3. Poisson Processes (9/10/04, cf. Ross)

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Exponential Distribution

Definition: The continuous RV X has the *exponential distribution* with parameter λ if its p.d.f. is $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

Facts: $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$.
$\mathsf{E}[X] = 1/\lambda$, $\mathsf{Var}(X) = 1/\lambda^2$,
$M_X(t) = \lambda/(\lambda - t)$, $t < \lambda$.

Theorem: The exponential distribution has the *memoryless property*. Namely, for $s, t > 0$,

$$\Pr(X > s + t | X > t) = \Pr(X > s).$$

Example: If $X \sim \text{Exp}(1/10)$, then

$$\Pr(X > 10 | X > 5) = \Pr(X > 5) = e^{-5/10} = 0.607.$$

Remark: The exponential is the only continuous distribution with this property.

Definition: For a continuous distribution, the *hazard function* (or *failure rate*) is

$$r(t) \equiv \frac{f(t)}{1 - F(t)}.$$

The hazard function is the conditional p.d.f. that X will fail at time t (given that X made it to t).

Why this interpretation?

$$\begin{aligned} r(t)\,dt &= \frac{f(t)\,dt}{1 - F(t)} \\ &\approx \frac{\Pr(X \in (t, t+dt))}{\Pr(X > t)} \\ &= \frac{\Pr(X \in (t, t+dt) \text{ and } X > t)}{\Pr(X > t)} \\ &= \Pr(X \in (t, t+dt) | X > t). \end{aligned}$$

Example: $X \sim \text{Exp}(\lambda)$ implies that $r(t) = \lambda$. This makes sense in light of the memoryless property. In fact, the $\text{Exp}(\lambda)$ is the only RV with constant $r(t)$.

Remark: $r(t)$ (uniquely) determines $F(t)$.

"Proof:"

$$r(t) \;=\; \frac{f(t)}{1-F(t)} \;==\; -\frac{d}{dt}\ell\mathsf{n}(1-F(t)),$$

so that

$$F(t) \;=\; 1-\exp\left(-\int_0^t r(s)\,ds\right). \quad \diamondsuit$$

Theorem: $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$ implies that

$$\sum_{i=1}^{n} X_i \sim \text{Erlang}_n(\lambda) \sim \text{Gamma}(n, \lambda).$$

Many proofs — m.g.f.'s, induction, you name it. $\diamondsuit$

Poisson Processes

Definition: Consider discrete (non-fractional) events occurring in continuous intervals (time, length, volume, etc.). The *counting process* $N(t) \equiv$ the number of events occurring in $[0, t]$. Let the *rate* $\lambda > 0$ be the average number of occurrences per unit time (or length or volume).

Examples: 1. Cars entering a shopping center (time); $\lambda = 5/\text{min}$.

2. Defects on a wire (length); $\lambda = 3/\text{ft}$.

3. Raisins in cookie dough (volume); $\lambda = 2.6/\text{in}^3$.

Definition: A counting process $N(t)$ satisfying the following three assumptions is called a *Poisson process* with rate λ (PP(λ)).

(A-1) Arrivals occur one-at-a-time,

(A-2) Independent increments, and

(A-3) Stationary increments.

Details follow. . .

(A-1) For any interval of sufficiently small length h,

(a) The prob of one arrival in that interval is

$$\Pr(N(t+h) - N(t) = 1) \;=\; \lambda h + o(h) \;\doteq\; \lambda h.$$

(b) The prob of no arrivals is

$$\Pr(N(t+h) - N(t) = 0) \;=\; 1 - \lambda h + o(h) \;\doteq\; 1 - \lambda h.$$

(c) The prob of more than one arrival is

$$\Pr(N(t+h) - N(t) \geq 2) \;=\; o(h) \;\doteq\; 0,$$

where $o(h)$ is a function that $\to 0$ faster than $h \to 0$.

(A-2) Independent Increments: The numbers of arrivals in two *disjoint* intervals are *independent*. I.e., if $a < b < c < d$, then $N(d) - N(c)$ and $N(b) - N(a)$ are independent.

(A-3) Stationary Increments: The number of arrivals in a time interval depends only on its length. Thus, $N(t + s) - N(t) \sim N(s) - N(0)$ for all t.

(A-4) (bonus assumption): $N(0) = 0$.

Example: People arriving to a restaurant is not a PP. Arrivals occur in groups (violating A-1), and arrival rates change throughout the day (violating A-3). $\diamondsuit$

Theorem: If $N(t)$ is a PP(λ), then $N(t) \sim$ Pois(λt). In particular, $N(1) \sim$ Pois(1). Further, E$[N(t)] =$ Var$(N(t)) = \lambda t$.

Proof: See any reasonable probability text. $\diamondsuit$

Remark: Be careful with units of time. For example, suppose that $X \sim$ Pois(3) is the number of phone calls in a one-minute period.

Then the number of calls in a 3-minute period is Pois(9) and the number in a 30-sec. period is Pois(1.5).

Poisson and Exponential Relationship

Definition: Let A_1 denote the time until the first arrival of a PP(λ).

For $i \geq 2$, let A_i denote the time between the $(i-1)$st and ith arrivals.

The A_i's are called *interarrival* times.

We can get the distributions of the A_i's. . .

First of all, consider A_1.

$A_1 > t$ iff no arrivals take place in $[0, t]$. Thus, for $t > 0$,

$$\Pr(A_1 > t) = \Pr(N(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t},$$

and so $A_1 \sim \text{Exp}(\lambda)$.

Now A_2. For $0 < s < t$, we have

$$
\begin{aligned}
&\Pr(A_2 > t | A_1 = s) \\
&= \Pr(\text{no arrivals in } (s, s+t] | A_1 = s) \\
&= \Pr(\text{no arrivals in } (s, s+t]) \\
&\quad \text{(by independent increments)} \\
&= \Pr(\text{no arrivals in } (s, s+t]) \\
&\quad \text{(by stationary increments)} \\
&= e^{-\lambda t} \quad \text{(by previous arguments).}
\end{aligned}
$$

Thus, $\Pr(A_2 > t | A_1 = s) = e^{-\lambda t}$ for all s.

So $\Pr(A_2 > t) = e^{-\lambda t}$, i.e., $A_2 \sim \text{Exp}(\lambda)$, independent of the value of A_1.

This can be generalized...

Theorem: $A_1, A_2, \ldots \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$.

Definition: The nth *arrival time* is $S_n \equiv \sum_{i=1}^{n} A_i$, $n \geq 1$.

Theorem: $S_n \sim$ Gamma$(n, \lambda) \sim$ Erlang$_n(\lambda)$. Thus, E$[S_n] = n/\lambda$ and Var$(S_n) = n/\lambda^2$.

Example: A UGA student can read $X \sim$ Pois(1) pages of *Dick and Jane* a day. (a) The expected time until the 10th page is completed is E$[S_{10}] = n/\lambda = 10$ days. (b) The prob that it takes > 2 days to read the 11th page is Pr$(A_{11} > 2) = e^{-2} \doteq 0.135$. $\Diamond$

Theorem: Consider a PP(λ), $N(t)$. Suppose arrivals are either Type I or II (e.g., male or female), with Pr(Type I) $= p$, Pr(Type II) $= 1 - p$. Let $N_1(t)$ and $N_2(t)$ denote the numbers of Type I and II events during $[0, t]$. Note that $N(t) = N_1(t) + N_2(t)$.

Then $N_1(t)$ and $N_2(t)$ are *independent* PP's with resp. rates λp and $\lambda(1 - p)$.

Proof: Long and tedious. See, e.g., Ross. $\diamondsuit$

Example: Suppose cars entering a parking lot follow a PP(5/min). Further, suppose the prob that a driver is female is 0.6. Find the prob that exactly 3 cars driven by females will enter the lot in the next 2 minutes.

The # of cars entering in two min. $\sim$ Pois($\lambda t = 10$).

The # driven by females is Pois($\lambda t p = 6$).

So the desired prob is

$$\Pr(\text{Pois}(6) = 3) \;=\; \frac{e^{-6}6^3}{3!} \;=\; 0.0892. \quad \diamondsuit$$

Conditional Distribution of Arrival Times

Theorem: $\text{Pr}(A_1 < s | N(t) = 1) = \frac{s}{t}$.

In other words, suppose we *know* that one arrival has occurred by time t, i.e., $N(t) = 1$. What's the (conditional) distribution of the time of that arrival? Answer: $\text{Unif}(0, t)$!

Proof: We have

$$\Pr(A_1 < s | N(t) = 1)$$

$$= \frac{\Pr(A_1 < s \text{ and } N(t) = 1)}{\Pr(N(t) = 1)}$$

$$= \frac{\Pr(1 \text{ arrival in } [0, s]; \text{ none in } (s, t])}{\Pr(N(t) = 1)}$$

$$= \frac{\Pr(1 \text{ arrival in } [0, s]) \Pr(\text{none in } (s, t])}{\Pr(N(t) = 1)}$$

(by independent increments)

$$= \frac{\Pr(N(s) = 1) \Pr(N(t) - N(s) = 0)}{\Pr(N(t) = 1)}$$

Proof (cont'd): Then

$$
\begin{aligned}
&\Pr(A_1 < s | N(t) = 1) \\
&= \frac{\Pr(N(s) = 1)\Pr(N(t-s) = 0)}{\Pr(N(t) = 1)} \\
&\quad \text{(by stationary increments)} \\
&= \frac{e^{-\lambda s}(\lambda s)^1}{1!} \, \frac{e^{-\lambda(t-s)}(\lambda(t-s))^0}{0!} \bigg/ \frac{e^{-\lambda t}(\lambda t)^1}{1!} \\
&= s/t. \quad \diamondsuit
\end{aligned}
$$

Can generalize above result. . .

Theorem: Given that $N(t) = n$, the joint probability distribution of the n arrival times $S_1, S_2, \ldots, S_n$ is the same as the joint distribution of n i.i.d. Unif$(0, t)$ RV's.

Thus, if we *know* that n arrivals have occurred by time t, the arrivals can be treated as if they were i.i.d. Unif$(0, t)$.

Bonus Theorem: Consider a PP(λ), $N(t)$. Suppose there are k possible types of arrivals. Further suppose that the prob that an arrival is of Type i depends on the *time* that it occurs — if it occurs at time t, then the probability that it's a Type i is $P_i(t)$, where $\sum_{i=1}^{k} P_i(t) = 1$. If $N_i(t)$ denotes the number of Type i arrivals by time t, then the $N_i(t)$'s, $i = 1, 2, \ldots, k$, are *independent* Poisson RV's with means

$$\mathsf{E}[N_i(t)] \;=\; \lambda \int_0^t P_i(s)\, ds.$$

Proof: Ross. $\diamondsuit$

Generalizations

Definition: The counting process $N(t)$ is a *nonhomogeneous PP* with *intensity function* $\lambda(t)$ if it satisfies:

(A-1′) Arrivals occur one-at-a-time. In particular,

$$\begin{aligned} \Pr(N(t+h)-N(t)=1) &= \lambda(t)h+o(h) \quad \text{and} \\ \Pr(N(t+h)-N(t)=0) &= 1-\lambda(t)h+o(h) \end{aligned}$$

(A-2) Independent increments.

(A-4) $N(0)=0$.

Note: We don't require stationary increments.

Fact: $N(t+s) - N(t) \sim \text{Pois}\left(\int_t^{t+s} \lambda(x)\,dx\right)$.

Remark: Arrivals may be more (or less) likely to occur as time progresses in a NHPP, since there is no stationarity requirement.

Example: Distribution of cars arriving to a lot as the day progresses is a NHPP. From 8:00–11:00 a.m., cars arrive at a steadily increasing rate: 5 cars/hr at 8:00 a.m. to 20 cars/hr at 11:00 a.m. I.e., $\lambda(t) = 5 + 5t$, $0 \le t \le 3$ hrs.

Number of arrivals between 8:30–9:30 a.m. is

$$N(1.5) - N(0.5) \sim \text{Pois}\left(\int_{0.5}^{1.5} (5 + 5x)\, dx\right) \sim \text{Pois}(10).$$

$\diamondsuit$

Definition: A stochastic process $X(t)$ is a *compound PP* if it can be written as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where $N(t)$ is a PP(λ) and the Y_i's are i.i.d. and independent of $N(t)$.

Facts: $\mathsf{E}[X(t)] = \mathsf{E}[N(t)]\mathsf{E}[Y_1] = \lambda t \mathsf{E}[Y_1]$ and $\mathrm{Var}(X(t)) = \lambda t \mathsf{E}[Y_1^2]$.

Example: If the Y_i's all equal 1, then $X(t) = N(t)$, the usual PP.

Example: Customers leave a market according to a PP. Let Y_i be the amount spent by customer i. If $X(t)$ is the total amount spent by time t, then $X(t)$ is a compound PP.

Example: A FB player makes Pois($\lambda = 2$) scores/game.

$$\Pr(\text{score} = x) \;=\; \begin{cases} 1/6 & \text{if } x = 1 \\ 1/3 & \text{if } x = 3 \\ 1/2 & \text{if } x = 6 \end{cases}$$

Let Y_i be the value of the ith score. $\mathsf{E}[Y_i] = \frac{25}{6}$, $\mathsf{E}[Y_i^2] = \frac{127}{6}$. Let $X(t)$ be the total points scored in t games.

$$\mathsf{E}[X(5)] \;=\; \lambda t \mathsf{E}[Y_1] \;=\; 125/3,$$

$$\mathsf{Var}(X(t)) \;=\; \lambda t \mathsf{E}[Y_1^2] \;=\; 635/3. \quad \diamondsuit$$