## Analysis of Barclay's Capital HPI-RPI Swap

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Barclay's Capital wishes to enter into a 15 year swap on the *difference* between the growth of the British Retail Price Index and the British Housing Price Index over the interval. Let the level of the two indices at time $t$ be denoted $P(t)$ and $H(t)$ respectively. The underlying for the swap is[1]

$$Z_T \equiv \frac{P_T}{P_0} - \frac{H_T}{H_0} \tag{1}$$

Current time is 0 and $T$ is 15 years. Note that $Z_0 = 0$. Payoff per unit of notional by Chubb at $T = 15$, under Option A of the IDM, is specified as

$$C(T, H_T, P_T) = \max\{ 0 , Z_T - k \} \tag{2}$$

where $k$ is a deductible (possibly 0) to be negotiated. The theoretical value of this option today is

$$C(0, H_0, P_0) = \mathbf{E}_0^* [ R_T C(T, H_T, P_T) \mid H_0, P_0 ] \tag{3}$$

in which $R_t \equiv e^{-\int_0^t r(s)\, ds}$ is the riskless rate discount factor over an interval $[0, t]$ and the conditional expectation indicated is under the risk-neutral probability measure over $r, H, P$ paths. We need some way of determining the distribution of $Z_T$ to compute the expectation in (3). Data available are time series of $H$ and $P$.

### 1. Simplifications and assumptions

First, define the ratio of the two indices as a new state variable

$$X_t \equiv \frac{H_t}{P_t} \tag{4}$$

The underlying of the swap can be written in terms of this as

$$Z_t = (1 - \frac{X_t}{X_0})\frac{P_t}{P_0} \tag{5}$$

The reason for doing this is that we can more plausibly conjecture processes for $X$ and $P$ than for $H$ and $P$ directly. As a general price level, we suppose that $P$ has no mean reverting tendency. We assume the simplest non-negative process for it, a lognormal diffusion. Letting lower case letters denote logs of state variables, $p_t \equiv \ln P_t$,

$$dp_t = \alpha\, dt + \sigma_1\, dz_1 \tag{6}$$

[1] The sign of the expression below is reversed form that in the IDM to simplify notation below. Also at this point, $H_T$ is taken as synonymous with $H(T)$, etc.

$X$ on the other hand is the relative price of housing to other goods, and is more likely to have a long run average level by arguments from standard demand theory. Hence, letting $x_t \equiv \ln X_t$, let us conjecture

$$dx_t \;=\; \kappa(\bar{x} - x)\, dt + \sigma_2\, dz_2 \tag{7}$$

with $\kappa > 0$ a coefficient of mean reversion, $\bar{x}$ the long-run (log) relative price of housing to other goods, and there being an instantaneous correlation coefficient $\rho$ between the two standard Weiner processes $dz_1, dz_2$.

At this level of analysis, we will assume that the riskless rate process $r_t$ is independent of these other state variables, so that the expected 15 year riskless rate discount factor can be taken outside the **E** operator. The current British 15 year zero coupon bond price should be used in its place. Although independence of nominal interest rates from inflation rates is clearly not the case, note that our simple process for $P$ implies a constant expected inflation over time. A two factor price level process would be required to accomodate fluctuating inflation expectations.[2]

The processes chosen here imply that $x_t, p_t$ will be joint normally distributed over intervals. This will make feasible the estimation of the six process parameters $\alpha, \kappa, \bar{x}, \sigma_1, \sigma_2, \rho$ and the simulation of swap payoffs in a consistent fashion.

It would have nice if the two state original state variables could have been compressed into one for all this. Unfortunately that is not possible unless either the general inflation rate $P_T/P_0$ over the period were known, in which case only the relative price $X$ would be stochastic, or the relative price of housing to other things $H_T/P_T$ were known (close to perfect correlation in the two indices), in which case only general inflation would be stochastic. Casual inspection of the data series suggests neither is true. And the swap payoff involves their product.

## 2. Parameter estimation

Without going into details here, let me assert that the distribution of $x, p$ over an interval of length $h$ satisfies[3]

$$p_{t+h} \;=\; p_t + \alpha h + \epsilon_1 \tag{8}$$

[2]An additional reason for defining $X$ the way it is is to put the less volatile state variable $P$ in the denominator, which should reduce any damage done by our mis-specification of the joint process for $H$ and $P$.

[3]See Robert Jones, "Estimating Correlated Diffusions", Simon Fraser University working paper (1999) and grind through details.

$$x_{t+h} \quad = \quad e^{-h\kappa}x_t + (1 - e^{-h\kappa})\bar{x} + \epsilon_2 \tag{9}$$

where the random components $(\epsilon_1, \epsilon_2)'$ are joint normally distributed with mean 0 and covariance matrix

$$\begin{bmatrix} h\sigma_1^2 & \frac{(1-e^{-h\kappa})}{\kappa}\rho\sigma_1\sigma_2 \\ \frac{(1-e^{-h\kappa})}{\kappa}\rho\sigma_1\sigma_2 & \frac{(1-e^{-2h\kappa})}{2\kappa}\sigma_2^2 \end{bmatrix} \tag{10}$$

These relations will be use in two ways: in estimating parameters from the time series data, with $h$ equalling 1.0 or .25 year for annual or quarterly data, and in simulating terminal price levels, with $h$ equalling 15 years.

From the above covariance matrix one can also compute the correlation between $p_{t+h}$ and $x_{t+h}$. This turns out to be

$$\rho\,\frac{(1 - e^{-h\kappa})2^{1/2}}{(1 - e^{-2h\kappa})(h\kappa)^{1/2}} \tag{11}$$

which approaches $\rho$ as $h \to 0$ and 0 as $h \to \infty$. Because of the latter, and because our interest is in the joint distribution for large $T$ (15 years), we will treat the realizations of $x$ and $p$ that far out as independent, and ignore $\rho$ in the estimation and simulation.

This reduces the estimation to performing two independent linear regressions. Estimate the coefficients in the following:

$$p_{t+1} \quad = \quad p_t + a + \epsilon_{1t} \tag{12}$$

$$x_{t+1} \quad = \quad bx_t + c + \epsilon_{2t} \tag{13}$$

where $t$ and $t + 1$ now denote successive observations in the data series. Note that logs have been taken of the raw price indices. The first regression is done by simply computing the sample average and standard deviation of $p_{t+1} - p_t$. Denote the estimates $\hat{a}, \hat{b}, \hat{c}, s_1^2, s_2^2$ respectively (the latter are variances of the residuals). Maximum likelihood estimates of the continuous time parameters obtain by solving the relations

$$\begin{aligned} a &= \alpha h \\ b &= e^{-h\kappa} \\ c &= (1 - b)\bar{x} \\ s_1^2 &= \sigma_1^2 h \\ s_2^2 &= (1 - b^2)\sigma_2^2/2\kappa \end{aligned}$$

where $h$ is the number of years between observations (e.g., 1.0 or .25).

Doing this gives

$$\begin{aligned}
\alpha &= \hat{a}/h \\
\kappa &= -\ln \hat{b}/h \\
\bar{x} &= \hat{c}/(1-\hat{b}) \\
\sigma_1^2 &= s_1^2/h \\
\sigma_2^2 &= -2\frac{s_2^2 \ln \hat{b}}{(1-\hat{b}^2)h}
\end{aligned}$$

I suggest doing this separately both with annual and quarterly observation intervals to see how well they agree (they should). In case of disagreement, indicating serial dependence of the random increments over time, I would favor the annual estimates data for our purposes.

## 3. Swap payoff simulation

Armed with the continuous time process parameters from above, simulate $P_T$ and $H_T$ as follows. Draw two independent standard normal variables $\nu_1, \nu_2$. Taking $h$ equal to 15, multiply the first by the square root of the first diagonal element of (10) and the second by the square root of the second diagonal element. Use these as $\epsilon_1$ and $\epsilon_2$ in equations (8) and (9) to give a random realization of $p_T, x_T$. I.e.,

$$\ln(P_T/P_0) = \alpha h + \epsilon_1 \tag{14}$$

$$\ln(X_T/X_0) = (1-e^{-h\kappa})(\bar{x}-x_0) + \epsilon_2 \tag{15}$$

Take the exponential of these two numbers (un-log them), and use the fact that $H_T = X_T P_T$ to get the realizations $P_T/P_0$ and $H_T/H_0$ needed to compute the swap payoff. Average the payoffs over a large number of simulations. Remember to discount the payoff by the 15 year British zero rate.

The option value so obtained will be based on the *historical* process for inflation and relative housing prices. There are possible problems here for valuation purposes. First, the Halifax housing price index may be constructed in a way that gives a consistent growth bias one way or the other (i.e., selection bias from it not being a fixed set of houses being valued). Second, relative housing prices appear (casually) to be procyclical, suggesting it has been a positive-$\beta$ asset class, and that the observed historical increase of $H$ relative to $P$ reflects an appropriate risk premium. Risk-neutral pricing of the swap should try to remove this. The above analysis does this to some degree by assuming a fixed historical relationship implied by $\bar{x}$. If we wish to further allow for a positive-$\beta$ type risk premium, that can be accomplished by *reducing* the $\bar{x}$ used in our simulation below that obtained from the historical estimation.