde{P}[\tau \leq T]$ is the risk-neutral default probability. If we set $\mu = r$, then we have simply $P[\tau \leq T] = \tilde{P}[\tau \leq T]$ and we can use our previous result.

The Term Structure of Credit Spreads



We set the parameters $r = 6\%$ per year, $V_0 = 1$, and $D = 0.7$. In the figure we plot credit spreads $S(0, T)$ as a function of maturity T for varying degrees of asset volatility σ. Short spreads are zero!

Empirical Plausibility

In a structural model bond investors are warned in advance when a default is imminent (we say the default event is *predictable*). This implies that spreads go to zero with maturity going to zero, which is empirically hardly plausible.

This spread property implies that bond investors do not demand a risk premium for bearing the default risk of the bond issuer for maturities up to a couple of weeks/months.

Moreover, contrary to what we observe in the bond markets, model bond prices converge continuously to their default contingent values.

Default Correlation

All firms are dependent on general (macro-) economic factors such as the stage of the business cycle, commodity prices, or interest rates. That means that different firms may not default independently of each other; we say that *defaults are correlated.*

Capturing these effects in a model is extremely important if one considers *portfolios of bonds*, as it is for example the case with a financial institution.

How can we model this in a simple and intuitive way? We can assume that firms' assets are correlated through time.

Joint Default Probabilities

In the classic Merton (1974) model with two correlated firms, joint default probabilities are given by

$$\begin{aligned} P[\tau_1 = T, \tau_2 = T] &= P[V_T^1 < F_1, V_T^2 < F_2] \\ &= P[W_T^1 < DD_1, W_T^2 < DD_2] \\ &= \Phi_2(\rho, DD_1, DD_2) \end{aligned}$$

where

- ρ is the asset correlation
- $\Phi_2(\rho, \cdot, \cdot)$ is the bivariate standard normal distribution with correlation ρ
- $DD_i = \frac{\ln(F_i/V_0^i) - (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i \sqrt{T}}$ is the std. distance to default of firm i

Implementation in Practice

The classic Merton model with correlated defaults can be implemented in practice, see KMV (1997) and JP Morgan (1997). These models are used by banks to assess the aggregated credit risk of their loan portfolios, which is also very important from a banking regulatory point of view.

A characteristic of these models is that they use a factor model for asset returns, i.e. they assume that the asset return of firm i is given by

$$\ln V_T^i = \sum_{j=1}^{n} w_{ij}\psi_j + \epsilon_i$$

where the ψ_j are (independently) normally distributed systematic factors, the w_{ij} are the factor loadings, and the ϵ_i are iid normal idiosyncratic factors.

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Intensity-Based Credit Risk Modeling

1. Idea
2. Default as Poisson event
3. Recovery Conventions
4. Time-Varying Intensities
5. General Intensities and Valuation
6. Default correlation

Main Idea

- The structural approach is based on solid economic arguments; it models default in terms of fundamental firm variables.
- The intensity based approach is more ad-hoc (reduced-form) in the sense that one does not formulate economic arguments about why a firm defaults; one rather takes the default event and its stochastic structure as exogenously given.
- In particular, one models default as some unpredictable Poisson-like event. This leads to a great deal of tractability and a better empirical performance, as we shall see.

Poisson Process

Let $T_1, \ldots, T_n$ denote the arrival times of some physical event. We call the sequence (T_i) a (homogeneous) *Poisson process with intensity* λ if the inter-arrival times $T_{i+1} - T_i$ are independent and exponentially distributed with parameter λ.

Equivalently, letting $N(t) = \sum_i 1_{\{T_i \leq t\}}$ count the number of event arrivals in the time interval $[0, t]$, we say that $N = (N(t))_{t \geq 0}$ is a (homogeneous) Poisson process with intensity λ if the increments $N(t) - N(s)$ are independent and have a Poisson distribution with parameter $\lambda(t - s)$ for $s < t$, i.e.

$$P[N(t) - N(s) = k] = \frac{1}{k!}(\lambda(t - s))^k e^{-\lambda(t-s)}$$

Poisson Default Arrival

In the intensity based approach, the default time is set equal to the first jump time of the Poisson process N. Thus $\tau = T_1$ is exponentially distributed with (intensity) parameter λ and the default probability is given by

$$F(t) = P[\tau \leq t] = 1 - e^{-\lambda t}$$

The intensity is the conditional default arrival rate given no default:

$$\lim_{h \downarrow 0} \frac{1}{h} P[\tau \in (t, t+h] \,|\, \tau > t] = \lambda$$

Letting f denote the density of F we can also write

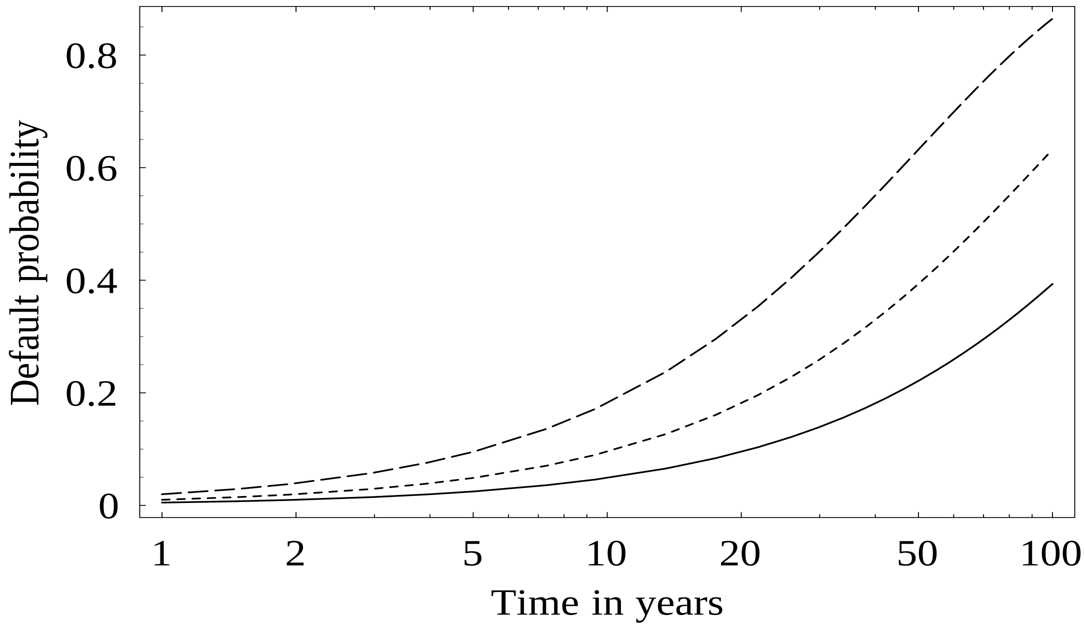
$$\lambda = \frac{f(t)}{1 - F(t)}$$

Properties of Defaults

In the structural approach a default is predictable, i.e. it can be anticipated.

Since the jumps of a Poisson process are totally *unpredictable* (they are complete surprises), in the intensity based approach the default is unpredictable as well. This has important consequences for the term structure of credit spreads, as we will see.

Default Probabilities



We plot default probabilities $F(T)$ as a function of horizon T for varying degrees of intensities $\lambda = 0.005, 0.01$, and 0.02. Clearly, $F(T)$ is for fixed T increasing in the default intensity λ.

Bond Pricing

Assuming constant interest rates $r > 0$, for a defaultable zero bond maturing at T with zero recovery we get

$$B_0^T = \tilde{E}[e^{-rT} 1_{\{\tau > T\}}] = e^{-rT} \tilde{P}[\tau > T] = e^{-(r+\tilde{\lambda})T}$$

That is, in the intensity based framework we can value a defaultable bond as if it were default free by simply adjusting the discounting rate. Instead of discounting with the risk-free interest rate r, we now discount with the default-adjusted rate $r + \tilde{\lambda}$, where $\tilde{\lambda}$ is the risk-neutral intensity.

Recovery Conventions

So far we have assumed that in the event of a default, bond investors loose all of their investment. In practice, however, investors frequently receive some recovery payment upon default. For modeling purposes, we consider the following recovery conventions:

- Constant recovery (recovery of face value)
- Equivalent recovery (recovery of an equivalent default free bond)
- Fractional recovery of market value

Recovery of Face Value

Suppose that in case of default investors receive a constant amount $R \in [0,1]$ at T. Then the defaultable zero bond has a value of

$$\begin{aligned} B_0^T &= \tilde{E}[e^{-rT}(1_{\{\tau > T\}} + R 1_{\{\tau \leq T\}})] \\ &= e^{-rT}(e^{-\tilde{\lambda}T} + R(1 - e^{-\tilde{\lambda}T})) \\ &= \bar{B}_0^T - \bar{B}_0^T(1-R)\tilde{P}[\tau \leq T] \end{aligned}$$

which is the value of a risk-free zero minus the value of the expected default loss.

Equivalent Recovery

Suppose that in case of default investors receive a constant fraction $R \in [0,1]$ of an equivalent default-free zero bond at default. Then

$$\begin{aligned} B_0^T &= \tilde{E}[e^{-rT} 1_{\{\tau > T\}} + e^{-r\tau} R \bar{B}_\tau^T 1_{\{\tau \le T\}}] \\ &= \tilde{E}[e^{-rT} 1_{\{\tau > T\}}] + R\bar{B}_0^T - R\tilde{E}[e^{-rT} 1_{\{\tau > T\}}] \\ &= (1-R)\tilde{E}[e^{-rT} 1_{\{\tau > T\}}] + R\bar{B}_0^T \\ &= (1-R)B_0^T \tilde{P}[\tau > T] + R\bar{B}_0^T \end{aligned}$$

which is the value of a fraction of $1-R$ of a zero recovery bond plus the value of a fraction of R of a risk-free zero bond.

Fractional Recovery

Suppose that in case of default investors receive a constant fraction $R \in [0,1]$ of the pre-default market value $B^T_{\tau-}$ of the defaultable bond. Then

$$\begin{aligned} B^T_0 &= \tilde{E}[e^{-rT} 1_{\{\tau > T\}} + e^{-r\tau} R B^T_{\tau-} 1_{\{\tau \leq T\}}] \\ &= e^{-(r+(1-R)\tilde{\lambda})T} \end{aligned}$$

which is the value of a zero recovery bond with *thinned* (risk-neutral) default intensity $\tilde{\lambda}(1-R)$.

Credit Spreads

Using our general spread definition, with a zero recovery convention we get

$$S(0,T) = -\frac{1}{T} \ln \frac{e^{-(r+\tilde{\lambda})T}}{e^{-rT}} = \tilde{\lambda}$$

meaning that the credit spread is in fact given by the (risk-neutral) intensity. With a constant intensity, the term structure of credit spreads is of course flat.

For richer spread term structures, we need more sophisticated intensity models.

Time-Varying Intensities

The simple Poisson process can be generalized to intensities which vary over time. N is called a inhomogeneous Poisson process with deterministic intensity function $\lambda(t)$, if the increments $N(t) - N(s)$ are independent and for $s < t$ we have

$$P[N(t) - N(s) = k] = \frac{1}{k!} \left(\int_s^t \lambda(u) du \right)^k e^{-\int_s^t \lambda(u) du}$$

The default probability is then given by

$$P[\tau \leq t] = 1 - P[N(t) = 0] = 1 - e^{-\int_0^t \lambda(u) du}$$

Examples

- Constant: $\lambda(t) = \lambda$ for all t (the homogeneous Poisson case)
- Linear: $\lambda(t) = a + bt$
- Piece-wise constant: $\lambda(t) = a_1 + a_2 1_{\{t \geq t_1\}} + a_3 1_{\{t \geq t_2\}} + \dots$

The parameters must be chosen such that $\lambda(t) \geq 0$ for all t.

Bootstrapping the Intensity

Let us calibrate a piece-wise constant intensity model to market prices of defaultable bonds. Suppose the issuer has liquidly traded zero recovery bonds with maturities $T_1 < T_2 < \ldots < T_n$ and respective price quotes $Q_1, Q_2, \ldots, Q_n$. We have

$$Q_i e^{rT_i} = \tilde{P}[\tau > T_i] = e^{-\int_0^{T_i} \tilde{\lambda}(u)du}$$

with

$$\tilde{\lambda}(t) = a_1 + a_2 1_{\{t \geq T_1\}} + a_3 1_{\{t \geq T_2\}} + \ldots + a_n 1_{\{t \geq T_{n-1}\}}$$

We can now determine the coefficient a_1 from Q_1, a_2 from Q_2, and so on, given an estimate of the risk-free interest rate r.

Example

Suppose $n = 3$ and $T_i = i$ years. We have

$$Q_1 e^{r} = e^{-\int_0^1 a_1 du} = e^{-a_1}$$

$$Q_2 e^{2r} = e^{-\int_0^2 (a_1 + a_2 1_{\{u \geq 1\}}) du} = e^{-2a_1 - a_2}$$

$$Q_3 e^{3r} = e^{-\int_0^3 (a_1 + a_2 1_{\{u \geq 1\}} + a_3 1_{\{u \geq 2\}}) du} = e^{-3a_1 - 2a_2 - a_3}$$

allowing us to compute first a_1, then a_2, and afterwards a_3. Now our piece-wise constant intensity model is fully calibrated to market bond prices.

Credit Spreads

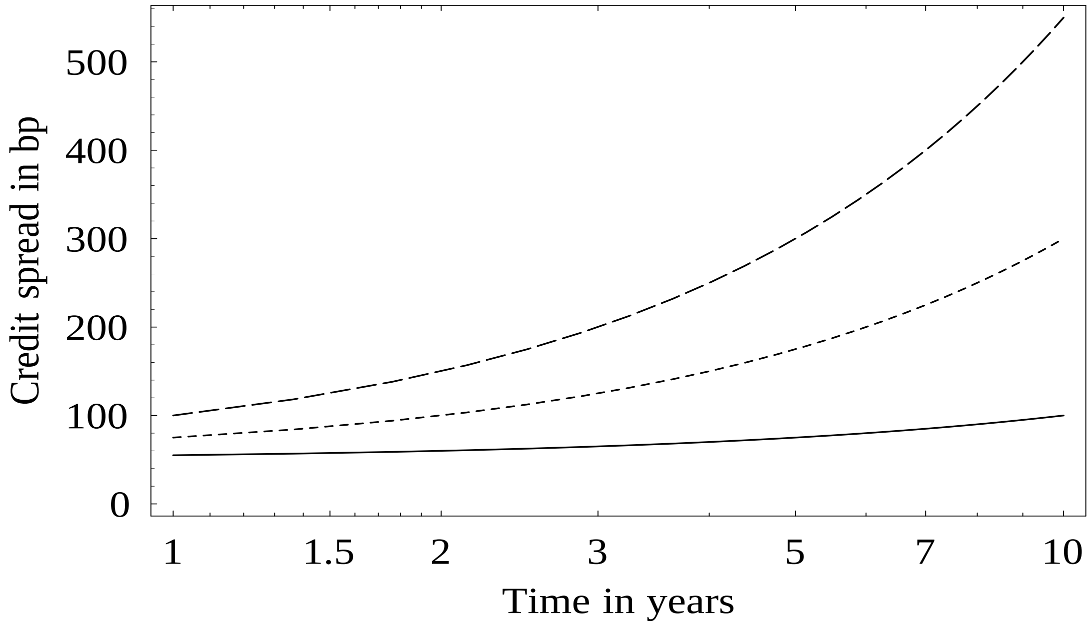
With time-varying intensities, zero-recovery bond prices are given by

$$B_0^T = \tilde{E}[e^{-rT} 1_{\{\tau > T\}}] = e^{-rT} \tilde{P}[\tau > T] = e^{-rT - \int_0^T \tilde{\lambda}(u) du}$$

Thus credit spreads are

$$S(0,T) = -\frac{1}{T} \ln \frac{e^{-rT - \int_0^T \tilde{\lambda}(u) du}}{e^{-rT}} = \frac{1}{T} \int_0^T \tilde{\lambda}(u) du$$

Term Structure of Credit Spreads



We assume that the intensity is linear: $\tilde{\lambda}(t) = a + bt$. Fixing the baseline intensity $a = 0.005$, we plot credit spreads $S(0,T)$ as a function of horizon T for varying intensity slopes $b = 0.001, 0.002$, and 0.01.

Cox Process

A Cox process N with intensity $\lambda = (\lambda_t)_{t \geq 0}$ is a generalization of the inhomogeneous Poisson process in which the intensity is allowed to be random, with the restriction that conditional on the realization of λ, N is an inhomogeneous Poisson process (for this reason N ia also called a conditional Poisson process or a doubly-stochastic Poisson process).

The conditional and unconditional default probability is given by

$$P[\tau \leq t \,|\, \lambda] = 1 - P[N(t) = 0 \,|\, \lambda] = 1 - e^{-\int_0^t \lambda_u du}$$

$$P[\tau \leq t] = E[P[\tau \leq t \,|\, \lambda]] = 1 - E[e^{-\int_0^t \lambda_u du}]$$

Stochastic Intensities: General Case

The most general definition of an intensity relies on the Doob-Meyer decomposition of the submartingale $1_{\{\tau \leq t\}}$: There exists an increasing predictable process A with $A_0 = 0$ (the compensator) such that the difference process defined by $1_{\{\tau \leq t\}} - A_t$ is a martingale. Now if A is absolutely continuous

$$A_t = \int_0^{\tau \wedge t} \lambda_s ds$$

for some non-negative process λ, then λ is called an intensity of τ. From this definition we can show that

$$P[\tau \leq t] = 1 - E[e^{-\int_0^t \lambda_u du}]$$

Intensity Models

- Credit rating driven: $\lambda_t = \lambda(X_t)$ where X is a vector of state variable which governs, among other things, also credit rating transitions (see Lando (1998), Jarrow-Lando-Turnbull (1997))
- Basic affine process

$$d\lambda_t = a(b - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + dJ_t$$

where W is a standard Brownian motion, J is an independent jump process with Poisson arrival intensity c and exponential jump size distribution with mean d, a is the mean-reversion rate, σ is the diffusive volatility, and the long-run mean of λ is given by $b + cd/a$. The famous Cox-Ingersoll-Ross model is the special case where $c = 0$. For affine intensities default probabilities are in closed-form.

Valuation for General Intensities

For general stochastic intensity processes the valuation of defaultable bonds is analogous to the case with constant intensities.

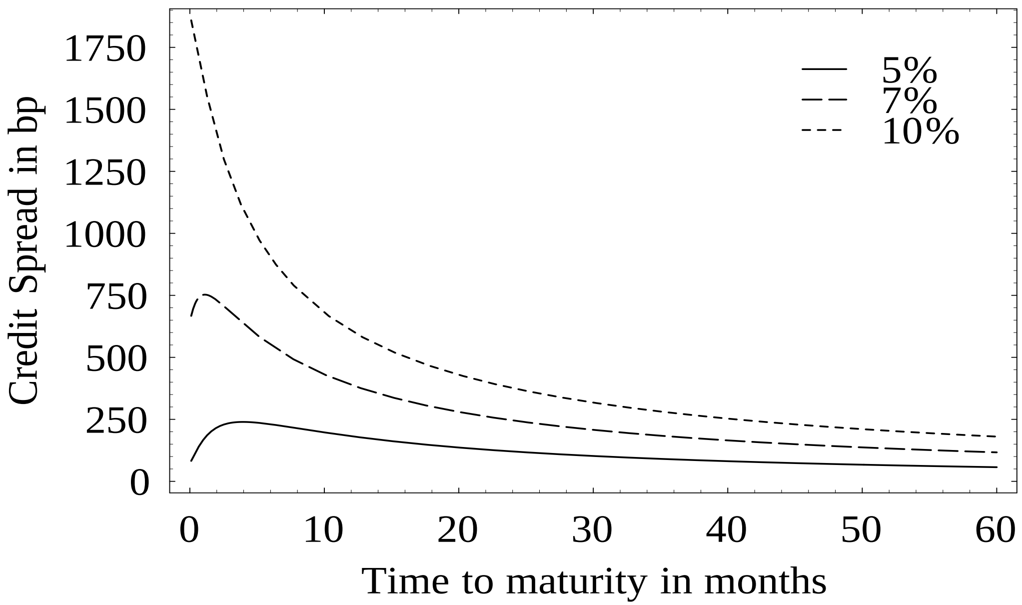
Consider a defaultable security paying off X at T if no default occurs and zero otherwise. For $X = 1$ this is a defaultable zero bond.

This security has (under technical conditions) a value given by

$$\tilde{E}\Big[e^{-\int_0^T r_s ds} X 1_{\{\tau > T\}}\Big] = \tilde{E}\Big[X e^{-\int_0^T (r_s + \tilde{\lambda}_s) ds}\Big]$$

where r is the risk-free short rate and $\tilde{\lambda}$ is the risk-neutral intensity process for default.

Credit Spreads for General Intensities



Credit spreads $S(0,T)$ as a function of horizon T for varying degrees of business risk (asset volatility). Spreads are strictly positive, which is empirically plausible. The short credit spread, i.e. the spread for maturities going to zero, is given by the intensity.

Default Correlation

To introduce default correlation, we can introduce correlation between individual firms' intensities. In such a model defaults are conditionally independent given the intensity.

To induce a stronger type of correlation, one can let the intensity of a particular firm jump upon the default of some other firm(s), corresponding to the idea of contagion among defaults. Another idea is to admit common jumps in intensities, corresponding to joint credit events.

Example: Joint Exponential Defaults

- 3 independent Poisson processes N_1, N_2, N with respective intensities λ_1, λ_2, λ
- 2 firms, where N_i leads to a default of firm i only and N leads to a simultaneous default of both firms
- Survival probability

$$P[\tau_i > t] = P[N_i(t) = 0]P[N(t) = 0] = e^{-(\lambda_i + \lambda)t}$$

- Joint survival probability

$$\begin{aligned} P[\tau_1 > t, \tau_2 > t] &= P[N_1(t) = 0]P[N_2(t) = 0]P[N(t) = 0] \\ &= e^{-(\lambda_1 + \lambda_2 + \lambda)t} \end{aligned}$$

Structural vs. Intensity Based Approach

- While the structural approach is economically sound, it implies empirically less plausible spreads. The intensity based approach is ad hoc, tractable, and empirically plausible.
- Structural and intensity based approach are not consistent; in the usual structural approach an intensity does not exist (this is due to the predicability of defaults).
- By introducing incomplete information in a structural model, both approaches can be unified (at least to some extent). This provides some economic underpinnings for the ad hoc nature of the intensity based framework.

Literature

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Credit Derivatives and Structured Credit: Basics

1. Definitions, terminology, key characteristics of contracts
2. The market and its participants
3. Regulatory considerations
4. Overview of instruments
5. Asset swaps

Structured Credit Market

The structured credit market encompasses a broad range of capital market products designed to transfer credit risk among investors through over-the-counter (OTC) transactions. These include mainly two classes:

1. Credit Derivatives, which are financial instruments whose value is derived from an underlying market instrument driven primarily by the credit risk of private or government entities
2. Structured Credit Products, which combine credit derivatives with cash transactions in a customized way

Key Characteristics

The key characteristic of credit derivatives is that they separate the credit risk from an underlying and thereby enable investors to gain or reduce exposure to credit risk.

Specifically, in the prospect of deteriorating credit and bond market illiquidity they allow to actively manage credit risk by

- buying protection to reduce credit risk
- selling protection to diversify credit risk

Credit Risk Management

Until recently, no tailored risk management products for credit existed. For loans, risk management consisted of diversification by setting limits in connection with occasional loan sales. Corporates often carried an open exposure, they were rarely able to buy insurance from factors.

These strategies are inefficient, because they do not separate the management of credit risk from the underlying asset; to lay off the risk one had to sell the asset entirely.

Credit derivatives separate the credit risk from an underlying, and allow an independent management of these risks. This implies efficiency gains through a process of market completion; risk can be efficiently taken.

Traditional Credit Protection

- Guarantee
- Letter of credit
- Loan participation (silent)
- Loan sale
- Credit insurance

These transactions share important features of credit derivatives. So why are credit derivatives so significant?

Short Credit Position

Credit derivatives allow to enter a short credit position, i.e. to load off credit risk, with reasonable liquidity and without the risk of a short squeeze.

They allow to reverse the risk-return profiles provided by credit instruments for hedging or speculating purposes.

Credit derivatives allow to take advantage of arbitrage opportunities (discrepancies in the pricing of the same credit risk through asset classes, maturities, or rating classes).

Reference Entity

The reference entity or entities need neither be a party to nor aware of the transaction.

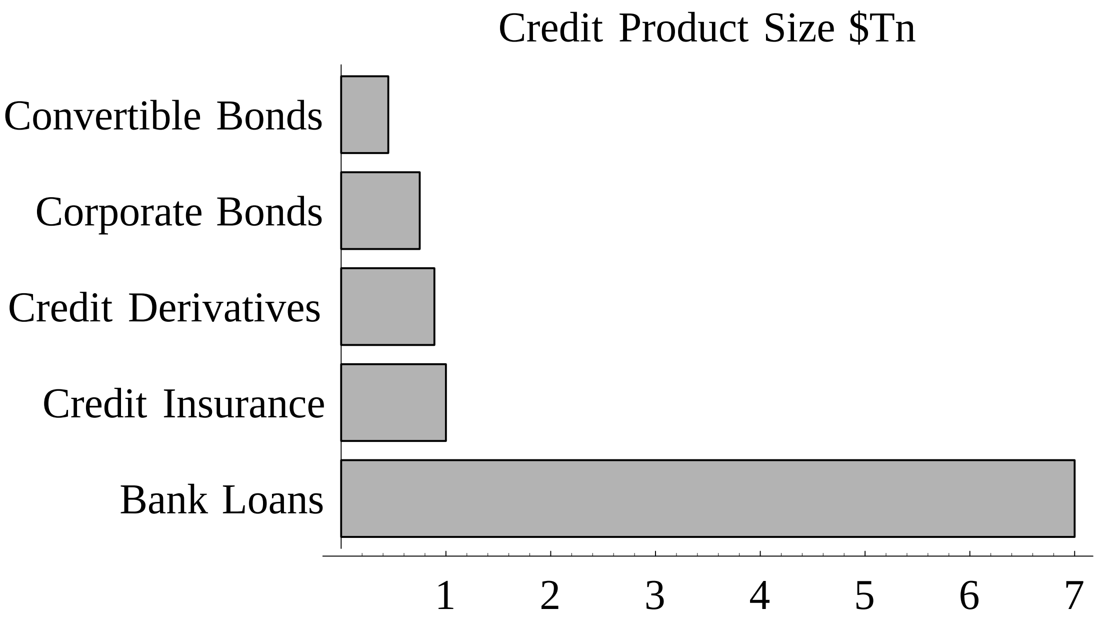
- Lay off credit risk without ceasing the customer relationship, i.e. manage risk discreetly
- Terms of the transaction can be negotiated freely; they can be customized to meet the needs of the involves parties
- Pricing discipline: credit derivative is objective market pricing benchmark for some specific credit risk

Off-Balance-Sheet Nature

Except when embedded with structured notes, credit derivatives are off-balance-sheet transactions, i.e. there is no payment of the underlying nominal.

- Flexibility in terms of leverage (risk vs. capital commitment); example: take on the risk-return profile of a bond without committing capital to buy the bond itself
- The more costly the balance sheet (i.e. the higher funding rates), the higher the appeal of the off-balance-sheet nature of the transactions
- New lines of counterparties for traditional credit instruments such as loans, which were previously not accessible

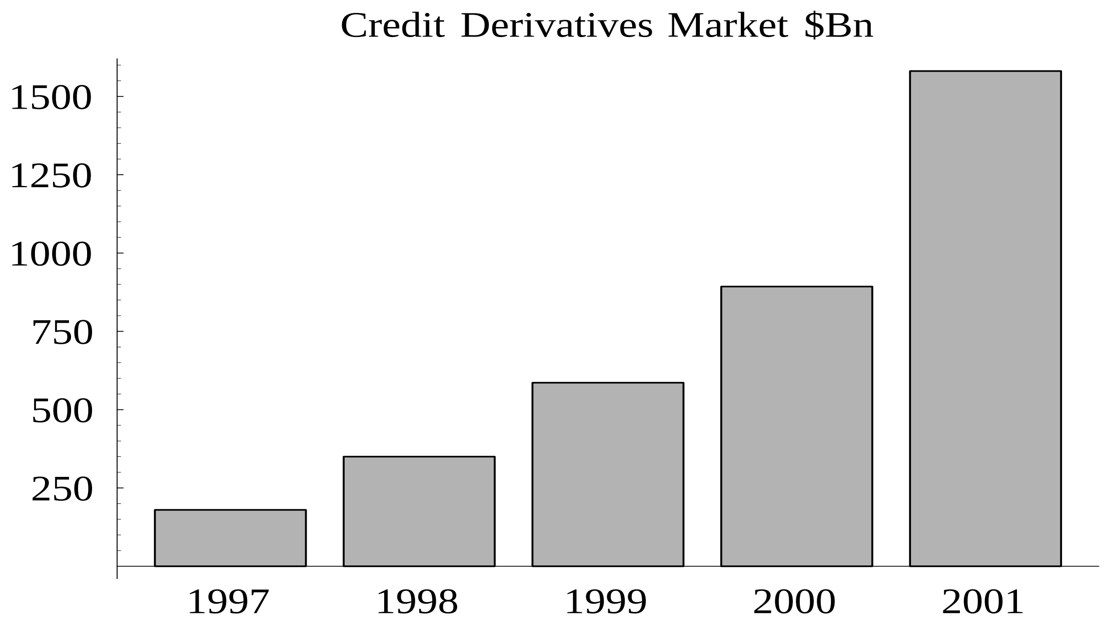
Global Credit Market, 2001



Source: Morgan Stanley Dean Witter

The Asian financial crisis in August '97 and the Russian default in August '98 were milestones in the development of the structured credit market.

Market Development



Source: Morgan Stanley Dean Witter

Emerging markets sovereign related, corporate name, and bank name each account for one-third of underlyings. 55% of volume is based on low investment grade, and 30% is based on sub-investment grade.

Regulatory Considerations

Banks have to set aside capital against their aggregated (credit) risks. Credit derivatives allow to lay off risks, and therefore provide the potential to reduce regulatory capital.

Currently, there is a 8% flat charge, irrespective of the risk of the credit. The buyer of protection via a credit derivative only gets a capital relief if the transaction can be considered very similar to a loan guarantee. The protection seller must account for the assumed credit risk as if it were a loan. The exact rules differ by country.

Basel II

From 2006 onwards, new capital adequacy rules (called the 'Basel II' guidelines, www.bis.org) will be effective, which prescribe a risk-adjusted capital charge. Reflecting their different nature, credit, market, and operational risk are treated differently, i.e. with different methods.

Basel II recognizes the ability of credit derivatives to mitigate credit risks. On the other hand, it acknowledges the need to assess the residual risks carefully. A significant residual risk is when the protection seller is highly correlated with the reference entity (for which protection is sought). This reduces the capital relief that can be gained with the transaction.

User Summary: Banks

- sell credit risk for regulatory capital management (improve return on equity (ROE))
- sell credit risk for economic capital management (increase diversification/reduce concentration while maintaining borrower relationship, improve return on equity)
- buy and sell credit risk for the trading book (leverage expertise on credit risk)

User Summary: Insurers

Insurers buy credit risk for credit insurance and re-insurance. Motivation:

- Low insurance spreads compared to capital market credit spreads
- Leverage extensive credit expertise (investors)
- Pre-existing control and technology infrastructure
- Comfortable with contract-to-contract risk

User Summary: Securities Firms

Securities firms sell credit risk. Motivation:

- Hedge extreme illiquidity and/or credit cycles
- Leverage pre-existing derivative infrastructure
- Leverage portfolio trading skills
- Leverage bond distribution capabilities

Investors in general buy credit risk profiles which are not available in the cash market.

Market Share

Protection Buyer/Seller	Buyer	Seller
Banks	64	54
Securities Firms	18	22
Corporations	7	3
Insurance Firms	5	10
Governments/Export Credit Agencies	4	1
Mutual Funds	1	4
Pension Funds	1	2
Hedge Funds	0	4

Documentation

To reduce legal exposure due to disagreements about documentation issues, the documentation of the terms of a credit derivative contract should be standardized.

This is done in the contract definitions completed by the International Swaps and Derivatives Association (ISDA, www.isda.org), which aims at minimizing the administrative effort required to close transactions.

Product Overview

Single-name instruments involve one reference entity, for example a corporate or sovereign bond. The following credit derivatives are popular in this class:

- Asset swap as basic building blocks
- Default swap (also called credit swap)
- Total return swap and index swaps
- Credit spread options

Structured credit products with a single reference entity include

- Credit linked note
- Synthetic revolver facility

Product Overview (2)

Multi-name instruments involve several reference entities, e.g. a portfolio of corporate or sovereign bonds. Basket default swaps are common credit derivatives in this class.

Structured credit products with a several reference entities include

- Repackaging programs
- Collateralized debt obligations (CDO's)

Asset Swaps

An asset swap can be viewed as the basic building block in the structured credit market. The transaction consists of two simultaneous steps:

- Purchase fixed-rate asset, e.g. a corporate coupon bond
- Enter into an interest rate swap with a bank that converts the fixed rate into floating, where the term of the swap matches bond maturity

Hence the investor extracts a floating rate from a fixed rate coupon bond. The spread of this rate over LIBOR is due to the credit risk of the bond; the asset swap strips this spread from the bond.

More Complex Asset Swaps

- bond is in foreign currency, the swap then includes both an IR and an FX component
- structured asset swap: strip away certain unwanted IR-features from the bond, which lead to a discount in its value
- generate enhanced yield for an exposure not available in the cash market; example: asset swaption, where the investor gets a fee for being liable to enter an asset swap if spread or interest rates reach a strike level

Advantages and Disadvantages

Asset swaps are simple, transparent, flexible, and isolate the credit risk of some underlying (bonds, loans, receivables). Its disadvantages led eventually to the development of many structured credit products:

- Many investors cannot enter in derivative transactions due to regulatory, accounting, or investment policy restrictions.
- Default of underlying security and swap in the package are not linked; after a default the investors remains in the swap.
- (Highly rated) bank has exposure to the investor, which is often unwanted (unless collateral is posted, which may affect the economics of the transaction).

Default Swaps

1. Structure of transaction
2. Synthesizing the default swap
3. Pricing the default swap
4. Example
5. Basis trade

Definition

A default swap (or credit swap) is a bilateral financial contract in which

- one party (the protection seller) makes a payment upon some specified credit event arrival before some stated maturity,
- and in return, the other party (the protection buyer) pays some periodic fee until the credit event or the contract's maturity, whichever is first.

That is, the protection seller is buying the credit risk of the reference credit, while the protection buyer is selling that risk.

Payments

The periodic fee is called the swap rate, swap spread, or swap premium. It is expressed in basis points (bp) per annum on the notional amount.

Normally, the contingent payment of the swap is $(1 - R)$ times the reference notional, where $R \in [0, 1]$ is the recovery rate.

If the contingent payment is some fixed pre-determined amount, we speak of a binary or digital default swap (DS).

If one leg is denominated in a currency other then the base currency of the reference bond, we speak of a quanto DS.

Settlement Procedures

The settlement mechanism prescribes the way in which the swap is closed out upon a default:

1. Physical delivery: protection buyer delivers the defaulted reference asset (i.e. the recovery value) in exchange for its notional value
2. Cash settlement: protection seller pays the difference between the notional and the value of the defaulted reference asset. The recovery value is determined by a dealer poll some specified time after default.

Remarks

Since no asset is transferred, the contract is not funded.

The reference credit does not need to be in a relation to any party.

The DS works like a default insurance contract, with the difference that in a classic insurance contract the insurance buyer has to own the object for which insurance is sought.

Sometimes the protection buyer is granted the right to cancel the contract before its maturity (this is a cancellable DS).

The DS is the most fundamental and most popular credit derivative.

Example

We consider a DS with the following characteristics:

- Swap parties: B (protection buyer), S (protection seller)
- Inception: March 1, 2000
- Maturity: 5 years
- Reference asset: DaimlerChrysler (DCX) bond XYZ
- Notional amount: 100 million Euro
- Credit event: default
- Swap rate: 90 basis points (=900.000 Euro) per annum, paid annually, starting March 1, 2001

Example (continued)

If DCX does not default before March 1, 2005: B pays $5 \cdot 900.000$ to S at the respective coupon dates and receives nothing from S.

If DCX does default on September 1, 2003: B pays to S $3 \cdot 900.000$ at the respective coupon dates and $0.5 \cdot 900.000$ at the default date and

- can deliver the defaulted DCX bonds to S, who pays 100 mio (=par value) for them (physical delivery)
- S pays 70 mio to B, given the dealer poll has resulted in an recovery estimate of $R = 30\%$, and B retains the defaulted bonds (cash settlement).

With either settlement procedure, B does not suffer a loss due to the default of DCX. This has its price for B, namely the fee paid to S.

Pricing a Default Swap

A DS involves two pricing problems:

- When making markets, one is interested in the fair swap rate at inception of the contract.
- When hedging or marking-to-market, one is interested in the market value of the swap, which need not be zero after origination due to changing interest rates and credit quality of the reference credit.

We will focus on the first problem.

Synthesizing a Default Swap

Suppose the underlying reference credit is a defaultable floating rate note (FRN, or floater), with floating rate $D_t = U_t + Q$ at date t, where U_t is the floating rate of some default-free FRN and Q is some constant contractually agreed spread.

Now consider the position of an investor who longs the default-free FRN and shorts the defaultable reference FRN. This portfolio is hold until the credit event or the swap's maturity (which is assumed to be that of the reference note). The investor pays D_t and receives U_t, and pays hence net Q.

Synthesizing a Default Swap (Continued)

If there is no default, both notes mature and there is no net cash flow.

If there is a default, the investor liquidates his positions and receives (at the following coupon date) the difference between the par value of the default-free FRN and the market value of the defaulted FRN (=R times notional). His net cash flow is $(1 - R)$ times notional.

In both scenarios, the position of the investor has the same cash flows as those to the protection buyer in the DS. Therefore, in the absence of arbitrage opportunities, the swap rate must be equal to Q, the spread on the reference FRN. (We have assumed here that the accrued swap rate is not paid.)

Shorting the Reference Note

The reference note can be shorted through a reverse repo and a cash sale. This involves receiving the note as collateral for a loan in a repo and a sale of the note in the cash market.

If the note is on special, then shorting requires an additional annuity, the term repo special S (this can be due to the 'scarcity' of the note). The investor's net position is then $U_t + Q + S - U_t = Q + S$. By our arbitrage argument, the swap rate must then be equal to $Q + S$. This is exact if the repo terminates at default (if not, the error is however small).

These arguments can also be extended to include transaction costs, accrued credit swap premium, and accrued interest on the reference note.

Hedging the Default Swap

We can hedge the DS by the replicating (synthesizing) portfolio strategy just described. If we are the protection seller (and are liable to the default of the reference FRN), we would just short the underlying defaultable FRN and long a default-free FRN.

Shorting Credit

Two possibilities: short bond via reverse repo or via buying protection in a DS. The former, however, involves the risk of a short squeeze.

Due to a market-wide in short activity, after inception of the repo the repo rate can decrease substantially, which increases the funding costs for the shorting investor, i.e. shorting becomes more expensive.

If one uses a DS to short a bond, the shorting costs are fixed at inception, i.e. there is no risk of a short squeeze.

Example

Consider an investor who wants to short 10 mio of face of some bond. The asset swap spread is 120 bp, and the repo rate at inception is LIBOR. 3 months after inception, short activity increases and the repo rate is now LIBOR minus 20 bp. The costs for a 6 months short are then $120bp \cdot 0.25years \cdot 10mio + 140bp \cdot 0.25years \cdot 10mio = 65.000$.

Alternatively, the investor can short the bond by buying protection in a DS on the bond with a spread of 120 bp. The total costs are then $120bp \cdot 0.5years \cdot 10mio = 60.000$.

Valuing the Fee Leg

Suppose the swap rate Q as a fraction of notional in bp/year is paid at dates $t_1 < t_2 < \ldots < t_n = T$ with $\Delta(t_{i-1}, t_i)$ representing the interval between payments dates (i.e. approximately 0.5 for semi-annually payments). Assuming that defaults are independent of interest rates, we get for the value f of the fee leg

$$f = \sum_{i=1}^{n} \tilde{E}[\bar{B}_0^{t_i} Q \Delta(t_{i-1}, t_i) 1_{\{\tau > t_i\}}] = Q \sum_{i=1}^{n} \Delta(t_{i-1}, t_i)(1 - p(t_i))\bar{B}_0^{t_i}$$

where $\bar{B}_0^t$ is the price of a riskless bond maturing at t and $p(t) = \tilde{P}[\tau \leq t]$ is the risk-neutral default probability.

Valuing the Contingent Leg

Suppose the recovery rate R is a random variable whose mean can be estimated, and suppose that R is independent of the default. Assuming that default can only occur at a set of discrete dates, the value of the contingent leg c is

$$
\begin{aligned}
c &= \tilde{E}[\bar{B}_0^\tau (1-R) 1_{\{\tau \le T\}}] \\
&= \tilde{E}[1-R] \int_0^T \bar{B}_0^u \tilde{P}[\tau \in du] \\
&\approx \tilde{E}[1-R] \sum_{i=1}^m [p(t_i) - p(t_{i-1})] \bar{B}_0^{t_i}
\end{aligned}
$$

Letting $m \to \infty$ the approximation is exact.

Swap Rate

The swap rate should be such that it compensates the protection seller for the potential contingent payment. That is, the swap rate should make the market values of the swap legs equal. From $c = f$ we find

$$Q = \frac{\tilde{E}[1-R]\sum_{i=1}^{m}[p(t_i) - p(t_{i-1})]\bar{B}_0^{t_i}}{\sum_{i=1}^{n}\Delta(t_{i-1}, t_i)(1 - p(t_i))\bar{B}_0^{t_i}}$$

For the computation of Q we hence need risk-neutral default probabilities $p(t)$. These can be implied out from defaultable bond prices.

Note that we assume here that the protection seller has no default risk.

Bootstrapping Default Intensities

Sometimes default probabilities are stripped of DS quotes instead of bond prices. These are then used to price more complex structures.

Suppose we observe price quotes of DSs of the same issuer but for various maturities $T_1, T_2, \ldots, T_m$. We can then calibrate an intensity based model with piece-wise constant intensity $\tilde{\lambda}(t) = a_i$ for $t \in [T_{i-1}, T_i]$ and $i \in \{1, 2, \ldots, m\}$. Then $p(t) = 1 - e^{-\int_0^t \tilde{\lambda}(u)du}$.

We now solve first for a_1 using the swap with maturity T_1, then for a_2 using a_1 and the T_2-swap quote, and so on.

Example

Suppose the following DSs of Daimler-Chrysler are quoted:

Maturity	Currency	Payments	Recovery	Market Swap Rate
0.5	EUR	Semi-annually	30%	50 bp
1	EUR	Semi-annually	30%	60 bp
3	EUR	Semi-annually	30%	80 bp
5	EUR	Semi-annually	30%	105 bp
7	EUR	Semi-annually	30%	120 bp
10	EUR	Semi-annually	30%	140 bp

Example (continued)

We let

$$\tilde{\lambda}(t) = \begin{cases} a_1 & : \quad 0 \leq t < 0.5 \\ a_2 & : \quad 0.5 \leq t < 1 \\ a_3 & : \quad 1 \leq t < 3 \\ a_4 & : \quad 3 \leq t < 5 \\ a_5 & : \quad 5 \leq t < 7 \\ a_6 & : \quad 7 \leq t < 10 \end{cases}$$

Example (Continued)

$$p(0.5) = 1 - e^{-\int_0^{0.5} a_1 du} = 1 - e^{-0.5a_1}$$

$$p(1) = 1 - e^{-\int_0^{0.5} a_1 du - \int_{0.5}^{1} a_2 du} = 1 - e^{-0.5a_1 - 0.5a_2}$$

$$p(3) = 1 - e^{-\int_0^{0.5} a_1 du - \int_{0.5}^{1} a_2 du - \int_1^3 a_3 du} = 1 - e^{-0.5a_1 - 0.5a_2 - 2a_3}$$

$$\ldots$$

Knowing that $Q(0.5) = 50bp$, with $p(0.5)$ we can solve our swap value formula for a_1 (given an estimate of riskless bond prices). Using that result, the fact that $Q(1) = 60bp$ and our expression for $p(1)$, we can solve for a_2, and so on:

$$a_1 = 0.7, \quad a_2 = 1.0, \quad a_3 = 1.25, \quad a_4 = 2.13, \quad a_5 = 2.4, \quad a_6 = 3.0$$

Default Swap Basis

The basis is the difference between DS spread and asset swap (AS) spread on the same bond.

A positive basis is typical:

- natural credit hedgers use the DS market
- shortage of underlying bond
- convertible bond issuance
- trading of structured bonds with unusual features
- delivery option of protection buyer

One can pick up the differential yield by a short basis trade: sell bond and simultaneously protection via a DS.

Default Swap Basis (Continued)

The basis can also be negative:

- firm has issued securities with different liquidity
- counterparty default risk of the protection seller
- residual risk in the AS for the investor if note defaults before the swaps maturity

One can pick up the differential yield by a long basis trade: buy bond and simultaneously protection via a DS.

Literature

- Duffie (1999) “Credit Swap Valuation,” Financial Analysts Journal, January-February, 73-87.

Basket Default Swaps

1. Contract definitions and characteristics
2. Copulas
3. Default correlation
4. Valuing a kth-to-default swap
5. First-to-default swaps

Definitions

Basket default swaps (BDS) have several underlying debt instruments; their payoffs depend on the joint performance of this pool of instruments.

The following specifications for the contingent swap leg are typical:

- kth-to-default: pay upon the kth default in the reference pool
- First k out of n to default: pay upon the first k defaults in the reference pool
- Last k out of n to default: pay upon the last k defaults in the reference pool

Typically, $5 \leq n \leq 15$.

Contract Characteristics

- Compared with single-name DS, BDS offer multi-name exposure without modifying the size of the contingent payment (depends on notional and recovery of the defaulted credit(s))
- Allow to take a view on default correlation between credits
- Default correlation is the fundamental driver of BDS premiums

Modeling Issues

- Need model for correlated defaults, i.e. for (risk-neutral) joint default/survival probabilities
- Calibration of correlation parameters from market data is difficult, which complicates the risk management for these products and leads to uncertainty in pricing
  - Structural: from equity price correlations
  - Intensity based: from DS spread/credit spread co-movements or DS spreads involving defaultable counterparties for example

Default Time Copula

Consider n firms with respective continuous default times τ_i. Suppose we have a model for risk-neutral joint survival probabilities s.

s can be uniquely represented by its *copula* C^τ and marginals s_i:

$$s(t_1, \ldots, t_n) = P[\tau_1 > t_1, \ldots, \tau_n > t_n] = C^\tau(s_1(t_1), \ldots, s_n(t_n)),$$

so that C^τ describes the complete *non-linear default dependence structure*. C^τ is a joint distribution function with standard uniform marginals.

One can efficiently simulate correlated default times from C^τ.

Copula Bounds

Define

$$C^L(u_1, \ldots, u_n) = \max(u_1 + \ldots + u_n - n + 1, 0)$$

$$C^\Pi(u_1, \ldots, u_n) = u_1 \cdots u_n$$

$$C^U(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$$

We have $C^L \leq C^\tau \leq C^U$ where

$C^\tau = C^L$ iff defaults are countermonotone

$C^\tau = C^\Pi$ iff defaults are independent

$C^\tau = C^U$ iff defaults are comonotone

Perfect Dependence

Comonotonicity means that defaults are perfectly positively correlated. In the bivariate situation, we then have $\tau_2 = F(\tau_1)$ with F increasing.

Countermonotonicity means that defaults are perfectly negatively correlated. In the bivariate situation, we then have $\tau_2 = F(\tau_1)$ with F decreasing.

Examples

The Clayton copula family is given by

$$C_\theta(u_1, \ldots, u_n) = (1 - n + u_1^{-\theta} + \ldots + u_n^{-\theta})^{-1/\theta},$$

where the parameter $\theta > 0$ controls the degree of correlation:

$$\lim_{\theta \to \infty} C_\theta = C^U \text{ and } \lim_{\theta \to 0} C_\theta = C^\Pi$$

The normal copula is given by

$$C_\Sigma(u_1, \ldots, u_n) = \Phi_\Sigma^n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where Φ_Σ^n is the n-variate standard normal distribution function with correlation matrix Σ and Φ^{-1} is the inverse standard normal distribution function.

Scalar-valued Default Correlation Measures

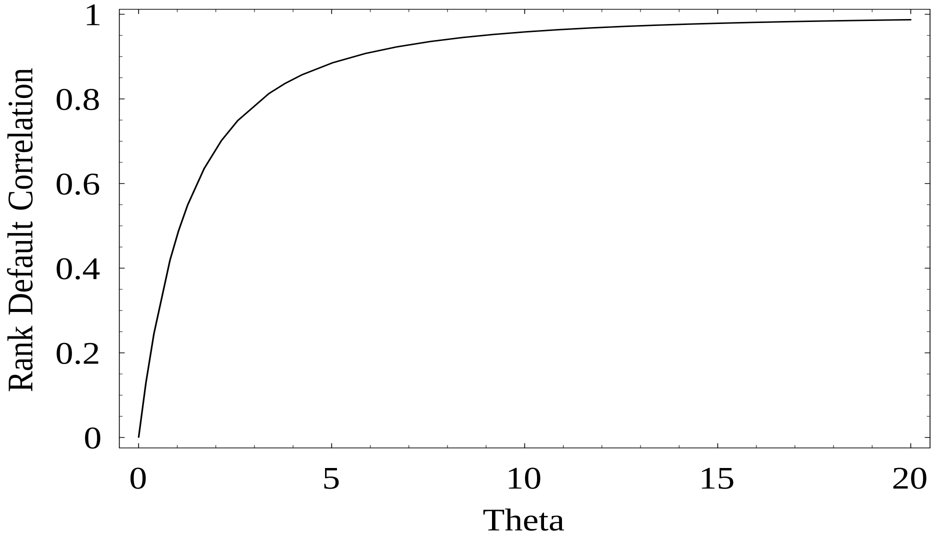
- Traditionally: *linear* indicator correlation $\rho(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$. But this can lead to severe misinterpretations of the dependence since covariance is only the the natural measure of dependence for joint elliptically distributed random vectors.

- Suggestion: *Rank* default correlation RC is copula based and does not share these deficiencies:

$$RC(\tau_i, \tau_j) = 12 \int_0^1 \int_0^1 (C_{ij}^{\tau}(u, v) - uv) du dv \in [-1, 1]$$

  where C_{ij}^{τ} is the two-dimensional default time copula.

Rank Default Correlation



The figure shows $RC(\tau_i, \tau_j)$ as function of θ if the default copula C_{ij}^{τ} is the two-dimensional Clayton copula with parameter θ. Clearly, we have $RC = -1$ iff $C^{\tau} = C^{L}$, $RC = 0$ iff $C^{\tau} = C^{\Pi}$, and $RC = 1$ iff $C^{\tau} = C^{U}$.

Valuing a kth-to-default Swap

Consider a kth-to-default swap, in which the swap rate Q (bp per year) is paid up to the kth default or the maturity T of the swap, whichever is first. Maintaining our assumptions of the previous lecture, the market value of the fee leg f is

$$f = \sum_{i=1}^{l} \tilde{E}[\bar{B}_0^{t_i} Q \Delta(t_{i-1}, t_i) 1_{\{T_k > t_i\}}] = Q \sum_{i=1}^{l} \Delta(t_{i-1}, t_i)(1 - p_k(t_i)) \bar{B}_0^{t_i}$$

where T_k is the kth default time, and $p_k(t) = \tilde{P}[T_k \leq t]$ is the risk-neutral probability that the kth default is before time t (taken from the multi-firm model).

Valuing a kth-to-default Swap (2)

The kth-to-default swap pays the random amount $(1 - R_i)$ upon T_k if $\tau_i = T_k \leq T$. The market value c of this contingent swap leg is

$$c = \tilde{E}[\bar{B}_0^{T_k} \sum_{i=1}^{n} (1 - R_i) 1_{\{\tau_i = T_k \leq T\}}]$$

The fair swap rate Q for a BDS can be calculated in the same manner as with a single-name DS: Q is determined through the condition $f = c$.

Other payoff structures can be handled analogously.

Usually Q is estimated using Monte-Carlo-Simulation.

First-to-default Intensity

Suppose one has bootstrapped the (risk-neutral) default intensities $\tilde{\lambda}^i$ of the individual names in the first-to-default (FTD) basket swap ($k = 1$). Then the FTD time $T_1 = \min(\tau_1, \ldots, \tau_n)$ has intensity

$$\sum_{i=1}^{n} \tilde{\lambda}^i$$

The FTD survival probability is then given by

$$1 - p_1(T) = \tilde{P}[T_1 > T] = \tilde{E}[e^{-\int_0^T (\tilde{\lambda}_t^1 + \ldots + \tilde{\lambda}_t^n) dt}]$$

This result can be used for the pricing of FTD swaps and the valuation of counterparty default risk in (single-name) DS.

Binary First-to-default Swap

We now consider a FTD swap, which pays at T the amount 1 (i.e. $R_i = 0$ for all i) if there is at least one default in the reference basket in return for a single up-front (i.e. $l = 1$) payment Q. Hence $f = Q$ and the fair rate is given by $Q = c$ with

$$\begin{aligned} c = \tilde{E}[\bar{B}_0^T 1_{\{T_1 \leq T\}}] &= \bar{B}_0^T \tilde{P}[T_1 \leq T] \\ &= \bar{B}_0^T (1 - \tilde{P}[\min_i(\tau_i) > T]) \\ &= \bar{B}_0^T (1 - \tilde{P}[\tau_1 > T, \ldots, \tau_n > T]) \\ &= \bar{B}_0^T (1 - C^\tau(s_1(T), \ldots, s_n(T)) \\ &= \bar{B}_0^T - \bar{B}_0^T C^\tau(s_1(T), \ldots, s_n(T)) \end{aligned}$$

Price Bounds for the FTD Swap

For fixed individual survival probabilities $s_i(T)$, the fair rate Q of the FTD swap is determined through the correlation between the defaults.

Since the 'smallest' default copula is the one for countermonotone defaults and the 'largest' the one where defaults are comonotone,

- Q is maximal if defaults are perfectly negatively dependent ($C^\tau = C^L$);
- Q is minimal if defaults are perfectly positively dependent ($C^\tau = C^U$)

and the FTD swap rate Q is *decreasing* in default correlation.

FTD Swap vs. Single-name DS

Setting interest rates equal to zero, we have

$$\underbrace{1-\min(s_1(T),\ldots,s_n(T))}_{\text{rate on riskiest name}} \leq Q \leq \underbrace{1-\max(s_1(T)+\ldots+s_n(T)-n+1,0)}_{\text{sum of rates on all names}}$$

where we note that, under our current assumptions, the swap rate on an individual name i is just $Q_i = 1 - s_i(T)$.

Hence, abstracting from transaction and documentation costs, a FTD swap makes sense only if the underlying credits are not perfectly correlated.

Example

Consider a homogenous basket with $n = 2$ and 2% individual one-year default probability, i.e. $s_1(1) = s_2(1) = 0.98$.

If the credits are perfectly negatively dependent,

$$Q = 1 - \max(s_1(1) + s_2(1) - 1, 0) = 1 - (2 \cdot 0.98 - 1) = 400bp,$$

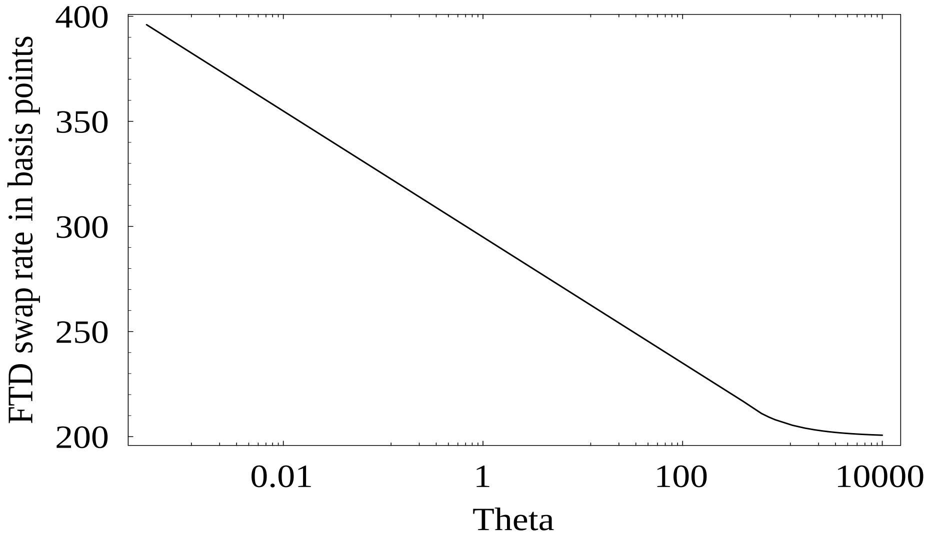
If the credits are independent, we get

$$Q = 1 - s_1(1)s_2(1) = 1 - 0.98^2 = 396bp,$$

With perfect positive dependence

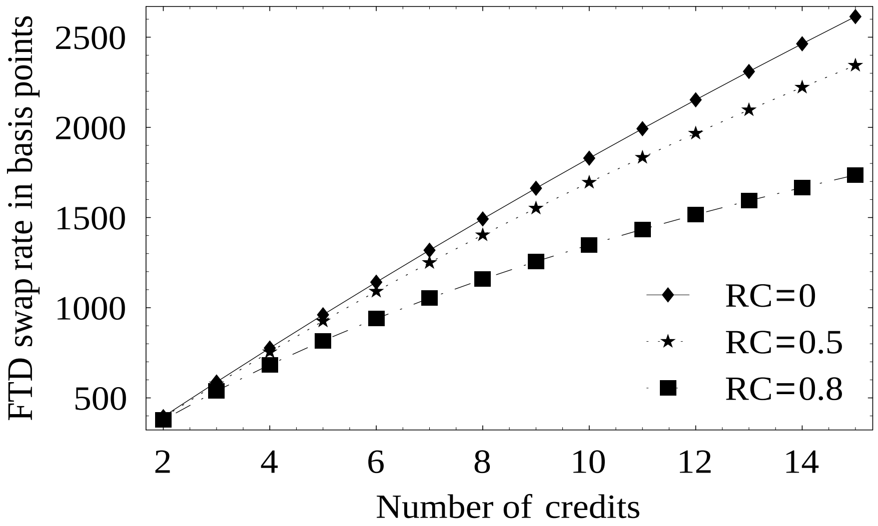
$$Q = 1 - \min(s_1(1), s_2(1)) = 1 - 0.98 = 200bp,$$

FTD Swap Rate vs. Default Correlation



We plot the fair FTD swap rate Q (we use the Clayton copula for C^{τ} and assume positive default correlation). With increasing default correlation, the probability of multiple defaults increases and the degree of default protection provided by a FTD swap is diminished, leading to lower premiums Q.

FTD Swap Rate vs. Number of Credits



We plot Q as a function of the number n of underlying credits for varying degrees of rank default correlation. With increasing number of underlyings, the probability of at least one default increases and so does the value of the FTD swap protection.

FTD Swap Characteristics

- Allow to hedge correlated defaults efficiently (i.e. with lower costs compared with hedging the underlying credits individually)
- While FTD protection seller is positively exposed to an increase in default correlation, the protection buyer is negatively exposed to an increase in correlation
- After termination of the contract upon the first default, the protection buyer remains unhedged on the remaining credits
  - Structure low correlation basket with investment-grade quality
  - Similar quality in the basket

Literature

- Nelsen (1999), An Introduction to Copulas, Springer-Verlag, New York.
- Embrechts, McNeil, Straumann, Correlation and Dependence in Risk Management, Working Paper, ETH Zurich. `http://www.math.ethz.ch/~baltes/ftp/papers.html`

Credit-linked Notes

1. Repackaging
2. Asset swap repackaging
3. Credit-linked notes: structure and mechanism
4. Motivations of the parties
5. Valuing a credit-linked note

Repackaging

Repackaging involves placing securities in an Special Purpose Vehicle (SPV), which then issues customized notes that are backed by the instruments in the SPV.

The goal is to take securities with attractive features that are nevertheless unappealing/inaccessible to many investors and repackage them to create viable investments that are not otherwise available to the investor.

A security may be unappealing/inaccessible because it is denominated in an foreign currency, does not trade locally, or because of onerous tax features.

Repackaging vs. Securitization

While repackaging is based on re-structuring securities, securitization involves the re-structuring of *non*-security financial assets, for example loan paper or accounts receivable originated by banks, credit card companies, or other providers of credit.

Repackaging is often less complex. Because of higher fixed costs, securitizations are typically larger than repackaging programs. While in repackaging programs the notes are available in a variety of documentation forms (medium-term notes, Schuldschein, commercial paper, loans) to meet investors' needs, in a securitization the notes are less customized (many investors typically participate).

Asset Swap Repackaging

Due to regulatory/investment policy restrictions, some investors cannot enter into an interest rate swap. Solution:

- Fixed-rate bonds are placed in an SPV
- SPV enters into an interest rate swap to exchange fixed for floating (LIBOR+credit spread) over the remaining term of the bonds
- SPV issues FRN to investor, who earns LIBOR+spread (lower than on comparable straight asset swap)
- In case of default, the transaction terminates, including the interest rate swap. Note investors receive any recoveries.

Credit-linked Notes

A credit-linked note (CLN) is a special form of repackaging that is often directly issued by a corporate issuer. Typical underlyings are individual debt instruments (loans, bonds, etc.), portfolios of debt instruments, or bond or emerging market indices.

The principal/coupon on the note is paid contingent on the occurrence of a credit event.

A CLN is a *funded* (i.e. on-balance sheet) alternative to a default swap (DS). Compared with a straight DS, the credit investment is de-leverd.

The reference credit is usually not aware of the transaction.

Mechanism of a CLN

A CLN is a structured note with an embedded DS, in which

- Issuer/SPV issues notes to investors which are backed by the reference assets, and receives the proceeds
- Issuer/SPV sells protection on the reference asset in a DS with a third party (bank), and receives the swap rate
- Issuer/SPV pays an enhanced coupon to the note investors

Instead of a DS, the CLN may also embed a first-to-default swap or a spread option.

Payoffs to Note Investors

The CLN terminates at default of the reference asset or the maturity, whichever is first.

- If there is no default, the DS terminates and the note investors receive the full note principal (like a straight bond position with enhanced coupon)
- In case of default, the SPV pays the default losses to the DS protection buyer. Note investors receive not the full principal repayment, but loose an amount equal to the default losses.

A (less-risky) principal-protected CLN is not terminated at default, but interest payments to the notes cease. At maturity, the full principal is returned to the note investors.

CLN Spreads

Note investors bear the potential losses due to default of the reference asset. In return, they pick up some yield. In essence, the position of the note investors is if they had sold default protection on the reference asset via a DS to the CLN issuer/SPV.

Thus, the spread picked up by the CLN investor must roughly equal the DS rate paid by the SPV. In practice, the CLN spread is somewhat higher than the DS spread, since the accrued swap rate is normally not paid in a CLN.

Protection Buyer in CLN

Usually, the SPV puts the proceeds of the note issuance in collateral. If the reference entity defaults, the SPV liquidates the collateral and simply pays less principal back to the note investors. This 'spared' amount is directly passed through to the protection buyer (a bank) in the DS.

Since the SPV has a very high rating (usually AAA), the bank is not exposed to a significant counterparty default risk in the DS (it were exposed if it had contracted the DS with the investors directly).

CLN Investor

An investor (e.g. an investment management fund) may be prohibited from buying bonds rated below AAA. The SPV's rating is AAA, which refers to its 'ability to pay ', but not its 'obligation to pay'. So the fund is allowed to invest into the CLN, although it is essentially investing into the riskier reference asset (it bears the losses associated with them).

Thus a CLN can be used to circumvent certain investment constraints in order to enhance yields.

Why are CLNs Attractive?

- Provides a funded credit derivative investment opportunity which is used by real-money DS investors such as mutual funds, investment departments of insurance companies, pension plans, which are not authorized to invest directly into credit derivatives
- Provides a yield pickup to other securities issued by the reference entity
- Provides customized maturity structures and credit features that are not otherwise available in the cash market
- If used by banks to lay off loan credit risks, a CLN provides access to funded credit opportunities that are not otherwise available

Mexican Default/Inconvertibility Structure

In 1995 following the financial crisis in Mexico, a short-term CLN was designed to allow investors to capitalize on their bullish view on Mexico (continue to meet obligations and to convert the Peso):

- A1+ issuer/SPV
- Up to USD 20 million notional
- Between 30 and 90 days maturity
- Issued at a discount (zero-coupon like structure)
- Credit event: any default on any reference security or inconvertibility of the Peso to USD

Mexican Default/Inconvertibility Structure

- Reference entity: Government of Mexico
- Reference securities: any issued or guaranteed by the reference entity
- Redemption: if there is no credit event, at par; if there is a credit event, delivery of any of the reference securities or its cash equivalent

The spread on the notes was around 350-450bp, depending on market conditions. Investors secured this yield enhancement for their willingness to risk principal repayment with a deterioration of the credit quality of Mexico.

Valuing a CLN

Suppose the CLN pays coupons at dates $t_1 < t_2 < \cdots < t_n = T$, where T is the maturity and the annual coupon rate is c. The notional is 1.

The value of the notes is $CLN = CLN_N + CLN_D$, the sum of the PV's of no-default payments and default payments. We have

$$\begin{aligned} CLN_N &= \tilde{E}[\sum_{i=1}^{n} \Delta(t_{i-1}, t_i) c \bar{B}_0^{t_i} 1_{\{\tau > t_i\}} + \bar{B}_0^T 1_{\{\tau > T\}}] \\ &= \underbrace{\sum_{i=1}^{n} \Delta(t_{i-1}, t_i) c \bar{B}_0^{t_i} (1 - p(t_i))}_{\text{PV of coupons}} + \underbrace{\bar{B}_0^T (1 - p(T))}_{\text{PV of principal}} \end{aligned}$$

where $p(t) = \tilde{P}[\tau \leq t]$ is the risk-neutral default probability.

Valuing a CLN (2)

Suppose the recovery rate is a constant $R \in [0,1]$. Then

$$\begin{aligned} CLN_D &= \tilde{E}[\bar{B}_0^{\tau} R 1_{\{\tau \leq T\}}] \\ &= R \int_0^T \bar{B}_0^u \tilde{P}[\tau \in du] \\ &\approx R \sum_{j=1}^{m} \tilde{P}[t_{j-1} \leq \tau < t_j] \bar{B}_0^{t_j} \\ &= R \sum_{j=1}^{m} [p(t_j) - p(t_{j-1})] \bar{B}_0^{t_j} \end{aligned}$$

Letting $m \to \infty$ the approximation is exact (in practice, one usually sets $t_j - t_{j-1} = 1$ day).

Pre-paid Default Swap

A pre-paid DS is another funded DS alternative. It is structured like a DS with the difference that the protection seller pays at inception the notional of the underlying to the protection buyer, who pays a periodic fee, composed of funding costs and the swap rate, to the seller.

The structure terminates at maturity or the credit event, whichever is first. If there is no default, the notional is returned to the seller. In case of a default, only the recovered amount on the reference asset is returned, so that the seller sustains the default losses on the underlying.

Credit Spread Options

1. Contract definition
2. Payoffs
3. Pricing
4. Synthetic lending facilities

Definition

A credit spread put (call) option involves the right, but not the obligation, to sell (buy) the defaultable reference bond at a given strike credit spread over treasury. The option can be European, Bermudean, or American.

Besides fixed-rate and floating-rate bonds, underlyings include default swaps and asset swaps (this is then called an asset swaption).

Options can survive a default or be knocked out upon default.

A variant is the downgrade option, which pays off if the reference asset is downgraded.

Settlement of the contract is the same as with default swaps (cash settlement, physical delivery).

Credit Spread Put

The payoff to a spread put with strike spread K and maturity $t < T$ (=maturity of the underlying bond) is given by

$$\max(0, \bar{B}_t^T e^{-K(T-t)} - B_t^T)$$

where $e^{-K(T-t)} \leq 1$ is the price of a bond with yield K over $[t, T]$, $\bar{B}_t^T$ is the price of a default-free bond, and B_t^T of a defaultable one, both with maturity T (note that $\bar{B}_t^T \geq B_t^T$).

The put involves the right to exchange a defaultable bond with price B_t^T at the maturity of the option t into $e^{-K(T-t)}$ default-free bonds with price $\bar{B}_t^T$.

Credit Spread Put

In terms of yields, we can also write for the put payoff

$$\max(0, e^{-K(T-t)} - e^{-S_t^T(T-t)})\bar{B}_t^T$$
$$= \max(0, e^{-(\bar{y}_t^T + K)(T-t)} - e^{-(\bar{y}_t^T + S_t^T)(T-t)})$$

where $\bar{y}_t^T$ is the non-defaultable yield, K is the strike spread, and S_t^T is the credit yield spread.

If the spread widens S_t^T to the level K (the bond price declines to the yield spread equivalent of K), the spread option comes into the money, as it provides the right to sell the defaultable bond at a price $e^{-(\bar{y}_t^T + K)(T-t)}$ corresponding to the spread level K.

Characteristics

- Allows to capitalize on/hedge against changes in credit spreads
- Allows to trade forward credit spread expectations separately from interest rate changes and provides an asymmetric payoff profile (credit spread vol counts)
- Addresses both credit and default risk, which makes it appealing to investors following a mark-to-market standard, who can hedge their mark-to-market exposure to fluctuation in spreads (here also the off-balance sheet nature of the deal is attractive)
- Interesting for portfolios that are forced to sell deteriorating assets: here options can reduce risk of sales at distressed prices

Default Swap vs. Credit Spread Put

Hedging against a significant spread change can also be accomplished with a default swap (DS).

Consider a DS and a long American spread put. In a default both DS and put will be in the money; i.e. the put can be used to hedge against default. The put has a higher price (making protection more costly), since it provides also protection to non-default credit events (spread widens beyond the strike).

At-market premiums on the DS will change with a spread widening as well–one can monetize this gain by unwinding the DS or selling protection on a comparable asset.

Example

An investor believes that the spread on an issuer narrows over a one-year period. He can monetize this view by selling the following spread put:

- Notional up to USD25 million
- Maturity one year
- Option premium 0.75% flat
- Current spread 85bp
- Strike spread 100bp

Example (2)

- Reference bond: ABC's 7.75% 10 year
- Reference Treasury: current US Treasury 10 year
- Reference spread: yield on ref bond minus yield on Treasuries
- Payoff: purchaser can put the ref bonds to the seller at the strike spread over the Treasury yield at maturity

The put is superior to a straight long bond position, since there is no interest rate risk and there is the possibility of pocketing the return on the high spread vol.

Pricing

The value of the spread put at time zero is given by

$$\bar{B}_0^T \max(0, e^{-K(T-t)} - e^{-S_t^T (T-t)})$$

so S_t^T, the credit spread on the reference asset, needs to be modeled. In the intensity based framework, assuming defaults to be independent of riskless rate, we have simply

$$S_t^T = -\frac{1}{T-t} \ln \tilde{P}[\tau > T \,|\, \mathcal{F}_t] = -\frac{1}{T-t} \ln \tilde{E}[e^{-\int_t^T \tilde{\lambda}_s ds} \,|\, \mathcal{F}_t]$$

where $\mathcal{F}_t$ can be thought of as the information available at time t. Hence, we need a model for the evolution of the risk neutral intensity $\tilde{\lambda}$ over time.

Normal Model

One can assume a mean-reverting Gaussian process for the intensity (called the extended Vasicek model):

$$d\tilde{\lambda}_t = (k(t) - a\tilde{\lambda}_t)dt + \sigma dW_t$$

where

- W is a standard Brownian motion
- σ is the volatility of the intensity
- $\tilde{\lambda}$ mean-reverts at speed a to the level $k(t)/a$

$\tilde{\lambda}$ is normally distributed, and becomes negative with (small) positive probability. $k(t)$ is calibrated from market swap spreads. In this model spreads S_t^T are available in closed form.

Log-Normal Model

An alternative specification is

$$\frac{d\tilde{\lambda}_t}{\tilde{\lambda}_t} = k(t)dt + \sigma dW_t$$

where

- W is a standard Brownian motion
- $k(t)$ (σ) is the mean (volatility) of the intensity

$\tilde{\lambda}$ is log-normally distributed (fat tails!), and is therefore guaranteed to stay positive. In this model spreads S_t^T are not available in closed form, which makes parameter calibration more difficult.

Remarks

- Extension to stochastic interest rates, correlated to intensity
- Extension to jumps in intensity, but calibration is difficult
- Implementation: trinomial tree with a branch for default
- Fitting to market swap spreads of different maturities by shifting the tree
- The higher the spread (intensity) volatility, the higher the option price (note: due to an increasing default probability, the effect is different from usual options for high vols)
- The higher the strike, the lower the option price (for out-of-the-money puts lognormal model yields higher prices than the normal model due to fat tails)

Synthetic Lending Facility

In a revolver credit a bank agrees to make loans up to a specified maximum up to a specified period. In addition to interest on the drawn amount, the bank receives a commitment fee from the borrower.

In a synthetic lending facility (or synthetic revolver), a bank (the investor) receives fee income in return for a forward commitment to purchase a reference security at a pre-specified spread level via an asset swap.

Mechanism

- Bank receives a commitment fee for agreeing to extend funds at a pre-specified rate to the SPV on demand
- SPV sells an asset swaption (an option to enter an asset swap) to a market maker and receives a fee; the market maker has the option to put the asset to the SPV
- If asset swaption is exercised, the SPV buys the the asset by drawing the loan facility from the bank, which then receives Libor + spread

Motivation for the Bank

- Higher fees than for funded revolving loan agreement
- Customized and liquid structure
- Transaction may benefit from regulatory capital arbitrage, because unfunded lending transactions qualify for a reduced capital charge in some countries (this is not the case for a default swap, which the bank could sell alternatively to achieve a similar investment profile)

Motivation for Market Maker

- Provide customized means to buy protection on the reference asset, when there are mostly requests to sell protection (i.e., investors want to buy protection)
- If spreads increase, demand on protection increases as well–now the market maker has the valuable option to buy this protection via the asset swaption at a pre-specified price and clear markets

Collateralized Debt Obligations

1. Definition and structures
2. Mechanics
3. Waterfall
4. Motivation and risks for investors

Definition

A Collateralized Debt Obligation (CDO) is a securitization in which a portfolio of securities is transferred to a SPV which in turn issues tranches of debt securities (notes) of different seniority and equity to fund the purchase of the portfolio.

A Collateralized Bond Obligation (CBO) involves mostly bonds, while a Collateralized Loan Obligation (CLO) involves mostly loans.

The debt tranches are typically rated based on portfolio quality, diversification, and structural subordination.

Characteristics

- CDO repackages the credit risk of the collateral pool
- Senior and mezzanine tranches concentrate 'good' risks, while the equity piece hosts 'bad' risks (equity absorbs losses up to some extent)
- Return on CDO securities depends on the joint default performance of the assets in the collateral pool

CDO Structures

CDOs can be classified as follows (purpose and credit structure):

- Balance-sheet CLOs
  - Cash-flow type: there is no trading of collateral; only defaults in the collateral cause loss risk for the investors
- Arbitrage CBOs
  - Cash-flow type
  - Market-value type: manager trades collateral assets; investors' loss risk is due to credit and market risk

The majority of structures is of the cash-flow type.

Arbitrage Structure

Typical collateral: high-yield bonds, leveraged loans, emerging market and investment-grade debt, derivatives, hedge funds.

The sponsor seeks to capture an arbitrage between yield on collateral acquired in the capital markets (largely sub-investment grade) and investment-grade notes issued to investors (the high-risk pieces are buffered by the equity tranche).

This provides note investors with sub-investment grade opportunities.

Balance-Sheet Structure

Typical collateral: investment-grade and leveraged loans, emerging market debt, ABS, project finance debt

- Used by banks (=sponsor) looking to securitize illiquid loans in the banking book
- Release risk-based capital through the sale of loan assets
- Shrink balance sheet and improve ROE
- Arbitrage: issue securities that are more highly rated than the sponsor

Note that the transfer of legal title of loans as well as keeping borrower confidentiality is quite costly.

Synthetic CLOs

In a synthetic CLO the credit risk of the collateral pool is transferred to the SPV by means of a credit derivative (e.g. a portfolio default swap), while the sponsor retains the assets on the balance sheet.

The SPV invests the proceeds from the note issuance in high-quality assets and uses interest and swap rate to pay coupons on the notes. In case of default, the SPV must sell collateral to pay in the credit derivative, up to the total amount of collateral. What remains is used to pay back the note investors.

Key problem: capital relief is not always granted

Collateral Coverage Tests

Collateral coverage tests are used to decide whether the current collateral is sufficient to cover the service payments on debt tranches.

- Haircut test: haircut collateral mark-to-market $\geq$ debt tranche par $+$ accrued interest (haircut rate depends on asset class)
- Over-collateralization (OC) test: ratio of the total par collateral value to the sum of par value of the tranche and all tranches senior to this tranche $\geq$ threshold
- Interest coverage (IC) test: ratio of total collateral interest to the sum of interest on the tranche and tranches senior to this tranche $\geq$ threshold

Mechanics of Market Value CDOs

The CDO-manager trades the collateral to take advantage of relative-value opportunities.

The collateral is marked-to-market periodically. If a tranche fails the haircut test, then collateral is sold and debt is repaid until the test passes. Alternatively, equity holders may contribute assets.

There is also a periodic check whether the value of equity ($\approx$ the excess of collateral market value over the par and accrued interest on all debt tranches) is below some threshold. If this is the case, collateral must be sold until all debt tranches are retired.

Some Issues in Market Value CDOs

- Credit quality of notes depends on the effectiveness of the haircut test (conservatism of rate vs. time between valuation)
- Volatility of the market value of collateral assets is affected by interest rate vol and credit spread vol
- Diversification of collateral assets lowers market value vol: restrictions with respect to concentrations by single-name, industry, geography, etc.
- Liquidity of collateral assets can be a factor
- Market valuation of collateral assets must be available

Mechanics of Cash Flow CDOs

The CDO collateral is not traded. The SPV collects any coupons, principal, and default recoveries on the collateral. This is used to service the issued securities. The credit quality of the tranches depends on the credit quality of the collateral and the protectiveness of the CDO structure.

Cash flow protection to investors is achieved by prioritizing the tranches:

- Priority in bankruptcy
- Priority in cash flow timing

Waterfall

Tranches are serviced according to a prioritization scheme:

1. Pay fees to trustee, asset manager, etc.
2. Pay interest to the most senior notes; if OC and IC coverage tests are not met, redeem notes until test is met
3. Pay interest to the next subordinated tranche; if OC and IC coverage tests are not met, redeem first most senior notes, and then, if necessary, this tranche until test is met
4. Service other tranches following this scheme
5. Pay down tranches according to their priority
6. Remains, if any, go to equity investors

Equity Piece

Credit markets are imperfect.

- Moral hazard of sponsor/CDO-Manager:
  - Selection of high-quality assets
  - Costly enforcement of debt covenants
- Adverse selection due to asymmetric information about collateral; investors demand a lemon's premium

In order to mitigate the effects of moral hazard and asymmetric information, the sponsor often signals commitment by retaining the equity piece.

Motivation for Investors

Notes:

- Higher yields as compared to corporates and asset-backeds of the same maturity and rating
- Narrower default loss distribution (benchmark: individual bond)
- Customized exposure to new asset class, allows to overcome investment restrictions

Equity:

- Leveraged position in collateral
- Sponsor in balance-sheet CDO: leverage collateral information
- Asset manager: increase assets under management and generate fees quickly

Primary Risks for CDO-Investors

- Credit risk of collateral: loss risk for investors (interest/principal)
- Interest-rate risk:
  - In arbitrage cash-flow mostly in the form of basis risk
  - Hedges are difficult and incomplete due to trading of assets
- Liquidity risk
  - Secondary market for CDO-notes is limited
  - Collateral is only limitedly liquid (concern in market-value structure)

Collateralized Debt Obligations

1. Analyzing a cash flow structure
2. Valuing the tranches
3. Moody's Binomial Expansion Technique
4. Infectious default model

Waterfall in a Cash Flow Structure

1. Pay fees to trustee, asset manager, etc.
2. Pay interest to the most senior notes; if OC and IC coverage tests are not met, redeem notes until test is met
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Collateral Pool

Consider a (cash flow) CDO with the following collateral:

- Face value M_i of asset i
- Annualized coupon rate C_i of asset i, pays n times a year
- $A(k)$ is the set of non-defaulted assets at coupon period k
- $B(k) = A(k-1) - A(k)$ is the set of assets defaulting between coupon periods $k-1$ and k
- L_i is the loss in face value at default of collateral asset i

Cash Flow to Collateral

The SPV collects any cash flows from the collateral pool: coupons, principal, and recoveries on defaulted assets. The total cash flow $Z(k)$ available in coupon period k is given by

$$Z(k) = \sum_{i \in A(k)} \frac{C_i}{n} M_i + \sum_{i \in B(k)} (M_i - L_i)$$

These cash flows are allocated to the prioritized CDO-tranches. There are several tranches of sinking fund bonds and an equity piece (most subordinated).

Sinking Fund Bond

- n coupon periods per year
- $F(k) = F(k-1) - D(k) - J(k)$ remaining principal at k, where $D(k)$ is the pre-payment of principal, and $J(k)$ is the contractual unpaid reduction in principal (prioritize tranches given defaults)
- c annualized coupon rate, so $F(k)\frac{c}{n}$ is paid at k
- $Y(k)$ actual interest paid at k; if $Y(k) < F(k)\frac{c}{n}$ (note that collateral may default) any difference is accrued at rate c
- $U(k)$ accrued unpaid interest at k,

$$U(k) = (1 + \frac{c}{n})U(k-1) + F(k)\frac{c}{n} - Y(k)$$

Assumptions

- Senior sinking fund bond, principal $F_1(0) = P_1$, coupon rate c_1
- Mezzanine sinking fund bond, principal $F_2(0) = P_2$, coupon c_2
- Equity with initial market value of $P_3 = P - P_1 - P_2$, where P is the principal value of the collateral, no guaranteed coupon
- Fixed maturity of the tranches ($=$ coupon date K)
- Excess cash flow from the collateral is deposited in a reserve account which earns interest at the risk-free rate $r(k)$ (alternatively, these funds may be used to purchase additional collateral)

Uniform-Prioritization Scheme

Let $W(k) = \sum_{i \in A(k)} \frac{C_i}{n} M_i$ be the collateral interest at k.

1. Interest to senior bond: $Y_1(k) = \min(U_1(k), W(k))$
2. Interest to mezzanine: $Y_2(k) = \min(U_2(k), W(k) - Y_1(k))$
3. Remains are deposited in a reserve account with value (before payments)

$$R(k) = (1 + \frac{r(k)}{n})[R(k-1) - Y_1(k-1) - Y_2(k-1)] + Z(k)$$

4. Let $H(k)$ be the losses since the previous coupon date, less collected and undistributed interest:

$$H(k) = \max\Big(0, \sum_{i \in B(k)} L_i - [W(k) - Y_1(k) - Y_2(k)]\Big)$$

Uniform-Prioritization Scheme (2)

5. Unpaid reductions in principal are applied in reverse priority order; for equity $J_3(k) = \min(F_3(k-1), H(k))$
6. Mezzanine: $J_2(k) = \min(F_2(k-1), H(k) - J_3(k))$
7. Senior: $J_1(k) = \min(F_2(k-1), H(k) - J_3(k) - J_2(k))$
8. There is no early redemption of principal, $D_i(k) = 0$ for $k < K$. At maturity K, remaining reserves are paid in priority order (wlog $Y_i(K) = 0$):

$$D_1(k) = \min(F_1(K) + U_1(K), R(K))$$

$$D_2(k) = \min(F_2(K) + U_2(K), R(K) - D_1(K))$$

$$D_3(k) = R(K) - D_1(K) - D_2(K)$$

Fast-Prioritization Scheme

1. Interest to senior bond: $Y_1(k) = \min(U_1(k), Z(k))$
2. Principal to senior bond: $D_1(k) = \min(F_1(k-1), Z(k) - Y_1(k))$
3. Interest to mezzanine: $Y_2(k) = \min(U_2(k), Z(k) - Y_1(k) - D_1(k))$
4. Principal to mezzanine:
   $D_2(k) = \min(F_2(k-1), Z(k) - Y_1(k) - D_1(k) - Y_2(k))$
5. Equity: residuals $D_3(k) = Z(k) - Y_1(k) - D_1(k) - Y_2(k) - D_2(k)$

For this scheme, there is no contractual reduction in principal, $J_i(k) = 0$.

Change of Prioritization Schemes

In practice, failure to pass certain over-collateralization tests triggers a change from uniform prioritization to some form of fast prioritization (to increase loss protection for the senior tranche).

If such a feature is in fact in place, then the spreads on the senior tranche applying with uniform prioritization would provide upper spread bounds, while those applying with fast prioritization would provide lower bounds.

Valuing the Tranches

The market value of the tranches can be estimated through simulation:

1. Simulate successive correlated defaults and the associated losses
2. Allocate cash flows to the tranches according to the chosen prioritization scheme
3. Discount cash flows for each tranche at risk-free rates
4. For each tranche, average the discounted cash flows over independently generated scenarios to obtain an estimate of the value of the tranche

The challenging task is here to generate correlated default times (see literature).

Loss Distribution

The joint default behavior of the collateral pool determines the returns of the CDO investors. In assessing the CDO, the quantity of interest is the distribution of the total (default) losses in the collateral pool, and associated risk measures (expected losses, variance, quantiles such as VaR, expected shortfall).

The loss distribution can be simulated as follows:

1. Simulate successive correlated defaults and the associated losses
2. Aggregate losses up to the maturity of the structure
3. Generate many independent scenarios to calculate the frequency distribution of aggregate losses

Moody's Binomial Expansion Technique

To assign a credit rating to the CDO debt tranches, Moody's applies the so-called Binomial Expansion Technique (BET) which aims at assessing the 'diversity' of the collateral pool.

Moody's considers the original collateral pool with n names as equivalent in terms of loss risk to an idealized comparison portfolio of the same total face value with

- $d \leq n$ *independent* firms with equal face value and default probability
- the same 'average' default probability as the collateral bonds
- the the same loss risk according to some risk measure.

The number d is called the diversity score.

Diversity Score

In the BET, Moody's assumes that firms in different industries default independently, while firms in the same industry are correlated. Moody's defines 32 industry sectors and the following diversity score table:

Number of Firms in Same Industry	Score
1	1.00
2	1.50
3	2.00
4	2.33
5	2.67
6	3.00
7	3.25
8	3.50
9	3.75
10	4.00
$>$10	case by case

Example

Consider a collateral portfolio of $n = 60$ bonds having equal (one-year, say) default probability q and face value with the following structure:

Number of Firms in Same Industry	1	2	3	4	5
Number of Incidents	2	7	6	4	2
Diversity Score	$2 \cdot 1$	$7 \cdot 1.5$	$6 \cdot 2$	$4 \cdot 2.3$	$2 \cdot 2.6$

Hence, the comparison portfolio has a diversity score of $d = 39$.

The maximal diversity is here $n = 60$, corresponding to independent collateral bonds.

Default Distribution

Since the defaults in the comparison portfolio are independent, the distribution of the number N of defaults in the comparison portfolio is binomial, i.e.

$$P[N = k] = \binom{d}{k} q^k (1-q)^{d-k}$$

where $\binom{d}{k} = \frac{d!}{k!(d-k)!}$. The expected number of defaults is $E[K] = dq$ and the variance is $Var[K] = dq(1-q)$.

Determining the Comparison Portfolio

A way to determine d and the individual (comparison) default probability $q(t)$ such that the first two moments of the law of

$$\sum_{i=1}^{n} 1_{\{\tau_i \leq t\}} \quad \text{and} \quad \frac{n}{d} \sum_{i=1}^{d} 1_{\{\sigma_i \leq t\}}$$

are equal for some fixed horizon t, say one year. The $1_{\{\sigma_i \leq t\}}$ are iid Bernoulli with parameter $q(t)$. This yields the relations

$$nq(t) = \sum_{i=1}^{n} p_i(t)$$

$$\frac{n^2}{d} q(t)(1 - q(t)) = \sum_{i=1}^{n} \Big(p_i(t)(1 - p_i(t)) + \sum_{j=1, j \neq i}^{n} (p_{ij}(t) - p_i(t)p_j(t)) \Big)$$

where $p_i(t)$ is the default prob. of i and $p_{ij}(t)$ is the joint default prob.

Infectious Default Model

Rather than to replace the original portfolio with some comparison portfolio in order to account for the correlation between defaults of collateral positions, one can also model interaction effects directly. Let $Z_i = 1$ if bond i defaults, and $Z_i = 0$ otherwise, and define

$$Z_i = X_i + (1 - X_i) \left(1 - \prod_{j \neq i} (1 - X_j Y_{ji}) \right)$$

where X_i and Y_{ij} are independent Bernoulli random variables with parameters q and c, respectively. That is, asset i may default 'directly' with probability q, or may be infected through a default of some other asset (there must be at least one defaulted asset j for which $Y_{ji} = 1$). c is called the infection parameter.

Infectious Default Distribution

The probability of k defaults out of the n assets in the pool is

$$P[N=k] = \binom{n}{k}\Big(q^k(1-q)^{n-k}(1-c)^{k(n-k)} + \sum_{i=1}^{k-1}\binom{k}{i}q^i(1-c)^{n-i}[1-(1-c)^i]^{k-1}(1-c)^{i(n-k)}\Big)$$

If $c=0$, then this is the Binomial distribution with parameter q. Also

$$E[N] = n[1-(1-q)(1-qc)^{n-1}]$$

and $Var[N]$ is available in closed-form as well.

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