Introduction to Computational Finance

Econ 482

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Probability, Brownian motion, Ito process . . .

Why continuous time?

It allows assumption of infinitesimal price changes over infinitesimal time intervals so that

- portfolios can be revised before next 'minishock' hits
- market prices can be treated as locally linear in the things that are varying (permits perfect hedging of risks)

Stochastic processes . . .

Brownian motion (Weiner process): stochastic process $z(t)$ such that

1. $z(0,) = 0$
2. $z(t_2) - z(t_1)$ is independent of $z(t_4) - z(t_3)$ for $t_1 < t_2 < t_3 < t_4$
3. $z(t_2) - z(t_1) \sim \mathsf{N}(0, t_2 - t_1)$

i.e., 0 start, Markov, Normal

Remarks:

$z(t)$ is continuous in t with probability 1
$z(t)$ is nowhere differentiable in t (very jiggly)

Martingale: stochastic process such that $\mathbf{E}[X(t_2)|X(t_1)] = X(t_1)$ (0 expected change)

Markov: stochastic process such that
$\mathbf{p}\{X(t_2) \in A|X(t_1)\} = \mathbf{p}\{X(t_2) \in A|X(t), t \leq t_1\}$

Ito process . . .

Ito process: stochastic process obtained by modifying a Brownian motion to have non-zero 'drift' and non-unit 'volatility'.

shorthand representation: $ds(t) = \alpha(s,t)\,dt + \sigma(s,t)\,dz(t)$
(stochastic d.e.) or just $ds = \alpha\,dt + \sigma\,dz$

short for

$$s(t) = s(0) + \underbrace{\int_0^t \alpha(s,\tau)d\tau}_{\text{regular integral}} + \underbrace{\int_0^t \sigma(s,\tau)dz(\tau)}_{\text{Ito integral}}$$

properties:

$$\begin{aligned} \frac{d}{d\tau}\mathsf{E}[s(\tau)]|_{\tau=t} &= \alpha(s(t),t) \\ \frac{d}{d\tau}\mathsf{Var}[s(\tau)]|_{\tau=t} &= \sigma^2(s(t),t) \end{aligned}$$

Ito's lemma: (used heavily in conts time econ and finance)

if $ds = \alpha\, dt + \sigma\, dz$
and $y(t) \equiv f(s,t)$
and $f(s,t)$ is 'nice'

then $y(t)$ follows an Ito process
and $$dy = \underbrace{(f_t + \alpha f_s + \frac{1}{2}\sigma^2 f_{ss})}_{\text{drift of } y}\, dt + \sigma f_s\, dz$$

E.g: With constant nominal interest rate r, the nominal value of a bond at time t is $B(t) = e^{rt}$. Suppose the price level follows the stochastic process

$$dP = \pi P\, dt + \sigma P\, dz$$

Real value of the bond at time t is $b(t) \equiv B(t)/P(t) = e^{rt}/P(t)$. Applying Ito's lemma, this real value follows

$$db = (r - \pi + \sigma^2) b\, dt - \sigma b\, dz$$

Hence the expected real interest yield on the bond is $r - \pi + \sigma^2$.

Arbitrage valuation ...

Let uncertain aspect of the world relevant for payoffs be described by $s(t)$ following Itô process

$$ds = \alpha\, dt + \sigma\, dz \tag{1}$$

Suppose there are two tradeable securities whose prices are functions of (s,t). Let $A(s,t)$ and $B(s,t)$ be the s-contingent prices. Applying Itô's lemma

$$\begin{aligned} dA &= (\tfrac{1}{2}\sigma^2 A_{ss} + \alpha A_s + A_t)\, dt + \sigma A_s\, dz \qquad (2) \\ dB &= (\tfrac{1}{2}\sigma^2 B_{ss} + \alpha B_s + B_t)\, dt + \sigma B_s\, dz \end{aligned}$$

Cash can be risklessly borrowed or lent at floating interest rate r. Note that r, α and σ can also be functions of (s,t).

Riskless portfolio: Construct portfolio of 1 unit of A and $-A_s/B_s$ units of B. Fund by borrowing amount $(A - A_s B/B_s)$. Let $P(t)$ denote value of this portfolio at time t. This value evolves as

$$dP = dA - \frac{A_s}{B_s} dB - (\text{borrowings})\, r\, dt \tag{3}$$

$$= \left(\tfrac{1}{2}\sigma^2 (A_{ss} - \frac{A_s}{B_s} B_{ss}) + \alpha(A_s - \frac{A_s}{B_s} B_s) + (A_t - \frac{A_s}{B_s} B_t) - r(A - \frac{A_s}{B_s} B) \right) dt$$

No arbitrage: Position is riskless and costs 0. Market equilibrium precludes riskless arbitrage opportunities. Hence $dP = 0\, dt$ for all (s, t). Rearranging gives

$$\frac{\frac{1}{2}\sigma^2 A_{ss} + \alpha A_s + A_t - rA}{A_s} = \frac{\frac{1}{2}\sigma^2 B_{ss} + \alpha B_s + B_t - rB}{B_s} \tag{4}$$

This value, common to all securities dependent on s, is the 'market price of s-risk'. Denote it $\lambda(s, t)$. Equating to left side of (4) gives

$$\tfrac{1}{2}\sigma^2 A_{ss} + (\alpha - \lambda) A_s + A_t - rA = 0 \tag{5}$$

This is the *fundamental valuation pde*. Can solve it for $A(s, t)$.

$$\tfrac{1}{2}\sigma^2 V_{ss} + (\alpha - \lambda) V_s + c + V_t - rV = 0$$

Remarks:

- if securities pay dividends at continous rate $c(s, t)$ then valuation pde modified as above [exercise: prove it]
- arbitrage doesn't determine λ (as in Fisher example); non-zero value indicates market risk aversion
- many functions satisfy the pde: added boundary conditions characterize the security and identify a particular solution
- multi-dimensional analogue when price depends on multiple factors (see Notes)
- obtain same pde if $\hat{\alpha} \equiv \alpha - \lambda$ is the objective drift and $\hat{\lambda} \equiv 0$. I.e., price is same as in a 'risk-neutral world' with different s process. $\hat{\alpha}$ termed the risk-adjusted drift.

$\frac{1}{2}\sigma^2 V_{ss} + (\alpha - \lambda)V_s + c + V_t - rV \;=\; 0$

- if $s(t)$ denotes price of a traded security (assume no dividends on *it*), then it too satisfies the pde. Since then $V(s,t) = s$ for all s, t, it follows that $V_s = 1$, $V_{ss} = V_t = 0$. Substituting into the pde gives

$$(\alpha - \lambda) - rs = 0$$

This pins down λ and implies that for all *other* derivatives of s

$$\tfrac{1}{2}\sigma^2 V_{ss} + rsV_s + c + V_t - rV \quad = \quad 0 \qquad (6)$$

This is the basis of the Black-Scholes formula, in which neither expectations α about the rate of change in s nor the level of risk aversion embodied in λ affect the value of the option (given s, r, σ).

Example boundary conditions

1. $s =$ stock price, $V =$ call option to purchase at price X at time T

$$V(s,T) = \max\{\ 0\ ,\ s(T) - X\ \} \qquad \forall s$$

BS model: $ds = (rs - d)\,dt + \sigma s\,dz$

2. $s \equiv r =$ short term interest rate, $V =$ bond with maturity value 1 at time T

$$V(r,T) = 1 \qquad \forall r$$

CIR model: $dr = \kappa(\bar{r} - r)\,dt + \sigma r^{1/2}\,dz$

3. $s = \$/£$ exchange rate, $V =$ exchange rate swap based on rate $\bar{s}$ (i.e., pay $\bar{s}$ \$/yr, receive 1 £/yr)

$$V(s,T) = 0$$

$$c(s,t) = s - \bar{s}$$

$ds = (r - r_f)s\,dt + \sigma s\,dz$

Example boundary conditions

4. $s =$ commodity price, $V =$ perpetual call option to purchase 1 unit at price K, combined with policy of exercising option when $s \geq \bar{s}$

$$V(\bar{s}, t) = s - K \qquad \forall t$$

Note: $\bar{s}$ optimal (max option value) $\Longleftrightarrow V_s(\bar{s}, t) = \frac{\partial}{\partial s}(s - K) = 1$

This is Merton 'high-contact' condition. Finding joint solution for V and $\bar{s}^*(t)$ is a 'free boundary value' problem. Arises with American options.

e.g., harvesting trees, opening a mine, buying equipment ...

Black-Scholes formula ...

Black and Scholes, *JPE* 1973; Merton, *Bell J.* 1973

Let $V(s, t)$ be value of European call option to purchase stock at exercise price x on date T. Current stock price is $s(t)$, pays no dividend, and follows risk-adjusted process $ds = rs\,dt + \sigma s\,dz$. Riskfree rate r and proportional volatility σ are constant.

Adopt the convention of letting t denote time *left* to maturity. This reverses sign on V_t in valuation pde. Maturity boundary value is $V(s, 0) = \max\{\ 0\ ,\ s(0) - x\ \}$. Solution to the pde is

$$V(s, T) = sN(d) - xe^{-rT}N(d - \sigma T^{1/2})$$

in which $N()$ denotes the cumulative normal distribution function and

$$d \equiv \frac{\ln(\frac{s}{x}) + (r + \frac{\sigma^2}{2})T}{\sigma T^{1/2}}$$

Black-Scholes formula . . .

Remarks:

- Value of put option can be obtained using *put-call parity*: Buying a call (price C) and shorting a put (price P), both with exercise price x, is identical to committing to buy the stock for price x at time T with certainty. That, in turn, is equivalent buying the stock now and borrowing an amount that grows to x at time T. Thus

$$C(s,T) - P(s,T) = s - xe^{-rT}$$

- If stock pays dividends at constant proportional rate q, replace s in V expression by se^{-qT} and r in d expression by $r - q$. (Hull ch. 14)
- American calls on non-dividend paying stocks are never rationally exercised early. American puts, calls on dividend paying stocks can be rationally exercised early. Hence values are greater than European.

Feynman-Kac formula . . .

Duffie (1992, 2001) chap 5.H

Relates the solution to the valuation pde to an expected value.

Suppose $ds = \alpha\, dt + \sigma\, dz$ and let f be the value of something that depends on s at a fixed future date. Define

$$F(s,t) \equiv \mathbf{E}[f(s(T))|s(t)]$$

Applying Ito's lemma, the expected drift in F is

$$\frac{\mathbf{E}[dF]}{dt} = \tfrac{1}{2}\sigma^2 F_{ss} + \alpha F_s + F_t$$

But if expectations are rational — i.e., follow the laws of conditional probability — then $\mathbf{E}[dF] = 0$. Hence

$$\tfrac{1}{2}\sigma^2 F_{ss} + \alpha F_s + F_t = 0$$

This is identical to the valuation pde without the $-rF$ term.

Feynman-Kac formula . . .

Now consider

$$G(s,t) \;\equiv\; e^{-r(T-t)}F(s,t)$$

Applying Ito's lemma, the drift in G is

$$\tfrac{1}{2}\sigma^2 G_{ss} + \alpha G_s + G_t = e^{-r(T-t)} \underbrace{[\tfrac{1}{2}\sigma^2 F_{ss} + \alpha F_s + F_t]}_{0} + r \underbrace{e^{-r(T-t)}F}_{G} = rG$$

This reduces to the same as the valuation pde:

$$\tfrac{1}{2}\sigma^2 G_{ss} + \alpha G_s + G_t - rG \;=\; 0$$

with boundary condition $G(s,T) = f(s)$. Hence the solution to the valuation pde can be interpreted as the expected discounted value of the terminal $f(s)$ conditional on the current level of s.

Feynman-Kac formula . . .

More generally, the solution to the valuation pde, even with both $r(t)$ stochastic and with dividend flows $c(s,t)$, is the following expectation:

$$V(s,t) \;=\; \mathbf{E}^* \left[f(s(T)) e^{-\int_t^T r\,d\tau} + \int_t^T c(s(\tau),\tau) e^{-\int_t^\tau r\,dv}\,d\tau \right]$$

By $\mathbf{E}^*$ here we mean the expectation over s paths where it follows the *risk-adjusted* process $ds = (\alpha - \lambda)\,dt + \sigma\,dz$. I.e., the arbitrage-free asset price is the expected present value of its cash flows, discounted at the risk-free short term interest rate, until it hits a stopping boundary.

Application: Monte Carlo valuation method

Futures prices: Cox, Ingersoll, Ross, *JFE* 1981

Let $F(s,t)$ be the futures price at time t for delivery at date T. Let $P(s)$ denote the *spot* price of the commodity at time T as function of s. Assume $ds = \alpha\, dt + \sigma\, dz$. Note F is not an asset price.

CIR trick: Invent an asset with price $V(s,t)$ which pays continuous dividend at rate $rV(s,t)$ and lump sum of $P(s)$ at time T.

Claim: $F(s,t) = V(s,t)\ \forall s,t$.

Verify by showing no arbitrage opportunity:

Borrow $V(s,0)$ now, buy 1 unit of V, short 1 futures contract at price $F = V$.

$dF = dV\ \forall s,t$ (assumed). If $dV > 0$, cash settlement on futures is $-dF$. Add to loan. Interest on loan is paid by dividend on V.

Final loan amount $V(s,0) + \int_0^T dV = V(s,T) = F(s,T)$ is paid off by maturity value of asset (spot-futures convergence).

Futures contract expires with no further settlements.

Hence there is net 0 final outcome, regardless of s path, verifying $F(s,t) = V(s,t)\ \forall s,t$ is a solution.

$\frac{1}{2}\sigma^2 V_{ss} + (\alpha - \lambda)V_s + c + V_t - rV \ = \ 0$

Remarks:

- Valuation pde for asset V, and hence for futures price F, is

$$\frac{1}{2}\sigma V_{ss} + (\alpha - \lambda)V_s + V_t = 0$$

  with boundary condition $V(s,T) = P(s)$, since dividend flow of $c(s,t) \equiv rV$ just offsets $-rV$ term.
- The Feynman-Kac formula then implies $F(s,t)$ equals the (undiscounted) *expected future spot price under the risk-adjusted process.*
- Further applications: computation of expected values of other functions of $s(T)$. E.g., mean, variance, probability $s < K$, ...

$$\frac{1}{2}\sigma^2 V_{ss} + (\alpha - \lambda) V_s + c + V_t - rV \ = \ 0$$

Interpretations of $V(s,t)$:

1. V is the price for which the security must trade in equilibrium for there to be no riskless arbitrage opportunities.

2. V is the expected present value, under the risk-neutral probability measure, of the cash flows from owning the security.

3. The instantaneous expected rate of return on holding V, under the risk-adjusted process, equals the riskfree interest rate.

4. V is the lump-sum amount that, if put into a portfolio, and used to trade in *other* securities whose values fluctuate with s and riskless bonds, can exactly replicate the cash flows of V. (self-financing replicating portfolio).

   e.g., Wells Fargo gold-linked term deposits . . .