A Heuristic Derivation of the HJM Result on Forward Rate Drift

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Let us accept the notion that, for any world with no riskless arbitrage opportunities, there exists an 'equivalent probability measure' over potential world histories under which the expected rate of return on any portfolio or asset at a point in time equals the instantaneous riskless interest rate at that same time.[1] I will use the term 'risk-adjusted' to describe relationships that are true when using these probabilities. An alternative statement is that the value of any portfolio deflated by the value of a money market account follows a martingale.[2]

Heath-Jarrow-Morton (HJM) models of interest rate related claims use the following important result: Under the risk-adjusted probability measure, the expected rates of change of forward interest rates are completely determined by the volatility structure of these rates. The object of this note is informally demonstrate why the relationship they use must hold.

Let $f(t,T)$ denote the forward interest rate at calendar time t on an instantaneously maturing loan to be delivered at time T. Let $P(t,T)$ denote the price at time t of a unit discount bond maturing at time T. The definition of forward interest rate implies

$$P(t,T) \equiv e^{-\int_t^T f(t,v)\,dv} \tag{1}$$

Consider the following trade: At time t, contract forward to lend one dollar at time T for a short time h. The rate that can be contracted is, as a first approximation, $f(t,T)$. Hold this position for a short time dt, then offset it at the then prevailing forward rate $f(t+dt,T)$. In the risk-adjusted world, since initial investment is 0, the expected gain from this transaction must be 0.

Let us see what this implies. Let the change in the forward rate and change in discount bond price over the interval be denoted by $\Delta f = f(t+dt,T) - f(t,T)$ and $\Delta P = P(t+dt,T) - P(t,T)$ respectively. Note that Δf and ΔP are random variables. Working at the level of first order approximations, the gain on the forward contract will be $-h\,\Delta f$. However receipt of the gain is deferred until the delivery date T. Thus the *current* market value of the gain on the transaction is

$$\Delta V = -(h\,\Delta f)P(t+dt,T) = -h\,\Delta f(P(t,T) + \Delta P) \tag{2}$$

[1] Harrison and Kreps (1979, *J. Econ. Theory*) provide the first formal general proof of this result. Note that the list of *possible* world histories, which includes the joint time paths of prices of all securities, interest rates, etc., is the same for the objective world as for the risk- adjusted world. 'Equivalent' is a technical term meaning that histories impossible under the objective probability measure are impossible under the risk-adjusted and vice versa.

[2] A *martingale* is a random variable whose expected future value, conditional on history to date, always equals its current value. I.e., expected change is 0.

Algebraic manipulation lets us rewrite this as

$$\Delta V = -hP(t,T)\,(1 + \Delta P/P)\,\Delta f \tag{3}$$

The definition of covariance implies that if x and y are random variables, then $\overline{xy} = \bar{x}\bar{y} + \mathrm{cov}(x,y)$. Overbar indicates expected values. Taking the expected value of both sides of the above, applying this relation, and dropping terms of higher order than dt such as $\Delta P \Delta f$, gives

$$\overline{\Delta V} = -hP(t,T)(\overline{\Delta f} + \mathrm{cov}(\Delta f, \Delta P/P)) \tag{4}$$

For this to be 0, as it must in the risk-adjusted world, the latter part of the righthand side must be 0. I.e., over the interval dt

$$\overline{\Delta f} = -\mathrm{cov}(\Delta f, \Delta P/P) \tag{5}$$

In words, the risk-adjusted expected change in any forward instantaneous rate must equal (minus) the covariance between that rate and the return on a discount bond of the same maturity. This result is not connected with any particular model of interest rates; it requires only lack of arbitrage opportunities, and that forward rates and bond prices are continuous processes to justify ignoring higher order terms.

Notationally, Heath-Jarrow-Morton (HJM) models take forward interest rates as the primitives whose stochastic processes are exogenously specified, rather than bond prices, spot interest rates, or other state variables. Forward rates are assumed representable as modified diffusions driven by a finite number n of independent standard Brownian motions $\{W_i(t)\}_{i=1,n}$ as follows:

$$df(t,T) = \alpha(t,T)\,dt + \sum_i \sigma_i(t,T)\,dW_i \tag{6}$$

It is understood that the drifts α and factor sensitivities σ_i can depend on the history of the world up to time t; this dependence is suppressed for notational convenience only. The current instantaneous interest rate r is simply the special forward rate

$$r(t) \equiv f(t,t) \tag{7}$$

The operational substance of the HJM result is that, in the *risk-adjusted* world relevant for security pricing, which exists when arbitrage opportunities are nonexistent, the forward rate drifts α are uniquely determined by the factor sensitivities.[3]

We make use below of the following expression of how bond prices change. This follows from the identity (1), assumed specification of forward rate changes (2), and Ito's lemma:

$$\frac{dP(t,T)}{P(t,T)} = (r(t) + b(t,T))\,dt + \sum_i a_i(t,T)\,dW_i \tag{8}$$

[3] When there is just one random factor ($n = 1$), then the factor sensitivity $\sigma(t,T)$ can be interpreted as that forward rate's (absolute) volatility.

in which $a_i \equiv -\int_t^T \sigma_i(t,v)dv$ and $b \equiv -\int_t^T \alpha(t,v)dv + \frac{1}{2}\sum_i a_i^2(t,T)$.

The covariance of df in equation (6) with dP/P in equation (8) will be the expected value of the product of their zero-mean random components (terms involving dW_i). Since the dW_i's are independent standard Brownian motions, $\mathbf{E}(dW_i dW_j)$ equals dt for $i = j$ and 0 for $i \neq j$. Thus

$$\mathrm{cov}(df, dP/P) = \sum_i \sigma_i(t,T)a_i(t,T)\,dt \tag{9}$$

Fitting this into equation (5), noting from (6) that the expected df equals $\alpha(t,T)\,dt$, gives the HJM result

$$\alpha(t,T) = \sum_i \sigma_i(t,T)\int_t^T \sigma_i(t,v)dv \tag{10}$$

When there is only one stochastic factor in the model, this further simplifies to

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,v)dv \tag{11}$$

This embodies the notion that, for the risk-adjusted expected rate of return on a *forward* position to be zero, then the expected rate of change of the forward interest rate must be as above.[4]

As a final observation, notice that the integral on the righthand side of (11) monotonically increases with T. Unless the volatility $\sigma(t,T)$ before it shrinks to 0 as T rises, the value of the expression becomes infinitely large as T rises. This means that distant forward rates must rise almost certainly in the risk-adjusted world, and to infinitely high levels in a short period of time. Since almost certain events in the risk-adjusted and the objective world must coincide (the probability measures are 'equivalent'), this means that distant forward rates must rise almost certainly in the objective world also. This embodies the claim by Dybvig, Ingersoll and Ross (*Journal of Business*, 1996) that 'Long Forward and Zero-coupon Rates Cannot Fall'. It also motivates using volatility functions that tail off to 0 with T if one believes that the real world has finite far distant forward rates.

Appendix

This appendix provides the steps in obtaining the stochastic process followed by bond prices (8) from the process followed by forward rates (6) and the definitional relation between bond prices and forward rates (1). Although it is 'just' an application of Ito's Lemma, it is a rather lengthy one. For notational simplicity we will work with the case of one stochastic factor only ($n = 1$) and hope it is apparent that having n factors would just give rise to the summation signs appearing in (8)-(10).

[4] The analogous result for *futures* prices, whose gains and losses are settled immediately, would be that the expected rate of change in futures prices is 0.

Recall that Ito's Lemma states that if a random variable x follows a modified Brownian motion[5]

$$dx = \beta\, dt + \omega\, dW \tag{12}$$

in which W is a standard Brownian motion, and another random variable y is a smooth function $F(\underline{x}, t)$ of x and its history up to t (denoted collectively by $\underline{x}(t)$ here), then y follows the modified diffusion

$$dF = (\frac{\partial F}{\partial t} + \beta \frac{\partial F}{\partial x} + \frac{\omega^2}{2} \frac{\partial^2 F}{\partial x^2})\, dt + \frac{\partial F}{\partial x}\, dW \tag{13}$$

Starting from equation (1), we see that the discount bond price $P(t, T)$ is a function of the vector of current forward rates $f(t, v)$ for $t \leq v \leq T$, which in turn are following diffusions pushed around by the common exogenous factor dW appearing in (6). In our case, we can specialize (12) to

$$dx = dW \tag{14}$$

in which case (13), with P playing the role of F and with $x = W$, simplifies to

$$dP = (\frac{\partial P}{\partial t} + \frac{1}{2} \frac{\partial^2 P}{\partial W^2})\, dt + \frac{\partial P}{\partial W}\, dW \tag{15}$$

The main task is now to find the necessary partial derivatives of P with respect to t and W. Repeating the definitional relation between bond price and forward rates,

$$P(t, T) \equiv e^{-\int_t^T f(t,v)\, dv} \tag{16}$$

note that t appears both as the lower limit of the integral and as an argument in the integrand f, and that W influences the level of the integrand along its whole length. Taking the partial derivative with respect to t,

$$\begin{aligned} \frac{\partial P}{\partial t} &= -P \frac{\partial}{\partial t} \int_t^T f(t, v)\, dv \\ &= P\Big(f(t, t) - \int_t^T \frac{\partial f(t, v)}{\partial t}\, dv\Big) \\ &= P\Big(f(t, t) - \int_t^T \alpha(t, v)\, dv\Big) \\ &= P\Big(r(t) - \int_t^T \alpha(t, v)\, dv\Big) \end{aligned} \tag{17}$$

[5] I am using the term 'modified' here to describe generalization of Brownian motion in two directions. First, the presence of a non-zero drift β and non-unit volatility ω, both of which may be functions of the *current* t and x, enlarge the set of possible processes. This would still leave x following a Markov process. But second, Ito permitted the drift and volatility more generally to depend in a well-behaved way on the *history* of W up to the current time, and still obtained his relationship. In such cases x and functions of it would not follow Markov processes. The application here makes use of this latter generalization.

The arguments (t, T) of P are suppressed but still there. The first equality follows from the rule for differentiating the exponential function plus the chain rule for derivatives; the second from rules for differentiating integral expressions; the third by obtaining the partial of f with respect to t from the specification (6) for the process followed by forward rates f; the last from (7) which noted that the spot rate was the forward rate for immediate delivery.

Turning to the partial derivatives of P with respect to $W(t)$, we similarly obtain

$$\begin{aligned} \frac{\partial P}{\partial W} &= -P \frac{\partial}{\partial W} \int_t^T f(t,v)\, dv \\ &= -P \int_t^T \frac{\partial f(t,v)}{\partial W}\, dv \\ &= -P \int_t^T \sigma(t,v)\, dv \end{aligned} \tag{18}$$

where the last equality comes from (6) for the process followed by forward rates. Differentiating this again with respect to $W(t)$, noting that only the P term in the last above is affected by W, gives

$$\frac{\partial^2 P}{\partial W^2} = P\Big(\int_t^T \sigma(t,v)\, dv\Big)^2 \tag{19}$$

Substituting these partial derivatives into equation (15) and dividing both sides through by the P common to all of them gives the monster

$$\frac{dP}{P} = \Big(r \underbrace{- \int_t^T \alpha(t,v)\, dv + \frac{1}{2}\Big(\int_t^T \sigma(t,v)\, dv\Big)^2}_{b(t,T)}\Big)\, dt + \Big(\underbrace{-\int_t^T \sigma(t,v)\, dv}_{a(t,T)}\Big)\, dW \tag{20}$$

Cleaning things up a bit by using $a(t, T)$ and $b(t, T)$ as indicated gives equation (8).