

Applying Extreme Value Theory to FinCat

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1. Jump-diffusion stock price model

Let $s(t)$ be the price at time t of a traded asset. Assume it follows a jump-diffusion process

$$ds(t) = (r + \lambda\bar{x})s dt + \omega s dz(t) - xs d\pi(t) \quad (1)$$

where $z(t)$ is a standard Brownian motion, ω a constant diffusion volatility, $\pi(t)$ the cumulative number of jumps by time t , and x the jump size conditional on a jump occurring. Jump arrival is a Poisson process with constant intensity λ . Jump size x is a random variable with mean \bar{x} . The instantaneous riskfree interest rate is r . Drift has been set to make the expected rate of return equal to the risk free rate assuming no dividends. (1) is thus an equilibrium process for s under the risk-neutral measure. Notice that jumps have been specified as *downward* moves for convenience below. The starting values of z and π are 0.

Applying Ito's Lemma, the transformed variable $\ln s(t)$ follows

$$d \ln s(t) = (r + \lambda\bar{x} - \omega^2/2) dt + \omega dz(t) - x d\pi(t) \quad (2)$$

Hence $\ln s(T)$ equals

$$\ln s(T) = \ln s(0) + (r + \lambda\bar{x} - \omega^2/2)T + \omega z(T) - \sum_{i=1}^{\pi(T)} x_i \quad (3)$$

in which x_i denotes the random size of jump i . The diffusion component $z(T)$ is normally distributed with mean 0 and variance T ; $\pi(T)$ is Poisson distributed with parameter λT . We leave unspecified the particular jump distribution, but will assume that the regularity conditions of extreme value theory hold so that its upper tail, conditional on the jump size exceeding some threshold u , can be approximated by a generalized Pareto distribution. Parameters of that distribution are its location u , shape k and scale σ .¹ Taking the exponential of both sides, (3) implies

$$s(T) = s_0 \exp((r + \lambda\bar{x} - \omega^2/2)T + \omega z(T) - \sum_{i=1}^{\pi(T)} x_i) \quad (4)$$

in which π is the number of jumps occurring in $[0, T]$, x_i are the corresponding jump sizes, and s_0 is the initial asset price.

¹These correspond to the parameters of the notes on extreme value theory.

2. Short maturity put option values

The theoretical value of a european put option with strike K maturing at T is

$$\begin{aligned} P(s_0, T) &= \mathbf{E}^*[e^{-rT} \max\{0, K - s(T)\}] \\ &= \mathbf{E}^*[e^{-rT} s_0 \max\{0, \frac{K}{s_0} - \exp(r + \lambda \bar{x} \omega z(T) - \sum_1^{\pi} x_i)\}] \end{aligned} \quad (5)$$

This section approximates (5) for short maturities. For small T treat the riskless rate as 0 and the drift as negligible. For small λ the probability of two jumps occurring in $[0, T]$ is negligible relative to the probability of one jump. The probability of no jumps occurring is $e^{-\lambda T} \sim 1 - \lambda T$. Value of the option per dollar of current s_0 is thus approximately

$$\frac{P}{s_0} = \mathbf{E}^*[\max\{0, \frac{K}{s_0} - e^{\omega z(T) - x}\}] \quad (6)$$

$$= (1 - \lambda T) \mathbf{E}^*[\max\{0, \frac{K}{s_0} - e^{\omega z(T)}\}] + \lambda T \mathbf{E}^*[\max\{0, \frac{K}{s_0} - e^{\omega z(T) - x}\}] \quad (7)$$

The first expectation of the right hand side is the standard Black-Scholes option value; the second is the value conditional on one jump occurring with x being the random jump size.

For near the money options and small λT , value will be dominated by the first term. For far from the money options, it will be dominated by the second. When estimating the tail distribution from daily data of large market declines, we were actually fitting the tail distribution to $x - \omega z(T)$ together.

Now limit attention to far out of the money options. For these the first term of (7) will be insignificant relative to the second. The option expires in the money only if

$$y = x - \omega z(T) > -\ln(K/s_0) \gg 0 \quad (8)$$

This is clearly in the *upper* tail of $x - \omega z$ realizations. Using the Generalized Pareto approximation to the density of this random variable, the value of the option is thus

$$\frac{P}{s_0} = \mathbf{p}(y > u) \int_{-\ln K/s_0}^{\infty} (\frac{K}{s_0} - e^{-y}) f(y|y > u) dy \quad (9)$$

$$\begin{aligned} &= \lambda T \int_{-\ln K/s_0}^{\infty} (\frac{K}{s_0} - e^{-y}) \frac{(1 - k(y - u)/\sigma)^{-1+1/k}}{\sigma} dy \\ &= \frac{\lambda T}{\sigma} \int_{1-K/s_0}^1 (h - 1 + \frac{K}{s_0}) (1 + k(u + \ln(1 - h))/\sigma)^{-1+1/k} \frac{1}{1 - h} dh \end{aligned} \quad (10)$$

in which $f(y)$ is the generalized Pareto density function. The second line takes $f(y)$ from the previous notes on the application of extreme value theory. The third line changes the variable of integration from y to $h = 1 - e^{-y}$ (one day proportional fall in the asset price). Evaluation of this integral is done numerically using Simpson's Rule.²

²The change of variable is done to give finite limits of integration to facilitate numerical solution.

3. The FinCat payoff structure

In the FinCat deal, payout is based on the proportion beyond .25 by which the S&P 500 Index falls over one day (close to close). Let $s(t)$ denote the level of the Index at time t . Let a_0 and a_1 denote the attachment and cap levels of the proportional Index change (currently values proposed are .25 and .50), and A the notional amount of the contract. Payout for day t is thus

$$q(t) = \max\{ 0, \min\{ a_1 - a_0, 1 - \frac{s(t)}{s(t-\delta)} - a_0 \} \} A \quad (11)$$

where δ is one trading day. Expected value of q is the value of the one day long put short call option that it represents. Expressed per dollar of notional A , it can be written similarly to (10)

$$\begin{aligned} \frac{\mathbf{E}^*[q]}{A} &= \frac{\lambda\delta}{\sigma} \left(\int_{a_0}^{a_1} (h - a_0)(1 + k(u + \ln(1 - h)))/\sigma)^{-1+1/k} \frac{1}{1 - h} dh + (a_1 - a_0)\mathbf{p}(h > a_1) \right) \\ &= \frac{\lambda\delta}{\sigma} \left(\int_{a_0}^{a_1} (h - a_0)(1 + k(u + \ln(1 - h)))/\sigma)^{-1+1/k} \frac{1}{1 - h} dh \right. \\ &\quad \left. + (a_1 - a_0)(1 + k(u + \ln(1 - a_1))/\sigma)^{1/k} \right) \end{aligned} \quad (12)$$

The expression for $\mathbf{p}(h > a_1)$ is $1 - G(y; \sigma, k)$, where G is the generalized Pareto cumulative distribution function from earlier Notes.

A maximum likelihood estimate of $\lambda\delta$, the probability of jump exceeding u in a given day, is simply n/N as in the earlier Notes. That is the fraction of days in the sample of size N where (log) market declines exceeded a specified threshold u (e.g., $u = .07$ for roughly 7 percent market decline). However note that this estimates the objective rather than risk neutral process parameter.

The integral in (12) is computed numerically in the fortran program ESTIMATE.FOR. Additionally, the price there is divided by $a_1 - a_0$ so that it is a price per dollar of maximum loss exposure and multiplied by 250 trading days/year to give the fair annualized rate of premium payment that would just compensate for the estimated empirical loss rate. This of course assumes each day is a new and independent draw of proportional market moves.

If premium is to be paid continuously over time, then no discounting of the daily options or the premiums is needed to find the fair premium rate. If paid up front for the 10 years, then discounting of the appropriate fair continuous stream at the current swap-curve rate would have to be done.

I believe the deal also caps our total exposure during the 10 year life. However premium would only be paid on remaining maximum exposure. If the effect is to reduce the notional

The last term in the integrand is the Jacobian dy/dh of the transformation required when changing variables.

underlying in proportion to our cumulative payout (i.e., a_0, a_1 are unchanged for the remaining notional), neither this nor any other scale changes should affect the premium rate required.