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MODELLING CREDIT: THEORY AND PRACTICE

Dominic O'Kane and Lutz Schlögl

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Summary

In recent years, there has been considerable interest in the application of credit models to the analysis and valuation of credit risk. In this article we present and analyse the most popular credit models, studying both the single issuer and the portfolio case. In doing so we provide practical examples of when and how these models should be used.

Dominic O'Kane
dokane@lehman.com
+44-20 7260 2628

Lutz Schlögl
luschloe@lehman.com
+44-20 7601 0011 ext. 5016

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Introduction to Credit Modelling

In recent years, the combination of low government bond yields, low issuance of government debt and the astonishing growth of the credit derivatives markets has attracted a significant flow of investors to higher yielding corporate and emerging market securities. Banks and other institutions which are in the business of taking on credit exposures are also looking for more sophisticated ways to reduce their credit concentrations, either through diversification or by securitising their portfolios into first and second-loss products.

As a result there is a growing need for credit models. These can be used for a variety of purposes - relative value analysis, marking-to-market of illiquid securities, computing hedge ratios, and portfolio level risk management. For marking to market, models need to be arbitrage-free to guarantee consistent pricing, and must be sufficiently flexible to reprice the current market completely. Furthermore, there is a need for models that can add value by providing some insight into the default process. This is especially so in view of the relative paucity of market data in the credit markets.

At a portfolio level, credit models are an invaluable aid to loan managers as they can quantify the marginal diversification of adding a new loan to the balance sheet. The introduction of new portfolio credit derivatives such as basket default swaps has also created a need for a better understanding of the default correlation between different defaultable assets. Within the growing field of debt securitisation, where the credit risk of a large pool of defaultable assets is tranched up into a number of issued securities, portfolio credit models have become an essential requirement to determine the rating and pricing of the issued securities.

However, credit modelling is a difficult business. Default is a rare event[1], such that there are barely enough observations for us to extract meaningful statistics. In addition, default can occur for many different reasons, ranging from the microeconomic, e.g. bad company management, to the macroeconomic, such as extreme currency or interest rate movements. In the case of sovereign debt, there is the added complication that default can occur not just because of the inability to pay but also because of an unwillingness to do so. The restructuring process initiated at default can be quite complicated and time consuming, resulting in considerable uncertainty about the timing and magnitude of recovery payments. Furthermore, insolvency and bankruptcy laws vary from country to country.

There are several contending approaches to credit modelling. As a means of classification, models can mainly be split into two groups - structural and reduced-form. The former type relates to models that have the characteristic of describing the internal structure of the issuer of the debt, so that default is a consequence of some internal event. The latter type - reduced-form- does not attempt to look at the reasons for the default event. Instead it models the probability of default itself, or more generally, the probability of a rating transition.

In this article we present a review of what we consider to be the most promising of all of these models. In doing so, we pay particular attention to how they are used in practice.

We also cover some of the portfolio credit risk models. These are an essential requirement for the pricing and analysis of portfolio default trades such as credit default baskets and collateralised debt obligations.

[1] *Moody's counted 106 defaults of Moody's rated issuers worldwide in the whole of 1999.*

Single Issuer Credit Models

Structural Models of Default

The structural approach to modelling default risk attempts to describe the underlying characteristics of an issuer via a stochastic process representing the total value of the assets of a firm or company. When the value of these assets falls below a certain threshold, the firm is considered to be in default.

Historically, this is the oldest approach to the quantitative modelling of credit risk for valuation purposes, originating with the work of Black/Scholes (1973) and Merton (1974). As the fundamental process being described is the value of the firm, these models are alternatively called **firm value** models. As the name implies, this approach is more suited to the study of corporate issuers, where an actual firm value can be identified, e.g. using balance sheet data. For sovereign issuers, the concept of a total asset value is much less clear-cut, though attempts have been made to adapt this approach to sovereign credit risk using national stock indices as proxies for firm values, c.f. Lehrbass (2000).

The Merton Model

Within the Merton model, it is assumed that the firm's capital structure consists of:

- Debt with a notional amount K, in the form of zero coupon bonds with maturity T and total value today (time t) equal to $B^d(t,T)$.
- Equity with total value today (time t) equal to $S(t)$.

At each time before the bonds mature ($t \leq T$), we denote the total market value of the firm's assets by $V(t)$. We shall refer to $V(t)$ as the firm or the **asset value** interchangeably.

Firms have limited liability. Therefore, by the fundamental balance sheet equation, the firm's total assets must equal the sum of its equity and its liabilities. This means that the stock price and the bond price are linked to the firm value via the equation

$$V(t) = S(t) + B^d(t,T) \tag{1}$$

The fundamental assumption of the Merton model is that default of the bond can only take place at its maturity, since this is the only date on which a repayment is due. The payoff at maturity is therefore

$$B^d(T,T) = \min(V(T), K) \tag{2}$$

If the firm value is greater than the redemption value of the debt, then the firm is solvent and the debt is worth par. If the firm value is less than the redemption value, the firm is insolvent and bondholders have first claim on its assets. This means that the shareholders are residual claimants with a payoff at maturity of

$$S(T) = \max(V(T) - K, 0) \tag{3}$$

The bond and equity payoffs are shown in Figure 1.

Valuation

In effect, the shareholders are long a European call option on the firm value. Subject to some assumptions[2], this can be priced just as in the Black/Scholes model.

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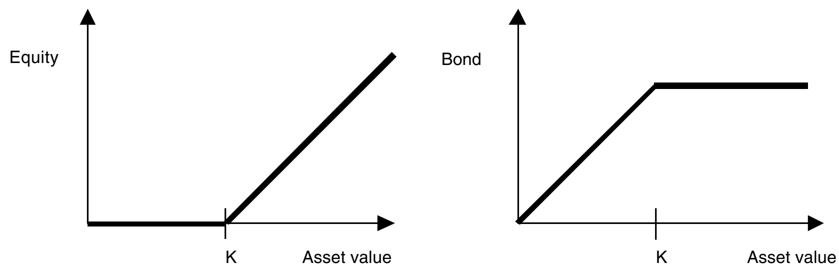
[2] The main assumptions are that the firm value evolves according to a stochastic lognormal process with constant volatility, and that interest rates are constant.

Figure 1. **Value of debt and equity at maturity as a function of asset value**



Using equation (1), we can imply out the value of the corporate bond. If $P(t)$ denotes the price of a put option on the firm value with a strike of K, and $B(t,T)$ is the price of a non-defaultable zero coupon bond with notional K and maturity T, basic put-call parity implies that

$$B^d(t,T) = V(t) - S(t) = B(t,T) - P(t) \tag{4}$$

The bondholders have sold the shareholders an option to put the firm back to them for the bond's notional value at maturity T. It is this additional risk which makes them demand a yield spread over the default-free zero coupon bond.

The market price $B^d(0,T)$ of the risky debt is calculated from the Black/Scholes option pricing formula. We introduce the quotient

$$d = \frac{K\exp(-rT)}{V(0)} \tag{5}$$

where r is the risk-free interest rate. This is the debt-to-assets ratio when the nominal value of the debt is discounted at the market's risk-free interest rate. It is one way of measuring the leverage of the firm. Clearly, a higher value of d leads to a greater degree of risk for the firm. Also, we define

$$h_1 := -\frac{1}{\sigma_F\sqrt{T}}\left(\frac{1}{2}\sigma_F^2 T + \ln d\right) \text{ and } h_2 := h_1 + \frac{2\ln d}{\sigma_F\sqrt{T}} \tag{6}$$

where σ_F is the volatility of the firm value. Then, the market value of risky debt in the Merton model is given by

$$B^d(0,T) = K\exp(-rT)\left(N(h_1) + \frac{1}{d}N(h_2)\right) \tag{7}$$

where N denotes the cumulative distribution function of the standard normal distribution. The definition of the T-maturity credit spread s implies that

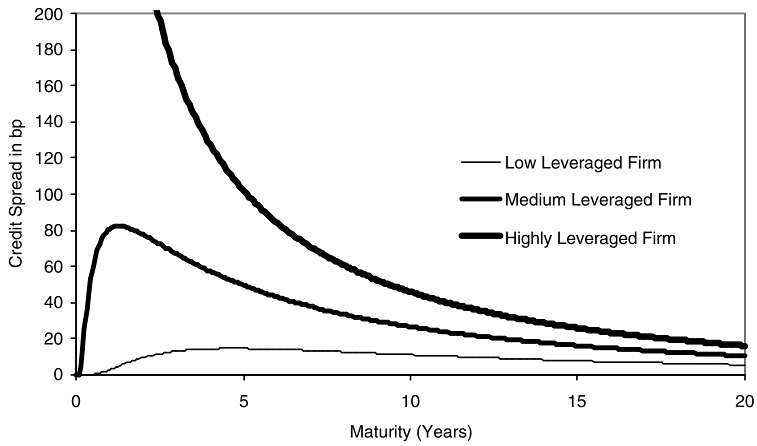
$$s = -\frac{1}{T}\ln\left(\frac{B^d(0,T)}{K}\right) - r \tag{8}$$

The spread s can be computed using equations (7) and (8) to give the curves shown in Figure 2.

Results

Figure 2 shows the three types of spread curve produced by the Merton model. They were calculated using a risk-free rate of 5%, a debt face value of 100, and

Figure 2 Term structure of credit spreads in the Merton model for three firms with different degrees of leverage



asset values of 140, 115 and 98 respectively. For a highly leveraged firm, where the face value of outstanding debt is greater than the current firm value, the credit spread is decreasing with maturity and actually explodes as the maturity of the bond goes to zero. Clearly, if the bond were to mature in the next instant, the firm would already be in default. In a sense, this behaviour results from the fact that the condition for default is imposed only at the maturity of the bond.

The hump-shaped curve for the firm with medium leverage is typical of the Merton model and can be interpreted as reflecting the fact that the credit quality of such a firm is more likely to deteriorate in the short term. However, should it survive, the credit quality is likely to increase.

Last of all, firms with low leverage, where the assets of the firm can easily cover the debt, are very unlikely to default, and can really only become more likely to do so over time. This results in a small but gradual increase in the credit spread until it is almost flat and the asymptotic behaviour of the spread becomes dominant.

For reasonable parameters[3] the credit spread tends to zero as the maturity of the bond goes to infinity. The present value of the outstanding notional of the bond falls in relation to the risk-free growth of the firm's assets, so that the default risk becomes negligible.

Calibration

The decisive pricing inputs of the Merton model are the volatility s_F of the firm value and the degree d of the firm's leverage. Though the book value of assets and the notional value of outstanding debt can be deduced from a firm's balance sheet, this information is updated on a relatively infrequent basis when compared to financial markets data. For pricing purposes we need the total market value of all the firm's assets. This cannot be observed directly, but must be estimated. Consequently, there is no time series readily available for it, and its volatility must also be estimated.

For a publicly traded company, we can use the model to imply out the firm value and its volatility from the notional of the outstanding debt and stock market data.

[3] *in particular in the case where* $s_F < \sqrt{2r}$

Recall that the stock is a call option on the firm value. As such, its price is given by the equivalent of the Black/Scholes formula. In the notation of equation (6) this is

$$S = VN(h_2) \quad K \exp(\quad rT) N(h_1) \tag{9}$$

Also, the stock's delta with respect to the firm value is given by $\quad = N(h_2)$. A simple calculation then shows that the volatility s_s of the stock price is given by

$$s_S = \frac{s_F V}{S} \tag{10}$$

Taking the outstanding notional K, as well as the stock price S and its volatility s_S as given, we simultaneously solve equations (9) and (10) for the firm value V and its volatility s_F. This has to be done numerically. We can then use these parameters as inputs for the valuation of debt.

Example 1: Assume that a firm has zero coupon debt outstanding with a face value of $100MM and a remaining maturity of 3 years. The riskless rate is 5%. Total stock market capitalization is $36MM, with a stock price volatility of 53%.

Using this information we can now determine the firm asset value and the asset volatility by performing a two-dimensional root search to solve equations (9) and (10) for V and s_F. We obtain a total market value V for the firm of $119.8MM with an asset volatility s_F of 17.95%. This implies a debt-to-assets ratio *d* of 71.85%, where *d* is computed as in equation (5). The market value of the debt is given by the difference between the firm value and the stock price, and is equal to $83.8MM. The spread *s* can be calculated from equation (8). It turns out to be 91bp.

In our previous classification of firms into those with low, medium and high leverage, the firm in this example qualifies as one with medium leverage. Its total market value is higher than the face value of the outstanding debt, but its credit spread is quite significant. The spread curve is similar to the middle one in Figure 2, with the maximum credit spread of 101bp being attained for a maturity of about 1.5 years.

For private companies, the whole estimation procedure is more involved, as there is no publicly available equity data.

Extensions of the Merton Model

The Merton model is the benchmark for all structural models of default risk. However, some of its assumptions pose severe limitations. The capital structure of the firm is very simplistic as it is assumed to have issued only zero coupon bonds with a single maturity. Geske (1977) and Geske/Johnson (1984) analyse coupon bonds and different capital structures in the Merton framework. The analytical valuation of bonds is still possible using methods for compound options.

Also, the evolution of the risk-free term structure is deterministic. Several authors have extended the model by combining the mechanism for default with various popular interest rate models. Among these are Shimko/Tejima/van Deventer (1993), who use the Vasicek (1977) specification for the (default) risk-free short rate. Using the techniques well-known from Gaussian interest rate models, credit spreads can be computed. The results are compatible with those in the deterministic case, and credit spreads are generally an increasing function of the short rate volatility and its correlation with the firm value.

The Passage Time Mechanism

It is clearly unrealistic to assume that the default of an issuer only becomes apparent at the maturity of the bond, as there are usually indenture provisions and safety covenants protecting the bondholders during the life of the bond. As an alternative, the time of default can be modelled as the first time the firm value crosses a certain boundary. This specifies the time of default as a random variable given by

$$t = \min\left\{ t \quad 0 \,|\, V(t) = K(t) \right\} \tag{11}$$

where $K(t)$ denotes some time-dependent, and possibly stochastic boundary. This passage time mechanism for generating defaults was first introduced by Black/Cox (1976), and has been extended to stochastic interest rates by, among others, Longstaff/Schwartz (1995) and Briys/de Varenne (1997).

The main mathematical difficulty in a passage time model is the computation of the distribution of default times, which is needed for risk neutral pricing. If the dynamics of the firm asset value are given by a (continuous) diffusion process, this is a reasonably tractable problem. However, if the paths of the firm asset value are continuous, this has an important practical consequence for the behaviour of credit spreads. If the firm value is strictly above the default barrier, then a diffusion process cannot reach it in the next instant - default cannot occur suddenly. Therefore, in a diffusion model, short-term credit spreads must tend towards zero; this is at odds with empirical evidence. One remedy is to allow jumps in the firm value, c.f. Schonbucher (1996) or Zhou (1997). The analytic computation of the passage time distribution, however, becomes much more complicated, and often recourse to simulation is the only option.

Allowing jumps in the firm value also introduces additional volatility parameters. The total volatility of the firm value process is determined by that of the diffusion component, as well as by the frequency and size of jumps. Qualitatively, it can be said that early defaults are caused by jumps in the firm value, whereas defaults occurring later are due primarily to the diffusion component. The additional variables give more freedom in calibrating to a term structure of credit spreads, but also pose the problem of parameter identification.

Empirical Testing of the Merton Model

There have been a number of empirical tests of the Merton model, which have attempted to analyse both the shape and the level of credit spreads. Compared to equity markets, data issues are much more challenging because of the relative lack of liquidity for many corporate bonds. Most empirical studies have been carried out with relatively small bond samples. At present, the empirical evidence is not wholly conclusive, in particular on the shape of credit spreads for medium and low quality issuers. The main studies are summarised as follows:

- Jones, Mason and Rosenfeld (1984) have studied monthly prices for the publicly traded debt of a set of 27 companies between 1975 and 1981, and found that, on the whole, the Merton model does not explain spreads very well, tending to overprice bonds.
- Based on monthly price quotes for corporate zero coupon bonds, Sarig and Warga (1989) have found that credit spreads in the market resemble the shapes produced by the model. However, this evidence is qualified by the small sample

used. Also, zero coupon bonds arguably do not constitute a representative sample of the corporate bond market.

- Helwege and Turner (1999) provide evidence for the fact that the hump-shaped and downward sloping credit spreads observed in empirical studies may just be a consequence of pooling issuers in the same rating class when constructing credit curves. They argue that issuers in the same rating class are not identical with respect to credit risk and that better-quality names tend to issue debt with longer maturities than the lower-quality ones in the same class. When considering individual issuers, they find that high yield debt exhibits an upward sloping term structure of spreads in the same manner as investment grade debt.

An industrial application based on the firm value approach is given by the Expected Default Frequencies (EDF) provided by the KMV corporation. KMV compute default probabilities for individual issuers. A default boundary is inferred from balance sheet data about the firm's liabilities. An approach based on the Merton model is used to infer the firm value and its volatility from equity prices; i.e. to "delever" the equity price movements. These data give a measure of what is called the "distance to default", which is then mapped to actual default frequencies via a very large proprietary database of corporate defaults. It has been argued by KMV that their model is a better predictor of default than credit ratings, cf. Crosbie (1998). Using EDF's to characterize credit quality, Bohn (1999) does find evidence that the term structure of credit spreads is hump-shaped or downward sloping for high yield bonds.

Practical Applications of Firm Value Models

The fact that firm value models focus on fundamentals makes them useful to analysts adopting a bottom-up approach. For example, corporate financiers may find them useful in the design of the optimal capital structure of a firm. Investors, on the other hand, can use them to assess the impact of proposed changes of the capital structure on credit quality. One caveat to this is that it is extremely difficult to apply the firm value model in special situations such as takeovers or leveraged buyouts, where debt might become riskier while equity valuations increase.

The calibration of a firm value model is very data intensive. Moreover, this data is not readily available. It is a non-trivial task to estimate asset values and volatilities from balance sheet data. If one follows the firm value concept to its logical conclusion, then it is necessary to take into account all of the various claims on the assets of a firm - a highly unfeasible task. Furthermore, fitting a term structure of bond prices would require a term structure of asset value volatilities and asset values, which is simply not observable

In terms of analytical tractability, one has to note that firm value models quickly become cumbersome and slow to compute when we move away from the single zero coupon bond debt structure. If instead we introduce a coupon paying bond into the debt structure then its pricing is dependent on whether the firm value is sufficient to repay the coupon interest on the coupon payment dates. Mathematically the form of the equations become equivalent to pricing a compound option. Similarly, if the issuer has two zero coupon bonds outstanding, the price of the longer maturity bond is conditional on whether the company is solvent when the shorter maturity bond matures. This also makes the pricing formulae very complicated. The pricing of credit derivatives with more exotic payoffs is beyond the limits of this model.

Finally, if a diffusion process is used for the firm value, default is predictable in the sense that we can see it coming as the asset price falls. This means that default is never a surprise. In the real world, it sometimes is. For example, the default of emerging market sovereign bonds is not just caused by an inability to pay, which can be modelled within a firm value approach, but also by an unwillingness to pay, which cannot.

Reduced-Form Models

Introduction

In contrast to structural models, reduced-form credit models do not attempt to explain the occurrence of a default event in terms of a more fundamental process such as the firm value or an earnings stream. Instead, the aim is to describe the statistical properties of the default time as accurately as possible, in a way that allows the repricing of fundamental liquid market instruments and the relative valuation of derivatives. This approach was initiated by Jarrow/Turnbull (1995) and has found wide application since then. The methodology used is closer to that of the actuarial sciences and of reliability theory than to the corporate finance methods used in structural models, and the pricing techniques are similar to those used in traditional models of the term structure, as opposed to the more equity-like structural models.

Modelling the Default Process

In a reduced-form model, default is treated as an exogenous event. The central object of the modelling procedure is the default counting process *N*. This is a stochastic process which assumes only integer values. It literally counts default events, with $N(t)$ denoting the number of events that have occurred up to time t. Each such event corresponds to the time of a jump in the value of *N*, Figure 3 shows a typical path of the process.

Usually, we are only interested in the time   of the first default. This can be written as

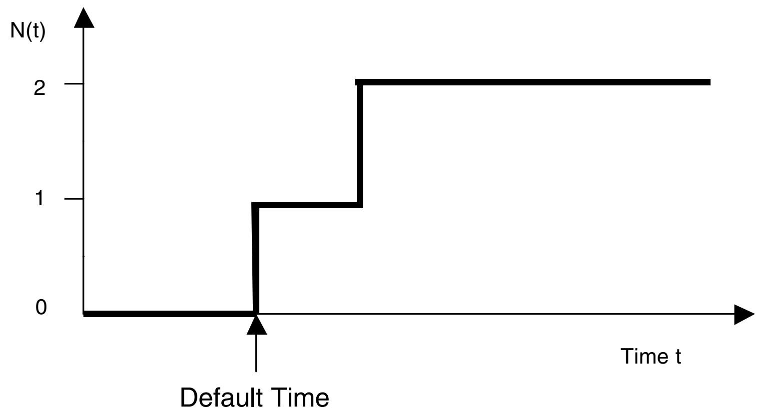
$$t = \min\{ t \quad 0 \mid N(t) \quad 1 \} \qquad \textbf{(12)}$$

Empirical evidence shows that the majority of defaults do not result in liquidation, cf. Franks/Torous (1994). Instead, the defaulting firm is often restructured, with creditors being compensated by new debt issues; the economic life of a security does not necessarily end with the first default event. On the other hand, defaults are relatively rare, and as most credit derivatives condition on the occurrence of any default, it is usually justified to focus just on the time   of the first default by any specific issuer.

The simplest example of a default counting process is that of a Poisson process. The stochastic behaviour of the process is determined by its **hazard rate** $l(t)$. It can be interpreted as a conditional instantaneous probability of default:

$$P[t \quad t + dt \mid t > t] = l(t)dt \qquad \textbf{(13)}$$

Figure 3. **Typical path of a default counting process**



Equation (13) states that, conditional on having survived to time t, the probability of defaulting in the next infinitesimal instant is proportional to $l(t)$ and the length of the infinitesimal time interval dt. The function l describes the rate at which default events occur, which is why it is called the hazard rate of N. Equation (13) can be integrated to give the survival probability for a finite time interval as

$$P[t > t] = \exp({}_0^t l(s)ds) \quad \textbf{(14)}$$

Note that the purpose of the model is the arbitrage-free valuation of default-linked payoffs. The probability measure P is therefore a **risk-neutral measure**, meaning that the survival probability under P is not directly related to historical default frequencies but where the default risk can be hedged in the market. Also, the intensity function $l(t)$ governs the behaviour of N under P, and must therefore incorporate the risk premium demanded by the market.

Risk-Neutral Pricing

In this framework, one uses the risk-neutral approach to compute contingent claims prices. Suppose that $X(T)$ is a random payoff made at time T if no default event occurs until then. The initial price $C^d(0)$ of this claim is given by the risk neutral expectation of the discounted payoff

$$C^d(0) = E \frac{X(T)}{b(T)} 1_{\{t > T\}} \quad \textbf{(15)}$$

where $$b(T) = \exp {}_0^T r(u)\,du \div$$

is the value of the money market account. If the default time t is independent of the random payoff $X(T)$ and the non-defaultable term structure, we can separate the two terms in the expectation in equation (15) to obtain

$$C^d(0) = E \frac{X(T)}{b(T)} P[t > T] \quad \textbf{(16)}$$

The remaining expectation is the price $C(0)$ of a claim to the payoff $X(T)$, which has no default risk; it is multiplied by the survival probability to time T. We have:

$$C^d(0) = C(0) P[t > T] \quad \textbf{(17)}$$

This shows that, under the independence assumption, the price of a defaultable claim is obtained by multiplying the price of the equivalent non-defaultable claim by the probability of survival until time T. Analogously, if $\overline{C}^d(0)$ is the price of a claim which pays $X(T)$ at time T if a default event **has** occurred before that time, we have

$$\overline{C}^d(0) = C(0) P[t \quad T]. \quad \textbf{(18)}$$

Another type of claim that is often encountered makes a random payment of $X(t)$ at the time of default, if this should occur before some time horizon T. Its initial price $\overline{D}(0)$ can be written as

$$\overline{D}(0) = E \frac{X(t)}{b(t)} 1_{\{t \quad T\}} \quad \textbf{(19)}$$

Example 2: Suppose that the hazard rate has a constant value λ. We consider a zero coupon bond with maturity T under the zero recovery assumption, i.e. the bond pays $1 if no default occurs until T and nothing otherwise. The survival probability is

$$P[\tau > T] = \exp(-\lambda T)$$

If y is the continuously compounded yield of the corresponding non-defaultable bond, we obtain from equation (17):

$$B^d(0,T) = \exp(-(y+\lambda)T)$$

This shows that in the zero recovery scenario, the yield spread of the defaultable bond is exactly equal to the hazard rate of the default process.

To compute this expectation, we need the probability density of the default time. Using the definition of conditional probabilities, equation (13) tells us that the probability of defaulting in the time interval from t to $t + dt$ is given by

$$P[t < \tau \le t+dt] = \lambda(t)P[\tau > t]dt = \lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)dt \qquad \textbf{(20)}$$

i.e. the probability of surviving to time t multiplied by the probability of defaulting in the next time interval dt. We obtain $\overline{D}(0)$ by integrating over the density of τ, so that

$$\overline{D}(0) = \int_0^T E\left[\frac{X(t)}{\beta(t)}\right]\lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)dt \qquad \textbf{(21)}$$

An important special case is the one where X is constant and equal to one. We denote the price of this claim by $D(0)$. It is an important building block for pricing bonds which recover a fraction of their par amount at the time of default. If $B(0,t)$ denotes the price of a non-defaultable zero coupon bond with maturity t, we see from equation (21) that

$$D(0) = \int_0^T B(0,t)\lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)dt \qquad \textbf{(22)}$$

We see that $D(0)$ is just a weighted average of non-defaultable zero coupon bond prices, where the weights are given by the density of τ.

Example 3: We price a defaultable zero coupon bond which pays a fraction $R = 30\%$ of its notional at the time of default. We assume a maturity *T* of 3 years, riskless interest rates are constant at $r = 4\%$, and the hazard rate is $l = 1\%$. If this were the hazard rate under the objective or real-world probability measure, it would translate into a default probability of 2.96% over the three year time horizon, which corresponds roughly to that of an issuer with a rating of *BBB*. However, note that we are considering the hazard rate under the risk-neutral measure, so that it also incorporates a risk premium.

The price of the defaultable bond is made up of the recovery payment and the payment at maturity, which is contingent on survival:

$$B^d(0,T) = RD(0) + B(0,T)P[t > T]$$

Because the hazard rate and interest rates are constant, we can explicitly calculate the integral in equation (22) to obtain

$$D(0) = \frac{l}{l+r}(1 \quad \exp(\quad (r+l)T))$$

Inserting the numbers gives a bond price of $B^d(0,T) = \$86.91$ on a \$100 notional. Using continuous compounding this translates into a credit spread of 68bp. If we compare this spread with actual credit curves, we see that it corresponds roughly to an issuer rating of single *A*. This reflects the risk premium investors demand for holding the bond.

Stochastic Hazard Rates

In this last example, we modelled the hazard rate process as being deterministic i.e. we know today how the hazard rate will evolve over time. For most instruments there is no need to model explicitly the dynamics of the hazard rate. This only becomes necessary when we wish to price instruments with an embedded spread-linked optionality or when we wish to examine the effect of the correlation between the hazard rate and other model factors such as interest rates and FX rates.

A general class of default counting processes with a stochastic hazard rate is given by the so-called **Cox processes**, introduced to financial modelling by Lando (1998). Here, the default counting process *N* is modelled in such a way, that, when conditioning on the complete information provided by the path of the stochastic hazard rate $l(t)$, it behaves just like a Poisson process. Typically, a factor model is used to drive the default-free term structure and the hazard rate . The formulae we have seen before for Poisson processes now hold conditionally. For example, the formula for the survival probability given in equation (14) now takes the form

$$P[t > t | (l(u))_{0 \; u \; t}] = \exp(\quad {}_0^t \, l(s)ds) \qquad \textbf{(23)}$$

By using a method known as iterated conditional expectations, it is possible to price derivatives in the Cox process setup in the same way as is used in ordinary term structure modelling. We consider a survival-contingent random payoff *X* to be made at time *T* if no default has taken place until then. As before, the initial price of this claim is given by equation (15). We first condition on the trajectory of the hazard rate up to time *T*, and then compute the unconditional expectation on the default process to obtain

$$C^d(0) = E\left[\frac{X}{\beta(T)} P\left[\tau > T \mid (\lambda(u))_{0 \le u \le T}\right]\right] = E\left[\frac{X}{\beta(T)} \exp\left(-\int_0^T \lambda(s)ds\right)\right]$$
$$= E\left[X \exp\left(-\int_0^T r(s) + \lambda(s)ds\right)\right] \quad \textbf{(24)}$$

This iteration of conditional expectations allows us to remove all explicit reference to the default event. The payoff is priced just as if it didn't have any default risk in a world where the short rate is adjusted by the hazard rate. If the hazard rate is independent of the payoff and the non-defaultable term structure, then, just as before, the price of the claim is obtained by multiplying the price of the equivalent non-defaultable claim by the probability of survival.

The pricing of a payment made at the time of default gives an analogous formula to the one we have presented in the deterministic case:

$$\overline{D}(0) = \int_0^T E\left[\frac{X(t)}{\beta(t)} \lambda(t) \exp\left(-\int_0^t \lambda(u)du\right)\right] dt \quad \textbf{(25)}$$

Valuing a Default Digital Payoff

Consider an instrument which pays \$1 immediately following default and zero otherwise. The corresponding process X is constant and equal to 1. If we also assume that the hazard rate and non-defaultable interest rates are independent, the formula in equation (25) simplifies to

$$D(0) = \int_0^T B(0,t) E\left[\lambda(t) \exp\left(-\int_0^t \lambda(u)du\right)\right] dt \quad \textbf{(26)}$$

This extends the result we obtained in equation (22). As the derivation of equations (25) and (26) is somewhat more involved than the deterministic case, we discuss it in the appendix.

Simulating Default

For credit contingent securities where there is no analytically tractable model and which do not have any American optionality, Monte-Carlo simulation is usually the favoured pricing methodology. It is both intuitive, flexible and extends easily to multiple factors. There are essentially two ways to simulate default within the Poisson framework we have laid out.

Method 1: Simulating using Bernoulli Trials

Suppose we have a simulation of the hazard rate process so that $\lambda(n,i)$ is the value of the hazard rate in simulation path number i and at time interval n. Assuming that the asset has not yet defaulted along this simulation path. To determine whether a default occurs at this time interval, we:

1. Perform a Bernoulli trial by drawing a random number u from a uniform distribution on the interval [0,1].
2. The condition for default is to test whether $u < \lambda(n,i)\Delta t$. If so we have a default, otherwise the asset survives to the next time step.

For example, consider the case when $\lambda(n,i)\Delta t$ equals 0.027 and we draw a random uniform equal to 0.29. As the random uniform is greater than 0.027 the asset does not default at this time. However, in another path at this time interval the random uniform draw equals 0.013. In this case the asset defaults.

This method is based on the idea that the hazard rate describes the rate at which defaults occur. It is very simple, and can be implemented in all situations. However, as the condition for default must be checked in every interval of the discretization, it is computationally intensive. As default is a rare event, we can be much more efficient by simulating the time of default directly.

If the default process is independent of the interest rate and has a deterministic hazard rate, we can take advantage of the fact that we know the distribution of the survival time explicitly. It is exponential as shown in Equation (14).

Method 2: Simulating the Time To Default

For each simulation trial we draw a random uniform $u\,(i)$. We then equate this to the survival probability function for the asset $P[\quad > t]$ in simulation trial *i* and solve for t such that

$$u(i) = P[t \; > t]$$

Searching for the value of t can be accomplished using a simple search down the monotonically decreasing survival probabilities. In the simple case where the hazard rate is flat, we have

$$P[t \; > t] = \exp(\quad l\, t)$$

so that solving for the realisation *t* of t is trivial:

$$t = \quad \frac{\log(u(i))}{l}$$

For example, suppose equals 0.03 and the random uniform equals 0.281, the time to default is then equal to 42.3 years. A simple check for this simulation is to verify that the average time to default is close to 1/ , to within simulation accuracy.

This method is very efficient in that it avoids the need to step through time in small increments and instead jumps directly to the time of the default event.

If the intensity is stochastic, we can still use a variant of the second method. This relies on equation (23), which states that, after conditioning on the path of the hazard rate, the distribution of the survival time is still exponential.

Method 3: Simulating the Time to Default with a Stochastic Hazard Rate

As before, for each simulation trial we draw a random uniform $u(i)$. We also simulate a discretized version of the path (n,i) of the hazard rate and approximate its integral (n,i), which is monotonically increasing, by the sum

$$(n,i) = \sum_{k=1}^{n} l(k,i) \quad t$$

As the conditional distribution of the default time is exponential, we can simulate it in the discretization by setting it equal to the first *n* such that

$$\exp(\quad (n,i)) = u(i)$$

Rating-Based Models

Up until now we have focused on the modelling of the default counting process. This approach is sufficient provided we are doing arbitrage-free pricing of defaultable securities which have no explicit rating dependency. The variations in the price of the security which occur as a result of market perceived deteriorations or improvements in the credit quality of the issuer, and which can lead to changes in credit ratings, can be captured by making the hazard rate stochastic. An increase in the stochastic hazard rate reflects a credit deterioration scenario and a decrease represents a credit improvement scenario.

However some securities do have an explicit credit rating dependency. Very recently, a number of telecom bonds have been issued with rating dependent step-ups on their coupons. And though they are a small fraction of the overall market, credit derivative contracts do exist which have payoffs linked to a credit rating[4]. Within the world of derivatives, ratings-linked models are also useful for examining counterparty exposure, especially if collateral agreements are ratings-linked. Other potential users of ratings-linked models include certain types of funds which are only permitted to hold assets above a certain ratings threshold. Finally, ratings-linked models have recently been given a new impetus by the fact that the Basle Committee on Banking Supervision has proposed allowing ratings-based methodologies for computing Bank Regulatory Capital.

Moreover, given the wealth of information available from rating agencies, a natural development has been to enrich the binary structure given by the default indicator to one incorporating transitions between different rating classes. The generally used approach is to model these transitions by a Markov chain; it was initiated by Jarrow/Lando/Turnbull (1997) and is described in the following section.

Description

Suppose that a rating system consists of rating classes 1,...,K, where K denotes default. The quantity to be modelled is the ratings transition matrix $Q(t,T) = \left(q_{i,j}(t,T)\right)_{i,j=1,\ldots,K}$, where the entry $q_{i,j}(t,T)$ denotes the probability that an issuer in rating class i at time t will be in rating class j at T. The default state is assumed to be absorbing, which means that an issuer never leaves the default state once it has been entered. Economically, this implies that there is no reorganisation after a default.

In this setup, the ratings transition process is Markovian, i.e. the current state of the credit rating is assumed to contain all relevant information for future rating changes. This implies that the transition probabilities satisfy the so-called Chapman-Kolmogorov equations. For $t \le T_1 \le T_2$, we have

$$Q(t,T_2) = Q(t,T_1)\, Q(T_1,T_2) \quad \textbf{(27)}$$

In other words, the transition matrix between time t and T_2 is the matrix product of the two transition matrices from time t to T_1 and T_1 to T_2. If Q is the one-year transition matrix, and the Markov chain is time-homogeneous, the transition probabilities for an n year period are given by $Q(0,n) = Q^n$. In the simplest continuous-time case, the transition matrix is constructed from a time-independent generator $\Lambda = \left(\lambda(i,j)\right)_{i,j=1,\ldots K}$ via the matrix exponential

$$Q(t,T) = \exp\left(\Lambda(T-t)\right) = \sum_{n=0}^{\infty} \frac{(T-t)^n \Lambda^n}{n!} \quad \textbf{(28)}$$

---

[4] *According to the British Bankers' Association 1998 survey of the credit derivatives market, derivatives conditioning on downgrades are not widely used, and have been phased out of the ISDA master documentation.*

For small time intervals, we can consider equation (28) up to first order to obtain

$$Q(t,t+dt) \quad I + \quad dt$$

and

$$q_{i,j}(t,t+dt) = \begin{cases} l(i,j)\,dt, & i \quad j \\ 1+l(i,i)\,dt, & i = j \end{cases} \qquad \textbf{(29)}$$

Equation (29) gives the natural interpretation of the generator matrix . For $i \quad j$, $l(i,j)$ is the transition rate between the rating classes i and j. Furthermore, $l(i,i)$ is negative, and can be interpreted as the exit rate from the ith class. As such, the generator   is the natural generalisation to a rating class framework of the hazard rate introduced in equation (13).

Table 1. Average one-year transition matrix for 1980-1999 by Moody's Investor Service, probabilities are conditional on rating not being withdrawn

	Aaa	Aa	A	Baa	Ba	B	Caa-C	Default
Aaa	89.31%	10.15%	0.50%	0.00%	0.03%	0.00%	0.00%	0.00%
Aa	0.96%	88.42%	10.04%	0.38%	0.16%	0.02%	0.00%	0.04%
A	0.08%	2.34%	90.17%	6.37%	0.81%	0.22%	0.00%	0.02%
Baa	0.09%	0.39%	6.42%	84.48%	6.92%	1.39%	0.12%	0.20%
Ba	0.03%	0.09%	0.50%	4.41%	84.25%	8.65%	0.52%	1.54%
B	0.01%	0.04%	0.17%	0.58%	6.37%	82.67%	2.98%	7.17%
Caa-C	0.00%	0.00%	0.00%	1.10%	3.06%	5.89%	62.17%	27.77%
Default	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	100.00%

Example 4: A typical example for a transition matrix is the average one-year matrix provided by Moody's for the period between 1980 and 1999 shown in Table 1.

Using the equation $Q(0,5) = Q^5$ we can matrix multiply the transition matrix to obtain the 5-year transition probabilities. By examining the likelihood of ending in the default state, we can compute the following 5-year default probabilities conditional on starting in the corresponding rating category. These are shown in Table 2.

Table 2. 5-year default probabilities for an issuer starting in a rating category using Moody's one-year transition matrix

Aaa	Aa	A	Baa	Ba	B	Caa-C
0.05%	0.28%	0.62%	2.97%	11.58%	31.23%	69.77%

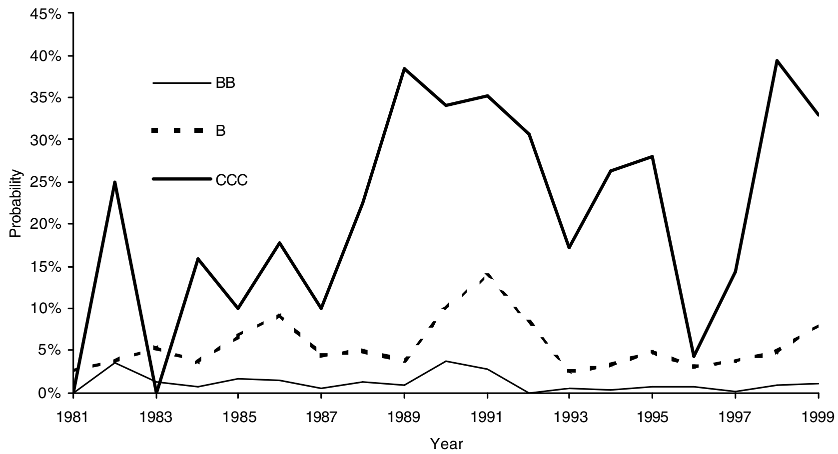
We stress the fact that only risk-neutral probabilities are relevant for pricing. Therefore, the risk premium must be modelled in some way, in order to relate model transition probabilities to historical transition probabilities such as those given above. Typically, one uses a tractable parametric form for a time-dependent generator and attempts to calibrate to the market prices of bonds.

Discussion

The Markov chain approach to the description of ratings transitions is elegant and tractable, but oversimplifies the actual dynamics, especially in the time-homogeneous formulation. Also, the relatively small amount of data available makes itself felt.

A closer look at empirical transition matrices shows this clearly. Standard and Poor's provide one-year transition matrices for all the years from 1981 to 1999,

Figure 4. **One-year default probability of ratings BB, B and CCC, conditional on rating not being withdrawn, based on Standard and Poor's rating transitions 1981-1999**



c.f. their annual study of long-term defaults and ratings transitions in Standard and Poor's (2000). In theory, if the dynamics of historical ratings changes were described by a homogeneous Markov chain, then all matrices should be the same up to sampling errors. Actually, the one-year default probabilities show huge variations, especially in the sub-investment grade sector, cf. Figure 4.

The small number of actual defaults means that many entries in the empirical transition matrix are zero, cf. Table 1. This leads to a slight problem when trying to extract a generator of the Markov chain via equation (28). It can be proven mathematically that an exact solution of equation (28), which satisfies the parameter constraints for a generator, does not exist for the matrix in Table 1. Therefore one must be satisfied with an approximate solution. For more on this topic see Israel/Rosenthal/Wei (2000).

On the other hand, the advantage of using a rating-based model for calibration is that it allows the construction of credit curves for issuers of different quality in a unified and consistent manner. This is particularly useful when there are only very few instruments available for calibration in each rating class. However, the calibration procedure is much more involved than with the standard hazard rate approach, due to the larger number of parameters and the internal consistency constraints imposed by the Markov chain framework.

It is well known that market spreads tend to anticipate ratings changes. However, stochastic fluctuations in the credit spread in between ratings changes cannot be modelled with a deterministic generator, it has to be made dependent on additional state variables. In particular, this is crucial for the pricing of payoffs contingent on credit spread volatility. For extensions along these lines, see Lando (1998) and the article by Arvanitis/Gregory/Laurent (1999).

In general, the case for implementing a rating-based model instead of a hazard rate model with a single intensity process is not completely clear due to the trade-offs mentioned above. Put simply, ratings-linked models are essential for evaluating ratings-linked contingencies in the real-world measure. It is therefore an approach which works best for risk managers and investors that have an explicit exposure to downgrade risk as opposed to spread risk. However, for pricing bonds and credit derivatives with no explicit ratings dependency, it is more natural to model the spread or hazard rate, especially if we are pricing within the risk-neutral measure.

Recovery Assumptions

Within the structural approach, the amount recovered by a bondholder in the event of a default emerges naturally from the model - it is simply the value of the assets of the firm at the bond's maturity. However, within the reduced-form approach, the recovery process must be modelled explicitly. Therefore to completely determine the price process of a security subject to default risk, the payoff in the event of default must be specified in addition to the mechanism describing the occurrence of default events. Currently, there are several conventions in widespread use, which we now survey.

Equivalent Recovery

Historically, the first assumption made is that of **equivalent recovery**, introduced by Jarrow/Turnbull (1995). Under this assumption, each defaulting security is replaced by a number $0 \le R \le 1$ of non-defaultable, but otherwise equivalent securities. Consider a defaultable zero coupon bond. For simplicity we assume independence between interest rates and the hazard rate. If the recovery rate is zero, its price $B_0^d(0,T)$ is given by

$$B_0^d(0,T) = B(0,T)P[\tau > T] \quad \textbf{(30)}$$

Note that we have added a subscript of zero to the bond price to emphasize that this is the price obtained under the zero recovery assumption. For a general recovery rate *R*, a simple static replication argument shows that the defaultable bond price $B^d(0,T)$ must be

$$B^d(0,T) = RB(0,T) + (1 - R)B_0^d(0,T) \quad \textbf{(31)}$$

One advantage of equivalent recovery is that it allows us to calculate implied survival probabilities from bond prices using equations (30) and (31) for a given recovery rate *R*. On the other hand, a fixed recovery rate implies an upper bound on the credit spread. If $y(0,T)$ and $y^d(0,T)$ denote the continuously compounded yield of the default-free and the defaultable zero coupon bond, respectively, then equation (31) implies that

$$\exp\left(-y^d(0,T)T\right) = B^d(0,T) \ge R\,B(0,T) = R\exp\left(-y(0,T)T\right) \quad \textbf{(32)}$$

so that

$$y^d(0,T) - y(0,T) \le \frac{1}{T}\ln\left(\frac{1}{R}\right) \quad \textbf{(33)}$$

This means that assuming an equivalent recovery imposes an upper bound on the credit spread based on the recovery rate. For example, if *R* is 50%, then for a maturity of 10 years, the maximum credit spread is 693bp. This constraint can become restrictive when one is modelling the senior bonds of a high-yield issuer, but is normally not a problem unless the bonds have a very long maturity.

Fractional Recovery

The **fractional recovery** assumption was introduced by Duffie/Singleton (1999) and extended to multiple defaults by Schonbucher (1998). The idea is that, at each default event, the bond loses a fraction *q* of its face value, and continues to trade after a restructuring. The strength of this approach is that it allows default-risky claims to be valued as if they were default-free and discounted by an adjusted interest rate. If r denotes the default-free short rate, it can be shown that the price of a defaultable zero coupon bond is given by

$$B^d(0,T) = E\left(\exp\left(-\int_0^T r(s) + q\,l(s)\,ds\right)\right) \quad \textbf{(34)}$$

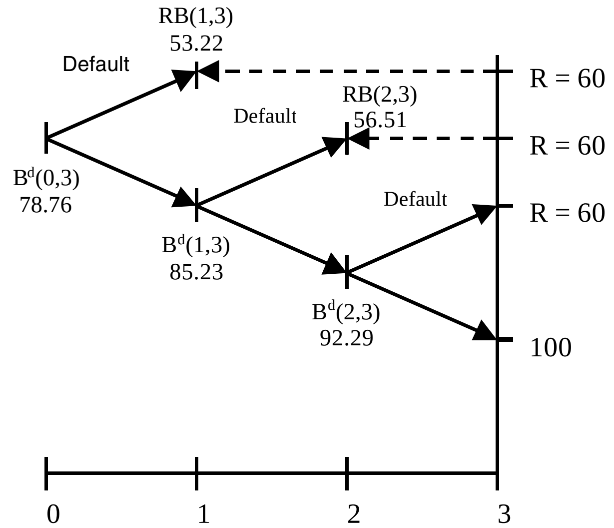
where the expectation is taken under the risk-neutral measure. This simplifies the modelling process for defaultable bonds, as only the loss rate $q\,l$ needs to be specified. In particular, the recovery rate does not impose any bounds on the credit spreads. However, knowledge of the default probabilities is necessary for the pricing of credit derivatives, e.g. digital default swaps. These cannot be directly inferred from defaultable bond prices under the fractional recovery assumption without specifying the stochastic dynamics of q and l.

Recovery of Face Value

Both the equivalent and the fractional recovery assumptions do not correspond to market conventions for bonds. When a real-world bond defaults, the bondholders recover a fraction R of the bond's principal value (and perhaps of the accrued interest since the last coupon date, but we will ignore this for expositional simplicity). The outstanding coupon payments are lost. In the literature, this convention is sometimes called **recovery of face value**. Similar to the case of equivalent recovery, the recovery rate does impose bounds on credit spreads under recovery of face value. However, the effects are more complex than in the equivalent recovery case and are best analysed on a case-by-case basis. In general, these constraints only become binding for long maturities or in extreme cases.

As an example for bond pricing under recovery of face value, consider a bond with a face value of one, paying a coupon of c at times $T_1 < ... < T_N$. Again, we denote by $D(0)$ the initial price of a payment of \$1 made at the time of default, if this should

Figure 5. **Equivalent recovery according to Jarrow/Turnbull. At default the bondholder receives an equivalent default-free bond with a face value of R.**



occur before T_N. The value of the risky redemption is then given by $B_0^d(0,T_N) + RD(0)$. The coupon payments are effectively a portfolio of risky zero coupon bonds with a zero recovery rate. The total price $P(0)$ of the bond is given by

$$P(0) = c\sum_{i=1}^{N} B_0^d(0,T_i) + B_0^d(0,T_N) + R\,D(0) \qquad \textbf{(35)}$$

Example 5: We price a defaultable zero coupon bond in a simple discrete time setting, highlighting the differences between the various recovery assumptions. In each case, we consider a 3-period model. The non-defaultable term structure is flat with a continuously compounded interest rate per period of 6%. The default probability per period is 5%, and we assume a recovery rate of 60%. Given that the bond has survived to the current period, it either defaults with a probability of 5%, or survives with a probability of 95% until the next period. The payoff in the default nodes is specified by the recovery assumption. In the survival nodes, the price of the bond is the discounted expectation of the next period's payoffs using these probabilities. At each node, we give the bond price for a face value of 100.

Figure 5 computes the bond price under the equivalent recovery assumption according to Jarrow/Turnbull. At each default node, the bond price is equal to the recovery rate multiplied by the price of a non-defaultable zero coupon bond that matures at the end of the third period. The payoff of the recovery rate at maturity is discounted back to each default node via the non-defaultable interest rate.

In Figure 6, we compute the defaultable bond price using the assumption of fractional recovery of face value according to Duffie/Singleton. At each default node, the bond price is equal to the recovery rate multiplied by the price of the defaultable bond at the survival node of the same period.

Finally, in Figure 7 the bond price is computed while assuming recovery of face value. In this case the bond's payoff is simply the recovery rate at each default node.

Table 3. **Bond yield under different recovery assumptions**

Recovery Assumption	Yield
recovery of face value	7.76%
equivalent recovery	7.96%
fractional recovery	8.02%

Each recovery assumption leads to a different yield for the bond. The bond's yield is lowest if we assume recovery of face value. This is because the redemption amount is paid out immediately upon default. Under equivalent recovery, the redemption payment is only made at maturity of the bond. Finally, the yield of the bond is highest if we assume fractional recovery. This is because the payment at default is a fraction only of the defaultable bond price at the corresponding survival node.

Figure 6. **Fractional recovery of market value according to Duffie/Singleton. At default the value of the defaulted security is a fraction R of its price at the corresponding survival node.**

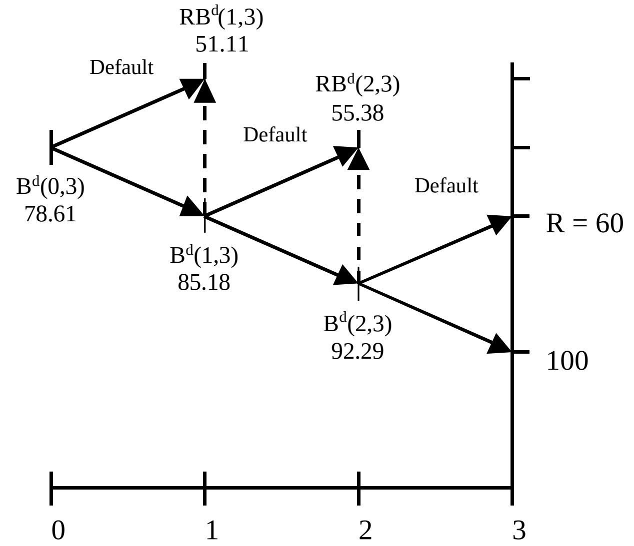
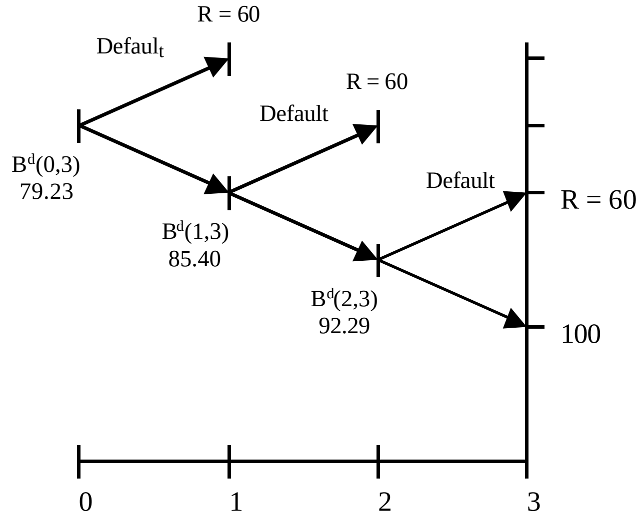


Figure 7. **Recovery of face value. At default the bondholder receives a fraction R of the bond's face value.**



Empirical Studies of Recovery Rates

An empirical analysis of the behaviour of recovery rates is hampered by the fact that defaults are relatively rare. The market standard source for recovery rates is Moody's historical default rate study, the results of which are plotted below in Figure 8.

The graph shows how the average recovery rate depends on the level of subordination. By plotting the 1st and 3rd quartiles, we have shown that there is a wide variation in the recovery rate. Note that these recovery rates are not the actual amounts received by the bondholders following the workout process, but represent the price of the defaulted asset as a fraction of par some 30 days after the default event. This is in line with the assumption of recovery of a fixed amount of the face value following default

.

Credit Curve Construction

The construction of a credit curve from the prices of liquid instruments is the prerequisite for the pricing of derivatives. One way of describing the credit curve is via the term structure of discount factors for a risky issuer under the zero recovery assumption. Inputs for the credit curve are typically bonds, whether fixed coupon non-callable bonds or floating rate notes, asset swap spreads, or default swap spreads.

In this section, we illustrate how to imply a deterministic intensity from a term structure of default swap spreads. For a complete discussion of mechanics of default swaps see the Lehman publication O'Kane (2000). Suppose that the payment schedule is given by $T(1) < ... < T(N)$. For a default swap with maturity date $T(n)$, the protection buyer pays a spread of $s(n)$ at each date, so long as default has not occurred. This is known as the **premium leg**. If a default event occurs during the lifetime of the swap, the protection seller pays 1 - R. For simplicity, we assume that this payment is made on the first schedule date immediately after the default. This is the **protection leg**. Also for simplicity, we have assumed that the

Figure 8. **Moody's historical recovery rate distributions, 1970-1999 for different levels of subordination. Each bar starts at the 1st quartile then changes colour at the average and ends at the 3rd quartile.**

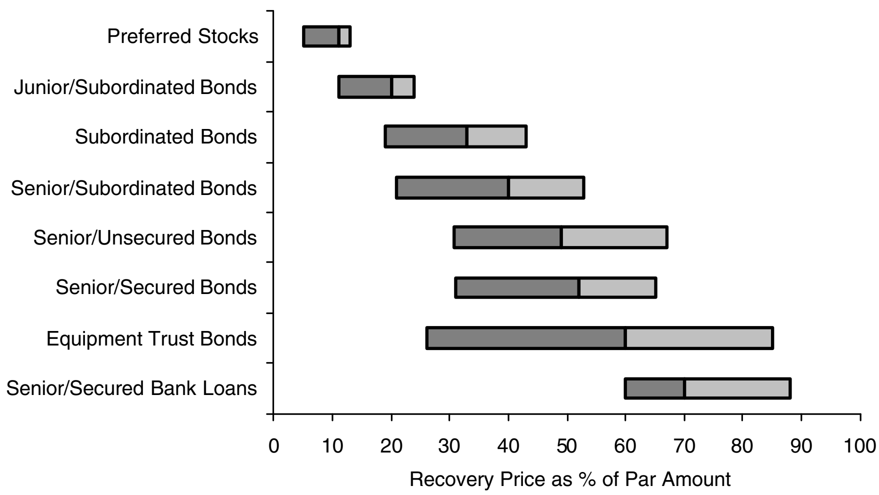
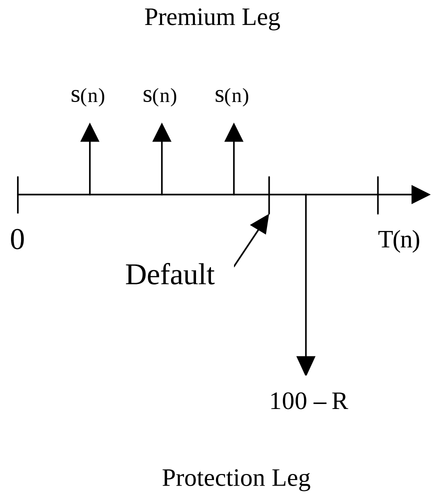


Figure 9. **Cashflows in a default swap**



protection buyer does not have to pay any premium accrued between the last payment date and the default event.

Valuing the Legs of the Default Swap

The value $prem(n)$ of the n-maturity premium leg is given by

$$prem(n) = s(n)\,q(n) \qquad \textbf{(36)}$$

where

$$q(n) = d \sum_{j=1}^{n} B(0,T(j))\,P[t > T(j)] \qquad \textbf{(37)}$$

is the issuer PV01 of the default swap and d denotes the year fraction between premium payments. The value $prot(n)$ of the n-maturity protection leg is

$$\begin{aligned} prot(n) &= (1-R)\sum_{i=1}^{n} B(0,T(i))\,P[T(i-1) < t \le T(i)] \\ &= prot(n-1) + (1-R)\,B(0,T(n))\,\big(P[t > T(n-1)] - P[t > T(n)]\big) \end{aligned} \qquad \textbf{(38)}$$

Each default swap spread $s(n)$ is determined by the **breakeven condition** $prot(n) = prem(n)$. Our goal is to recursively calculate the survival probability $P[t > T(n)]$. Applying some simple algebra to equation (38) shows that

$$\begin{aligned} &(d\,s(n) + 1 - R)\,B(0,T(n))\,P[t > T(n)] \\ &= (s(n-1) - s(n))\,q(n-1) + (1-R)\,B(0,T(n))\,P[t > T(n-1)] \end{aligned} \qquad \textbf{(39)}$$

This equation makes it easy to solve for $P[t > T(n)]$. If we assume that the hazard rate has a constant value of $\lambda(n)$ on $[T(n-1), T(n)]$, it can be calculated from

$$P[t > T(n)] = P[t > T(n-1)]\exp\big(-l(n)\,(T(n) - T(n-1))\big) \qquad \textbf{(40)}$$

Credit Curve Slope Restrictions

The fact that the hazard rate should be non-negative imposes a constraint on the term structure of default swap spreads. Indeed, if the term structure slopes downward too sharply, this could theoretically lead to negative implied probabilities of default. The monotonicity condition is that $P[t > T(n)] \leq P[t > T(n-1)]$. Inserting this into equation (39) and simplifying gives the following inequality for $s(n)$:

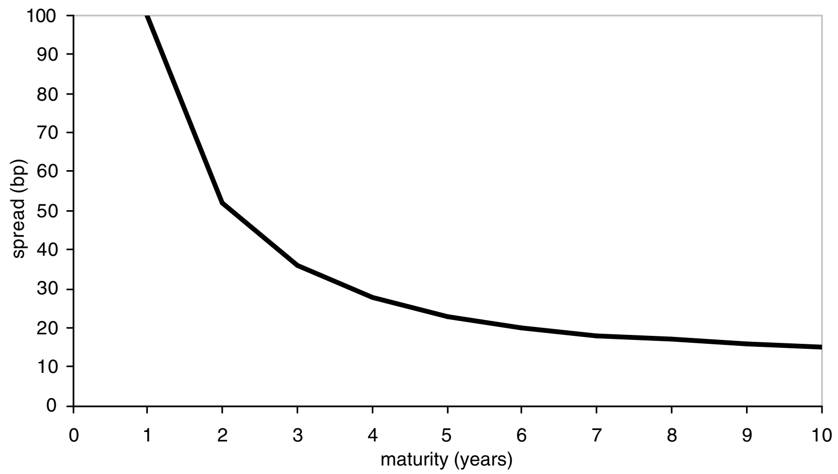
$$s(n) \geq \frac{q(n-1)\, s(n-1)}{q(n-1) + B(0,T(n))P[t > T(n-1)]} \quad \textbf{(41)}$$

In actual practice, the problem is not very acute, as the following example shows.

Example 6: We consider a term structure of default swap spreads with annual maturities out to ten years. We start out with a spread of 100bp for a maturity of one year. For each year thereafter, we calculate the minimum spread according to equation (41) and assume the actual spread is the smallest whole basis point amount greater than this value. We assume a recovery rate of 30% and constant interest rates of 5%.

The spread term structure is plotted in Figure 10. We see that we can quite comfortably accommodate decreasing spreads at the front end of the term structure. Also, the minimum spreads prove to be quite insensitive to the recovery rate.

Figure 10. **Minimum default swap spread which ensures non-negative hazard rates**



Example of a Credit Curve Construction

The basis of our example is the LIBOR swap curve in Euros from the 11th October 2000. The discount factors $B(t,T(i))$ are obtained by bootstrapping the swap curve.

Strictly speaking, the swap curve is not a risk-free curve, as swap rates are quoted for counterparties of *AA*-rated credit quality. Despite this, there are several good reasons for using the swap curve. Since AAA corporate issuers trade close to

Figure 11. **Libor Swap Curve**

LIBOR, it can be argued that the swap spread is more a measure of liquidity than credit risk. In any case, given the collateralization and netting agreements in place for swaps, the credit component must be seen more as a measure of systemic risk in the banking sector than as a credit spread for a specific issuer. Finally, when pricing derivatives, it is necessary to take into account the funding costs associated with the corresponding hedge. As the typical participants in the market for default swaps are banks that fund at LIBOR or close to it, using the swap curve is a viable way of including these costs.

For our example, we assume a term structure of default swap spreads as shown in Figure 12 for a fictitious issuer with a recovery rate of 20%. On each default swap, the premium is paid quarterly. For simplicity, we take each accrual factor as 0.25.

From the default swap spreads we have implied out the hazard rate as a function of time, as discussed above. It is shown in Figure 13.

Figure 12 . **Term Structure of Default Swap Spreads**

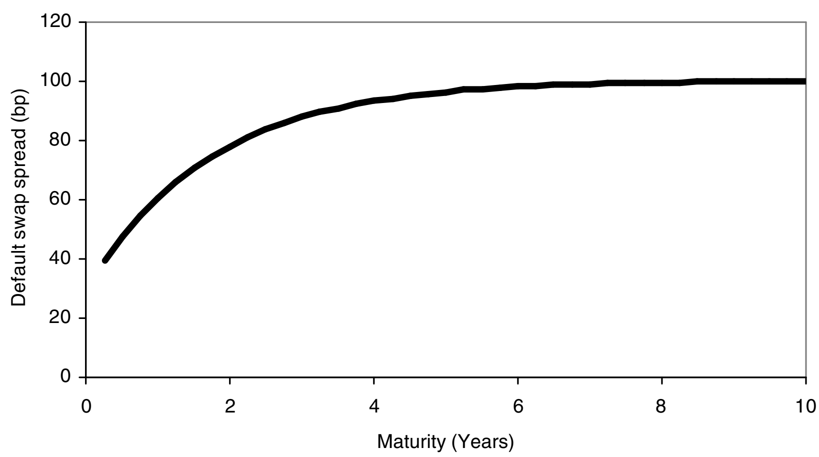
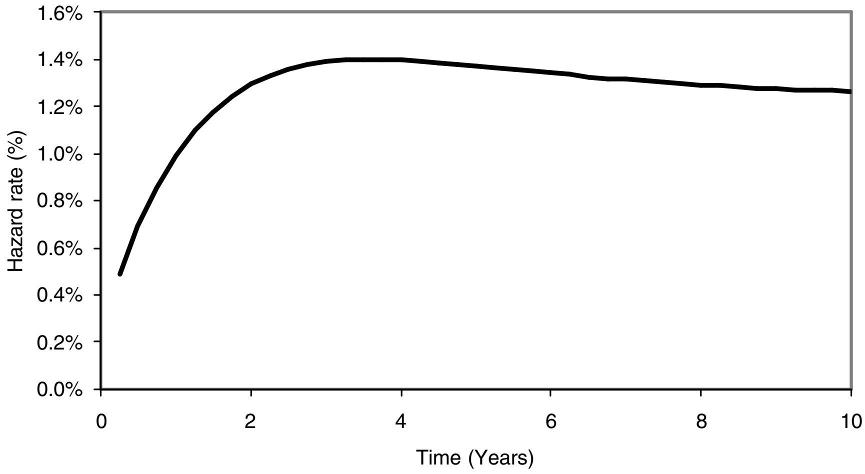


Figure 13. **Hazard Rate implied from default swap spreads**



The relationship between Figure 12 and Figure 13 becomes clearer by interpreting the deterministic hazard rate as the conditional forward default probability. In this sense, it is related to spreads in much the same way as the forward curve is related to the yield curve in the interest rate world. The easiest way to see this is to consider a defaultable zero coupon bond with zero recovery and a maturity of *T*. Using equation (17), we find that the spread $s(T)$ of this bond over the risk-free rate is given by

$$s(T) = \frac{1}{T}\int_0^T \lambda(u)\,du \qquad \textbf{(42)}$$

This is analogous to the way the yield of a risk-free bond is determined as the average over forward rates, and implies that the hazard rate is related to the slope of the spread curve via

$$s + T\frac{\partial s}{\partial T} = \lambda \qquad \textbf{(43)}$$

Portfolio Credit Models

Introduction

Up to now, we have discussed the modelling of default risk on a single-issuer basis. However, tools for describing correlated defaults are essential. For one, due to the limited upside and the potential for large losses, credit risk should be managed on a portfolio basis. Also, derivatives conditioning on the payoff of multiple credits, such as basket default swaps and collateralized debt obligations (CDO's), are increasingly used by investors to create credit risks which match their risk appetites. In these sorts of structures, the factors that determine the credit risk profile of the portfolio are:

- The number of assets and their weighting in the portfolio
- The credit quality of the assets
- The default correlation between the assets

Clearly, the more assets in the portfolio, the less exposed is the investor to a single default. Equally, the higher the credit quality of the assets, the less likely a default and the lower the expected loss on the pool of assets. Default correlation plays a significant role in these structures. In simple terms, the default correlation between two assets is a measurement of the tendency of assets to default together. If the assets in a portfolio have a high default correlation then, when assets default, they do so in large groups. This can significantly affect the credit risk profile, making large portfolio losses more likely, fattening the tail of the loss distribution.

It is natural to expect that assets issued by companies which have common dependencies would be more likely to default together. For example, companies in the same country are exposed to a common interest rate and exchange rate. Companies within the same industrial sector have the same raw material costs, share the same market and so could be expected to be even more strongly correlated. On the other hand, the elimination of a competitor may be beneficial to companies in the same sector.

Default Correlation

Empirical Evidence of Default Correlation

Empirical analysis of default correlation is limited by the lack of default events. One study, Lucas (1995), which computed the default correlation between assets in different rating categories, has made two particularly interesting observations. The first is that default correlation increases as we descend the credit rating spectrum. This has been attributed to the fact that lower rated companies are more vulnerable to an economic downturn than higher rated companies and so are more likely to default together.

Table 4. Empirical measurement of the 10-year default correlation as a function of credit rating. (D.J. Lucas, The Journal of Fixed Income, March 1995)

	10-Year Default Correlations (%)					
	Aaa	Aa	A	Baa	Ba	B
Aaa	1					
Aa	2	0				
A	2	1	2			
Baa	2	1	1	0		
Ba	4	3	4	2	8	
B	9	6	9	6	17	38

The second observation is that default correlation is horizon-dependent and it has been postulated that this may be linked to the periodicity of the economic cycle.

As mentioned above, computing industry-industry default correlations is difficult due to the shortage of default events. In practice, default correlation is often proxied using some other quantity such as credit spread correlation or stock price correlation. While this may appear a reasonable assumption, strictly speaking there is no model-independent mathematical relationship that can link the two together.

Defining Default Correlation

Before we can model default correlation, we need to define it mathematically. Let us denote by p_A the probability that asset A defaults before some time T. Likewise, p_A is the probability that asset B defaults before T. The joint probability that both asset A and asset B default before T is p_{AB}. Using the standard definition of the correlation coefficient, we can compute the pairwise default correlation between the two assets A and B as:

$$r_D = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1 - p_A)}\sqrt{p_B(1 - p_B)}} \quad \textbf{(44)}$$

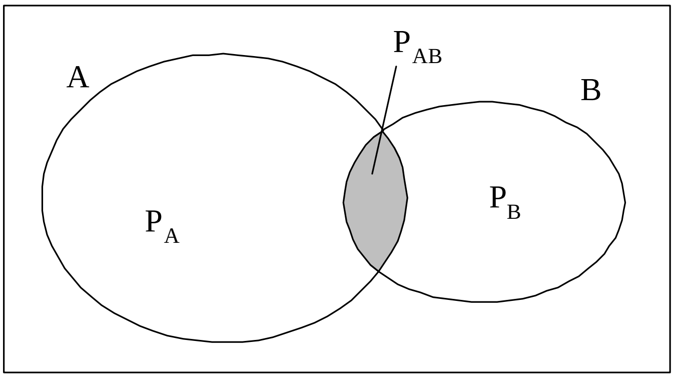
It is important to note that the default correlation will typically depend on the time horizon T considered.

Default Correlation and Basket Default Swaps

The simplest portfolio credit derivative is the basket default swap. It is like a default swap but for the fact that it is linked to the default of more than one credit. In the particular case of a first-to-default basket, it is the first asset in a basket whose credit event triggers a payment to the protection buyer. As in a default swap, the protection buyer pays a fee to the protection seller as a set of regular accruing cash flows, in return for protection against the first-to-default.

To see how default correlation affects pricing, consider a T-maturity first-to-default basket with two assets A and B in the basket. Pictorially one can represent the outcomes of two defaultable assets using a Venn diagram as shown in Figure 14. Region A encompasses all outcomes in which asset A defaults before the maturity of the basket. Its area equals the probability p_A. Similarly, region B encompasses all scenarios in which asset B defaults before the maturity of the basket and its

Figure 14. **Venn Diagram representation of correlated default for two assets**



area equals p_B. The shaded overlap region corresponds to all scenarios in which both asset A and asset B default before time T and so has an area equal to the probability of joint default p_{AB}.

As we have shown in the section on curve construction, the market-implied probabilities of asset A defaulting and the probability of asset B defaulting are easily derived from a knowledge of where the individual default swaps trade.

A first-to-default basket pays off as soon as one of the assets in the basket defaults. So in order to value the protection, we need to calculate the probability of this occurring. It is given by the area contained within the outer perimeter of the two regions since this contains all scenarios in which one or the other or both assets default before the maturity of the basket. This area equals

$$\Omega(\rho_D) = p_A + p_B - p_{AB} \quad \textbf{(45)}$$

To understand the effect of default correlation, consider the following scenarios.

Independent Assets

When the assets are independent, the default correlation is zero and the probability of both assets defaulting before time T is given by $p_{AB} = p_A p_B$. The probability that the first-to-default basket is triggered is then given by

$$\Omega(0) = p_A + p_B - p_A p_B \quad \textbf{(46)}$$

The last term is usually very small compared to the first two since it is the product of two probabilities.

Example 7: If $p_A = 5.0\%$ and $p_B = 3.0\%$, then we have $\Omega(0) = 7.85\%$

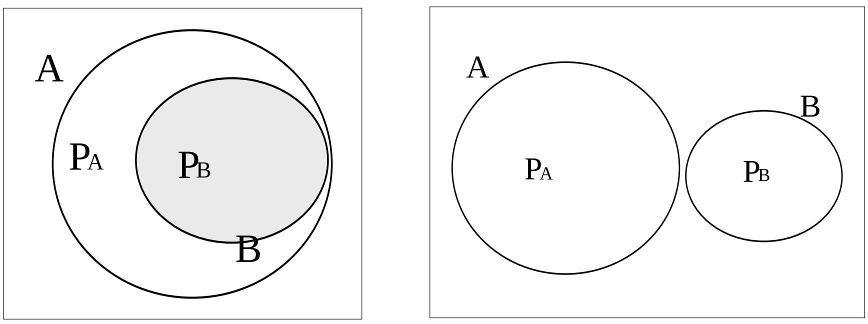
Maximum Default Correlation

At the highest default correlation, the default of the stronger asset B always results in the default of the weaker asset. The weaker asset A (the one with the higher default probability), can still default by itself. This is best shown graphically on the left side of Figure 15. In this limit the joint default probability is given by the default probability of the stronger asset, so that

$$p_{AB} = \min[p_A, p_B] \quad \textbf{(47)}$$

As we have chosen A to be the weaker asset $p_A > p_B$, and the joint probability $p_{AB} = p_B$ so that $\Omega(p^{MAX}) = p_A$.

Figure 15. **Venn Diagram for maximum (left) and minimum (right) default correlation**



Example 8: Using the above probabilities, we find that at the maximum default correlation, $= \max[\ 5.0\%, 3.0\%] = 5.0\%$.

Minimum Default Correlation

As the default correlation falls, there comes a point when there is a zero probability of both assets defaulting together before time T - the default events for assets A and B are mutually exclusive. Graphically, there is no intersection between the two circles and we have $p_{AB} = 0$, cf. the right-hand side of Figure 15. The probability that the first-to-default basket is triggered is simply the sum of the two areas A and B

$$\left(r_D^{MIN}\right) = p_A + p_B \qquad \textbf{(48)}$$

Example 8 continued: Using the above values for the probabilities we find that at the minimum correlation

$$= \max[\ 5.0\%, 3.0\%] = 5.0\%$$

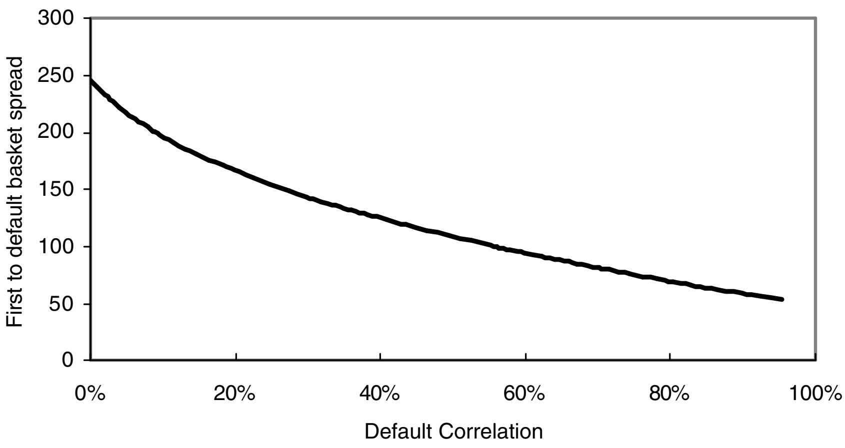
This is when the first-to-default basket is at its riskiest and so protection is at its most expensive.

More Than Two Assets

To complete the pricing, we need to discount payoffs, which means that we need to have access to the time at which defaults occur. Also, as we increase the number of assets in the basket, we increase the number of possible default scenarios and the number of inputs required. One approach is to simulate the default times of the assets in a similar manner to the one described on page 15.

While we do not typically know the form of the joint density distribution for the default times, it is possible to impose one with the correct marginal distributions using the method of copulas. This technique is explained in Li (2000) and we refer the reader to that paper for details. Using such a model, we plot in Figure 16 the first to default basket spread as a function of the default correlation for a 5-asset basket, each asset in the basket has a 5-year default swap spread of 50bp. At zero correlation the basket price is slightly less than 250bp, which is the sum of the spreads. As the

Figure 16. First-to-default basket spread as a function of default correlation for a 5 asset basket



maximum default correlation is approached, the basket spread tends to the maximum of the individual issuer spreads, which (as all spreads are the same) is 50bp.

Such a model also makes it easy to compute the price of second, third and *n*th-to-default baskets.

Modelling and Valuing Collateralized Debt Obligations

Introduction

A **collateralized debt obligation (CDO)** is a structure of fixed income securities whose cash flows are linked to the incidence of default in a pool of debt instruments. These may include loans, revolving lines of credit, emerging market corporate and sovereign debt, as well as subordinate debt from structured transactions.

In a CDO, notes are issued in several tranches, with the senior tranches having coupon and principal repayment priority over the mezzanine and equity tranches. The income from the collateral is paid to the most senior tranches first as interest on the notes. The remaining income is then paid to the notes issued on the mezzanine tranche, followed by payments to those in the equity tranche. As the senior note is paid first, it has a lower credit risk than the other notes and can often achieve a AAA credit rating.

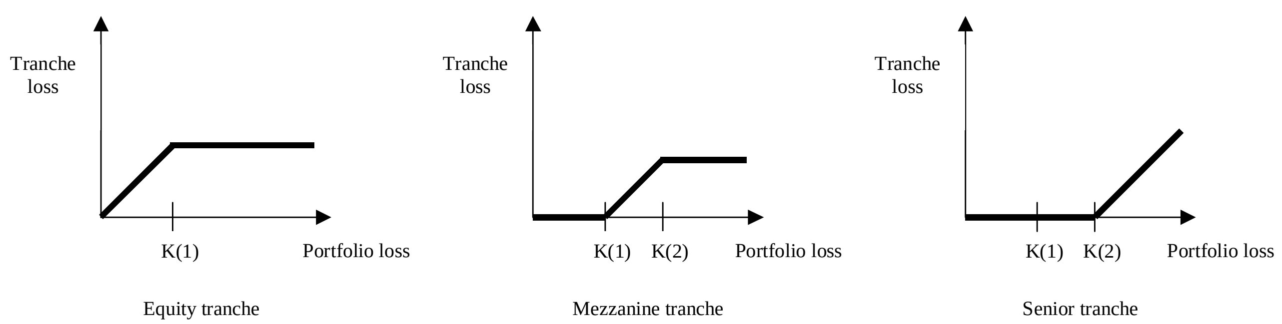
The likelihood of losses depends on the individual quality of the constituent collateral and the degree of diversification across the pool. The more diversified the collateral, the lower the risk of the senior tranches. Indeed, one of the main determinants of the riskiness of a CDO is the correlation between the collateral in the pool. In this sense a CDO is closely related to the default basket, where the equity tranche is like a first-to-default basket, and higher tranches are like second, third and fourth-to-default baskets, respectively.

In each situation, the key issue is modelling the joint default probability given the default probabilities for each individual issuer.

Models for Introducing Default Correlation

In the simplest case, defaults of issuers occur independently. If a portfolio consists of m bonds with the same notional and the default probability of each issuer over some time horizon is p, then the distribution of the percentage loss L of the portfolio is binomial, with

Figure 17. **Tranche losses in CDO as a function of portfolio loss**



$$P\left(L = \frac{k}{m}\right) = \binom{m}{k} p^k (1-p)^{m-k} \tag{49}$$

In practically all applications the defaults of different issuers will be correlated to some extent, and assuming independence will overestimate the portfolio diversification. In the rating of CDO's, Moody's applies the technique of a **diversity score** to deal with this problem, cf. Moody's (1999). The distribution used to compute the portfolio losses is still the binomial one. However, the number m is not the actual number of credits in the portfolio, but is adjusted downwards to take the effects of correlation into account. This is done for credits within an industry sector, different sectors are treated as independent.

The Firm Value Approach to Modelling Correlated Default

Conceptually, the asset return of the ith issuer in a portfolio can be described using a standard normally distributed random variable $A(i)$. Rating transitions, and in particular default, occur when the asset return $A(i)$ crosses certain thresholds. The default threshold $C(i)$ is implied from the issuer's default probability $p(i)$ via the equation

$$p(i) = P[A(i) < C(i)] = N(C(i)) \tag{50}$$

where, as before, N denotes the cumulative distribution function of the standard normal distribution.

In the single-issuer case, this is merely an exercise in calibration, as the default thresholds are chosen to produce default probabilities which reprice market instruments. The concept of asset returns becomes meaningful when studying the joint behaviour of more than one issuer. In this case, the returns are described by a multivariate normal distribution. Using the threshold levels determined before, it is possible to obtain the probabilities of joint rating transitions.

As standardized asset returns are used, it is only necessary to estimate the correlation structure of asset return. One reasonable approximation is provided by the correlation between equity returns.

With a completely general correlation structure, the calculation of joint default probabilities becomes computationally intensive, making it necessary to resort to Monte Carlo simulation. However, substantial simplifications can be achieved by imposing more structure on the model. An important concept in this context is that of conditional independence. We assume that the asset return of each of the m issuers is of the form

$$A(i) = b(i)Z + \sqrt{1 - b(i)^2}\, Z(i) \tag{51}$$

where $Z, Z(1), \ldots, Z(m)$ are independent standard normal random variables. The variable Z describes the asset returns due to a common market factor, while $Z(i)$ models the idiosyncratic risk of the ith issuer, and $b(i)$ represents the correlation of $A(i)$ with the market.

The advantage of this setup is that, conditionally on Z, the asset returns are independent. This makes it easy to compute conditional default probabilities. Let us assume that the portfolio is homogeneous, i.e. that $b(i)$ and $C(i)$ are the same for all assets. The ith issuer defaults in the event that $A(i) < C$. Using equation (51), we see that this is equivalent to

$$Z(i) < \frac{C - bZ}{\sqrt{1 - b^2}} \qquad \textbf{(52)}$$

The conditional default probability $p(Z)$ of an individual issuer is given by

$$p(Z) = N\left(\frac{C - bZ}{\sqrt{1 - b^2}}\right) \qquad \textbf{(53)}$$

Calculating the Loss Distribution

If we also assume that the exposure to each issuer is of the same notional amount, the probability that the percentage loss L of the portfolio is k/m is equal to the probability that exactly k of the m issuers default, which is given by

$$P\left(L = \frac{k}{m}\right) = \binom{m}{k} N\left(\frac{C - bZ}{\sqrt{1 - b^2}}\right)^{k} \left(1 - N\left(\frac{C - bZ}{\sqrt{1 - b^2}}\right)\right)^{m-k} \qquad \textbf{(54)}$$

In other words, the conditional distribution of L is binomial. Again, this distribution becomes computationally intensive for large values of m, but we can use methods of varying sophistication to approximate it.

Large Portfolio Limit

One very simple and surprisingly accurate method is the so-called "large homogeneous portfolio" approximation, originally due to Vasicek. After conditioning on Z, the asset returns are independent and identically distributed. By the law of large numbers, the fraction of issuers defaulting will tend to the probability given in equation (53). Therefore, one can assume that the percentage loss given Z is approximately equal to $p(Z)$. In particular, the expected loss of the portfolio is equal to the individual default probability p. Using (53), we see that

$$L \le q \Leftrightarrow Z \ge p^{-1}(q) \qquad \textbf{(55)}$$

Equation (55) states that the percentage loss of the portfolio will not exceed the level q if and only if the market return Z is sufficiently high. Because $p(Z)$ is a monotonically decreasing function, the upper bound for L is translated into a lower bound for Z. The unconditional distribution for L follows immediately from (55):

$$P[L \le q] = N\left[-p^{-1}(q)\right] \qquad \textbf{(56)}$$

More involved calculations show that the distribution of L actually converges to this limit as m tends to infinity. For a given individual default probability, the loss distribution is quite sensitive to the correlation parameter b. In the limit of $b = 0$, the portfolio loss is deterministic, as can be seen from the conditional loss given in equation (53). As b tends to 100%, the loss distribution becomes bimodal, as the portfolio effectively only consists of one credit.

Example 9: Consider a portfolio where the individual default probability is 7.17%, which corresponds to the default probability of a single-*B* rated issuer over a one year time horizon, cf. Table 1. Figure 18 shows the loss distribution if we assume a market correlation of 20%, whereas Figure 19 shows the loss distribution if we assume a market correlation of 55%. At the higher correlation the tail of the distribution is fatter than for the lower correlation as there is now a greater tendency for multiple defaults. To ensure that the expected loss remains constant, this also means that the likelihood of fewer defaults also increases.

Note that the correlation between issuers is actually the square of the market correlation, cf. eq. (51). Values of 20% and 55% for the market correlation correspond to issuer correlations of 4% and approximately 30%, respectively.

Figure 18. **Loss distribution of a large homogeneous portfolio with *B* rated issuers and a market correlation of 20%**

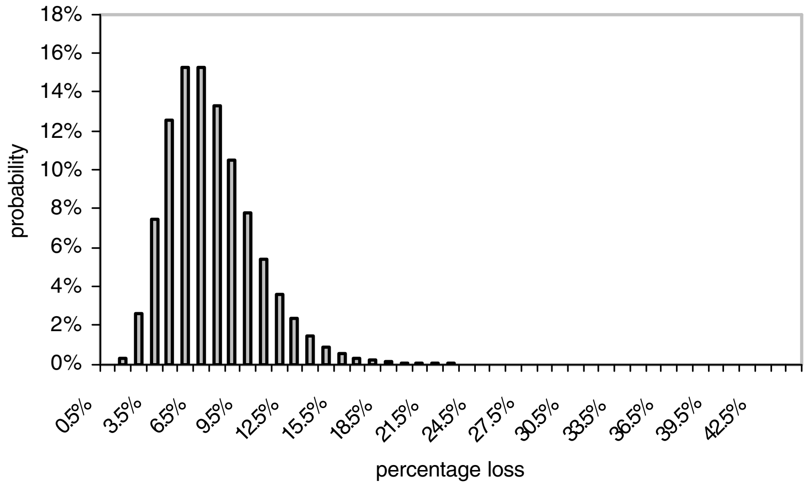
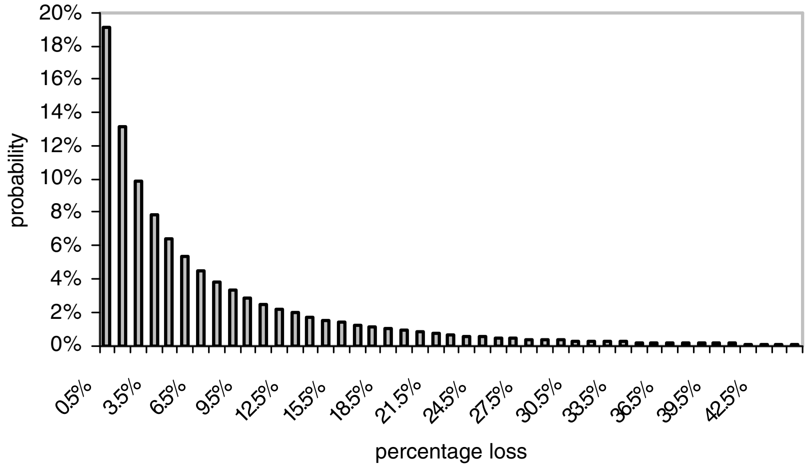


Figure 19. **Loss distribution of a large homogeneous portfolio with *B* rated issuers and a market correlation of 55%**



The Tranche Loss Distribution

We can use the portfolio distribution to derive the loss distributions of individual tranches. If $L(K(1),K(2))$ is the percentage loss of the mezzanine tranche with boundaries $K(1)$ and $K(2)$, then

$$L(K(1),K(2)) = \frac{\max(L \quad K(1),0) \quad \max(L \quad K(2),0)}{K(2) \quad K(1)} \tag{57}$$

Note that the percentage loss is scaled by the width of the tranche. If $q < 1$, then

$$L(K(1),K(2)) \quad q \qquad L \quad K(1)+q(K(2) \quad K(1)) \tag{58}$$

Equation (58) shows that we can easily derive the distribution of $L(K(1),K(2))$ from that of L. The distribution of $L(K(1),K(2))$ is discontinuous at the edges of $[0,1]$, due to the probability of the portfolio loss falling outside the interval $[K(1),K(2)]$. This discontinuity becomes more pronounced when the tranche is narrowed.

Example 10: We consider a mezzanine tranche of the same portfolio of *B* rated issuers we examined earlier. We suppose that all portfolio losses between 2% and 15% are applied to this tranche. Figure 20 shows the loss distribution assuming a correlation of 20%, while in Figure 21 the assumed correlation is 55%. Again, we see that the loss distributions are fundamentally different.

Figure 20. **Loss distribution of mezzanine tranche taking losses between 2% and 15% for a market correlation of 20%**

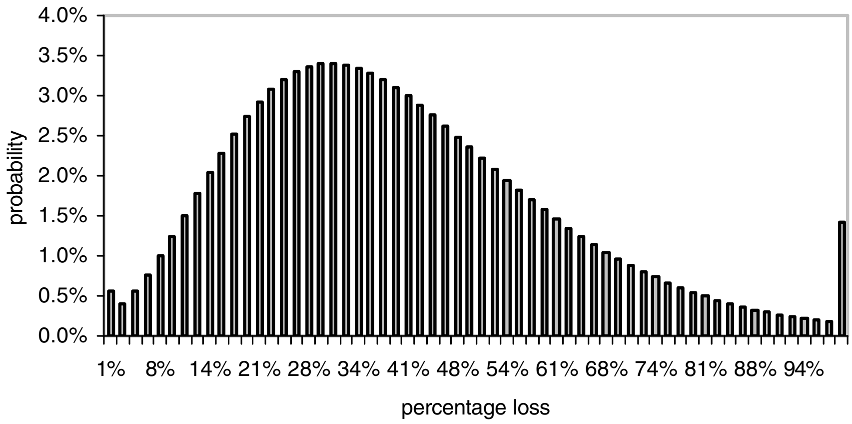
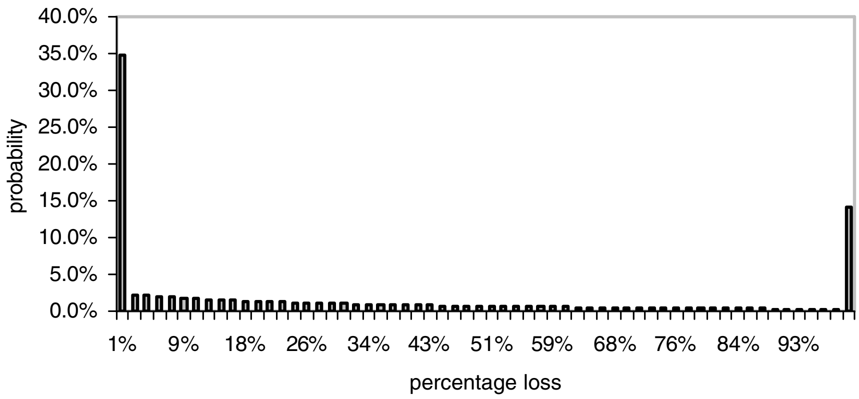


Figure 21. **Loss distribution of mezzanine tranche taking losses between 2% and 15% for a market correlation of 55%**



We can compute the expected loss of a senior tranche analytically. For any K in $[0,1]$, we denote by $L(K)$ the senior payoff

$$L(K) = \max(L - K, 0) \tag{59}$$

If we denote the density of the portfolio loss distribution by $g(L)$, the expectation of this payoff is given by

$$E[L(K)] = \int_K^1 (s - K) g(s) ds \tag{60}$$

After deriving the explicit form of the density from (56) and some algebra, it is possible to rewrite the expectation in terms of the distribution function N_2 of a bivariate normal distribution[1]:

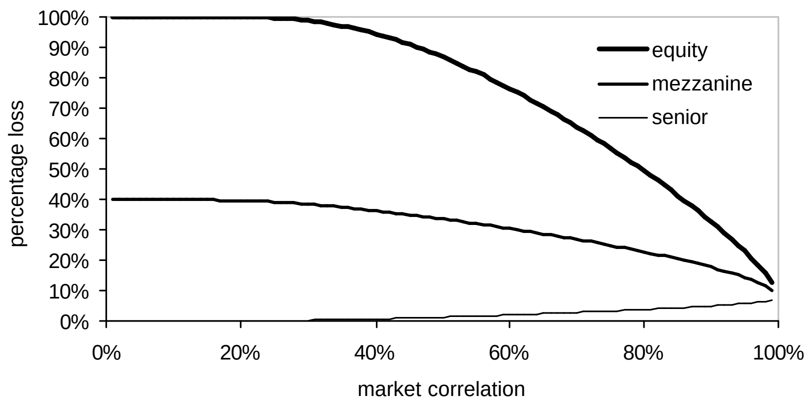
$$E[L(K)] = N_2\left(-N^{-1}(K), C, -\sqrt{1-\beta^2}\right) \tag{61}$$

From equation (60), we immediately conclude that the expected loss of a mezzanine tranche is given by

$$E[L(K(1), K(2))] = \frac{N_2\left(-N^{-1}(K(1)), C, -\sqrt{1-\beta^2}\right) - N_2\left(-N^{-1}(K(2)), C, -\sqrt{1-\beta^2}\right)}{K(2) - K(1)} \tag{62}$$

Example 11: Again, we consider our portfolio of *B* rated issuers. We divide the portfolio into three tranches. The equity tranche absorbs the first 2% of all portfolio losses, the mezzanine tranche absorbs all losses between 2% and 15%, while the senior tranche assumes all losses above 15%. Using equation (62) we compute the expected loss of each tranche as a function of the market correlation parameter. The expected losses are plotted in Figure 22. We see that the equity and the mezzanine tranches actually benefit from a higher market correlation, i.e. less diversification in the portfolio. This is because the expected portfolio loss of 7.17% is well within the range of the mezzanine and beyond the equity tranche. Therefore, these tranches benefit from the all-or-nothing gamble that is taken if the correlation becomes very large. Of course, the senior note holders only stand to lose if the diversification of the portfolio is significantly reduced.

Figure 22. **Expected tranche loss as a function of market correlation**



[1] *An approximation formula for the cumulative distribution function of the bivariate normal distribution is given in Hull (1997)*

Further Modelling Approaches

In addition to the asset-based approach presented in detail here, other methods have been used to analyse correlated defaults. Intensity models can be generalized to the multi-issuer case by specifying the joint dynamics of the individual intensities. This is another application of the conditional independence concept; after conditioning on the joint realizations of the intensities, the default counting processes are independent. For the simulation of correlated defaults in this intensity framework see Duffie and Singleton (1998). An application to the analysis of CDO's is presented in Duffie and Garleanu (1999).

Davis and Lo (1999) have developed a model in this framework which explicitly models default correlation. Each issuer can either default idiosyncratically or via contagion from another issuer. The model is static in the sense that only idiosyncratic defaults are transmitted via contagion, i.e. there are no "domino effects". The effect of contagion fundamentally alters the distribution of losses. For a given number of expected defaults, it exhibits fatter tails than the base case binomial distribution, making both very small and very large losses more likely. Unfortunately, the model involves intensive combinatorial calculations for more complex portfolios.

Another approach, based on the firm value model of Merton, is implemented in the third party system CreditMetrics , cf. Gupton/Finger/Bhatia (1997). It attempts to assess the possible losses to a portfolio due to both defaults and rating changes, where the rating system can be either internal or one provided by a rating agency.

A very useful technique for generating correlated default times is the method of copula functions. This effectively lets one separate the modelling of the dependence structure from that of the univariate distributions of the individual default times. For an introduction to the application of copula functions to finance, see Li (2000).

Summary

Multi-issuer credit derivatives are assuming an increasing importance in the market. The pricing of first and second loss products requires the use of models which induce a default correlation between assets. In these models, the specification of the independence structure is of primary importance. This was shown clearly in the discussion of the first-to-default basket and the pricing of CDO tranches.

Conclusions

As stated at the beginning, modelling credit is a difficult task for a wide variety of reasons. Nonetheless, credit models have become an essential requirement in the analysis, pricing and risk management of credit. An understanding and appreciation of the advantages and disadvantage of the various models is therefore necessary to anyone wishing to apply a more quantitative approach.

Structural models were reviewed and shown to be best for performing a risk assessment of publicly traded companies, or where a better understanding of the effect of the capital structure of a firm is needed. Structural models are not however the preferred choice for pricing and hedging credit derivatives in a risk-neutral framework. This is where reduced form models win out since they are powerful enough to price even the more exotic credit derivatives and have the flexibility to exactly reprice the observable market instruments.

We presented two approaches for studying portfolio credit default, looking first at a simple two asset first-to-default basket and then at the large portfolio limit. In between these limits we favour the use of the copula method to generate correlated default times, which is described for example in Li (2000). The valuation results that we presented show the absolutely crucial role played by correlation parameters when valuing multi-issuer credit derivatives.

The models presented here represent the current state of quantitative credit modelling. There are still many questions to be addressed, especially in the multi-issuer context. As yet, valuation models do not play the same role in the credit world as they do for the pricing of exotic interest rate derivatives. However, as the credit derivatives markets continue to mature, the scope and usage of quantitative models will undoubtedly increase.

Appendix

Valuation of Payments made at the Time of Default

In the Cox process framework we consider a payment that is made at the time of default, should this occur before T. To be more precise, we assume that X is a predictable process[2], and that a payment of $X(\tau)$ will be made at time t if default occurs before T. The initial value $D(0)$ of this claim is given by

$$D(0) = E\left[\frac{X(\tau)}{\beta(\tau)} 1_{\{\tau \le T\}}\right] \qquad \textbf{(63)}$$

If we define the default indicator process H by $H(t) := 1_{\{\tau \le t\}}$, we have

$$\frac{X(\tau)}{\beta(\tau)} 1_{\{\tau \le T\}} = \int_0^T \frac{X(u)}{\beta(u)} dH(u) \qquad \textbf{(64)}$$

The process M given by $dM(u) = dH(u) - \lambda(u) 1_{\{u \le \tau\}} du$ is a martingale by virtue of the definition of the intensity, therefore

$$E\left[\int_0^T \frac{X(u)}{\beta(u)} dH(u)\right] = E\left[\int_0^T \frac{X(u)}{\beta(u)} \lambda(u) 1_{\{u \le \tau\}} du\right] \qquad \textbf{(65)}$$

Using iterated conditional expectations as before and changing the order of integration gives

$$E\left[\int_0^T \frac{X(u)}{\beta(u)} \lambda(u) 1_{\{u \le \tau\}} du\right] = \int_0^T E\left[\frac{X(u)}{\beta(u)} \lambda(u) \exp\left(-\int_0^u \lambda(v) dv\right)\right] du \qquad \textbf{(66)}$$

The price of the default contingent claim is therefore

$$D(0) = \int_0^T E\left[\frac{X(u)}{\beta(u)} \lambda(u) \exp\left(-\int_0^u \lambda(v) dv\right)\right] du \qquad \textbf{(67)}$$

If the process X is determined by the non-defaultable interest rates and their evolution is independent of the intensity, this can be rewritten as

$$D(0) = \int_0^T E\left[\frac{X(u)}{\beta(u)}\right] E\left[\lambda(u) \exp\left(-\int_0^u \lambda(v) dv\right)\right] du \qquad \textbf{(68)}$$

One payoff that is particularly important is the case when $X \equiv 1$; i.e. a payoff of par at the time of default. Then, equation (68) becomes

$$D(0) = \int_0^T B(0,u) E\left[\lambda(u) \exp\left(-\int_0^u \lambda(v) dv\right)\right] du \qquad \textbf{(69)}$$

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6 Predictability is a measurability property of stochastic processes which ensures that martingale properties are preserved under stochastic integration. A good example of a predictable process is one which is adapted and has continuous paths. The exact technical definition of predictability is beyond the scope of this overview.

References

Arvanitis, Gregory, and Laurent (1999): "Building models for credit spreads", The Journal of Derivatives, Spring issue: 27-43

Black and Cox (1976): "Valuing corporate securities: some effects of bond indenture provisions", Journal of Finance 31:351-367

Black and Scholes (1973): "The pricing of options and corporate liabilities", Journal of Political Economy, 81:637-54

Bohn (1999): "Characterizing credit spreads", Haas School of Business, University of California, working paper.

Briys and de Varenne (1997): "Valuing risky fixed rate debt: an extension", Journal of Financial and Quantitative Analysis 32:239-247

Crosbie (1998): "Modelling default risk", KMV, working paper.

Davis and Lo (1999): "Infectious defaults", Tokyo-Mitsubishi International plc, working paper.

Duffie and Garleanu (1999): "Risk and valuation of collateralized debt obligations", Graduate School of Business, Stanford University, working paper.

Duffie and Singleton (1998): "Simulating correlated defaults", Graduate School of Business, Stanford University, working paper.

Duffie and Singleton (1999): "Modeling term structures of defaultable bonds", Review of Financial Studies 12:687-720

Franks and Torous (1994): "A comparison of financial recontracting in distressed exchanges and Chapter 11 reorganizations", Journal of Financial Economics 35:349-370

Geske (1977): "The valuation of corporate liabilities as compound options", Journal of Financial and Quantitative Analysis 12:541-552

Geske and Johnson (1984): "The valuation of corporate liabilities as compound options: a correction", Journal of Financial and Quantitative Analysis 19:231-232

Gupton, Finger and Bhatia (1997): CreditMetrics, Technical Document, Morgan Guarantee Trust, New York

Helwege and Turner (1999): "The slope of the credit yield curve for speculative-grade issuers", Journal of Finance 54:1869-1884

Hull (1997): "Options, Futures, and other Derivatives", 3rd edition, Prentice Hall International

Israel, Rosenthal, Wei (2000): "Finding generators for Markov chains via empirical transition matrices, with applications to credit ratings", forthcoming in Mathematical Finance

Jarrow, Lando and Turnbull (1997): "A Markov model for the term structure of credit spreads", Review of Financial Studies 10:481-523

Jarrow and Turnbull (1995): "Pricing derivatives on financial securities subject to credit risk", Journal of Finance 50:53-86

Jones, Mason and Rosenfeld (1984): "Contingent claims analysis of corporate capital structures: an empirical investigation", Journal of Finance 39:611-625

Lando (1998): "On Cox processes and credit risky securities", Review of Derivatives Research 2:99-120

Lehrbass (2000): "A simple approach to country risk", in eds. Franke, Haerdle, Stahl, "Measuring Risk in Complex Stochastic Systems", Lecture Notes in Statistics, Springer, Berlin

Li (2000): "On default correlation: a copula function approach", Journal of Fixed Income, March 2000

Longstaff and Schwartz (1995): "A simple approach to valuing risky fixed and floating rate debt", Journal of Finance 50:789-819

Lucas (1995): "Default correlation and credit analysis", Journal of Fixed Income, March 1995

Merton (1974): "On the pricing of corporate debt: the risk structure of interest rates", Journal of Finance 29:449-470

Moody's Investor Services (1999): "The binomial expansion method applied to CBO/CLO analysis"

O'Kane (2000): "Introduction to default swaps", Analytical Research Series, Lehman Brothers

Sarig and Warga (1989): "Some empirical estimates of the risk structure of interest rates", Journal of Finance 44:1351-1360

Schonbucher (1996): "Valuation of securities subject to credit risk", University of Bonn, working paper

Schonbucher (1998): "Term structure modelling of defaultable bonds", Review of Derivatives Research 2:161-192

Shimko, Tejima and van Deventer (1993): "The pricing of risky debt when interest rates are stochastic", Journal of Fixed Income 3:58-65

Standard and Poor's (2000): "Ratings performance 1999: stability and transition", special report

Vasicek (1977): "An equilibrium characterization of the term structure", Journal of Financial Economics 5:177-188

Zhou (1997): "A jump-diffusion approach to modelling credit risk and valuing defaultable securities", Federal Reserve Board, working paper

Publications—L. Pindyck, A. DiTizio, B. Davenport, W. Lee, D. Kramer, S. Bryant, J. Threadgill, J. Borwick, R. Madison, A. Acevedo

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