Efficient Portfolio Choice

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This note sets out the solution to the problem of choosing a mean-variance efficient portfolio in a one period context. This problem was first set out and solved by Harry Markowitz (1952) and James Tobin (1958). It forms the basis for the Capital Asset Pricing Model of Sharpe (1965) and Lintner (1967).

1. Decision environment

Consider an investor who must allocate wealth between a risk-free asset and n risky assets for one period. Return on the risk-free asset is r per period. Let us decompose the return on risky asset i into

$$R_i \equiv r + p_i + \epsilon_i \tag{1}$$

where p_i denotes the expected return on asset i relative to the risk-free rate, and ϵ_i is a random variable with expected value 0. Let $p = (p_i)$ and $\epsilon = (\epsilon_i)$ be the column vectors of these variables. The covariance matrix of risky asset returns is

$$\Omega \equiv [\omega_{ij}] = \mathbf{E}[\epsilon\epsilon'] \tag{2}$$

in which ϵ' denotes the transpose of ϵ.

Assume the investor cares only about expected wealth and its variance as of the end of the one period. Let α_i denote the share of initial wealth invested in asset i, and α the vector of such shares. Any wealth not invested in a risky asset is invested in the risk-free asset. The expected return and variance of the resulting portfolio are

$$\mathbf{E}[\text{return}] = r + \alpha'p \tag{3}$$

$$\mathbf{Var}[\text{return}] = \alpha'\Omega\alpha \tag{4}$$

The investor's problem is to choose α to minimize the variance of portfolio return for any given level of expected return. Let a candidate level of expected return be denoted by g.

2. Solution to the problem

Formally, the investor's problem is choose α to minimize $\alpha'\Omega\alpha$ subject to the constraint $g = r + \alpha'p$. Define the Langrangian expression

$$L = \alpha'\Omega\alpha + \lambda(g - r - \alpha'p) \tag{5}$$

The first order conditions for a minimum are

$$L_\lambda \quad = \quad g - r - \alpha'p = 0 \tag{6}$$

$$L_\alpha \quad = \quad 2\Omega\alpha - \lambda p = 0 \tag{7}$$

Solving (7) for α gives

$$\alpha = \frac{\lambda}{2}\, \Omega^{-1}p \tag{8}$$

Multiplying on the left by p' and substituting $g - r$ for $p'\alpha$ from equation (6) allows one to solve for λ. Substituting this back into (7) gives the optimal portfolio shares

$$\alpha^* = \frac{g - r}{p'\, \Omega^{-1}p}\, \Omega^{-1}p \tag{9}$$

The minimized portfolio variance achieved is

$$\sigma^2 \equiv \mathbf{Var}^* = \frac{(g - r)^2}{p'\, \Omega^{-1}p} \tag{10}$$

3. Solution properties

The optimal portfolio shares have several noteworthy properties.

- Regardless of the value of g, risky assets are held in the same relative proportions, given by the vector $\Omega^{-1}p$. The risky asset portfolio shares are directly proportional to $g - r$. At $g = r$, α^* is the zero vector and all is put in the risk-free asset. As g is increased, more is put into this 'mutual fund' of risky assets. As g is increased still further, the sum of the portfolio shares will exceed 1, indicating negative amounts in the risk-free asset and hence a leveraged portfolio.

- The minimized portfolio *standard deviation* is the square root of $\mathbf{Var}^*$, and hence directly proportional to $g - r$. The graph of expected portfolio return against this standard deviation is thus a straight line with vertical intercept at r and slope $(p'\Omega^{-1}p)^{-1/2}$ — the so-called security-market line.[1]

- The relative proportions in which risky assets are held are directly proportional to their respective excess returns p_i. If the risky asset returns are uncorrelated (diagonal Ω), an asset's share is inversely proportional to its variance.

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[1]Note: The solution to the problem of maximizing expected return for a given variance of return is the same as minimizing variance for given expected return. This line portrays the solution to either problem. The maximum g attainable for given σ is $r + (p'\Omega p)^{1/2}\sigma$.

4. Probability returns are below some threshold

If we make further assumptions about the joint distribution of asset returns, we can also compute the probability of realized returns falling below any desired threshold. For example, if asset returns are joint normally distributed, then the portfolio return is normally distributed with mean g and standard deviation σ^*. The probability of return being below some level k is then

$$F(\frac{k-g}{\sigma^*}) \tag{11}$$

where F is the standard normal cumulative distribution function. Note that this will not be the probability if joint asset returns have some different distribution.

5. Spreadsheet illustration

The Excel spreadsheet EfficientPortfolio.xls computes the efficient portfolio shares of four risky assets plus the riskless asset. User enters asset characteristics and target expected excess return in the fields colored yellow. $\mathbf{E}[R]$ refers to $g - r$, the desired expected excess return over and above the riskless rate. All data can be thought of as annualized rates. Risky asset characteristics are their individual excess expected returns (these are the p_i), their individual *volatilities* of return (these correspond to $\omega_{ii}^{1/2}$), and the *correlations* between pairs of asset returns. Volatilities here must be stricly positive.

   Outputs consist of some intermediate results, the optimal relative proportions in which the risky assets should be held, the optimal proportions of the initial portfolio to invest in each risky asset and the riskless asset, and the resulting standard deviation of the overall portfolio's return. Negative share for the riskless asset means borrowing to purchase more risky assets. Negative share for any risky asset means short-selling that security.

   Also computed is the probability of returns falling below any given cutoff level. Cutoff level here is a level of realized *excess* returns. It corresponds to $k - r$ above. Hence for a cutoff of 0 gross return, enter a value of $-r$.

6. Presence of non-adjustable assets or liabilities

Let us extend the basic problem by introducing some assets or liabilities whose positions cannot be adjusted. Their quantities are assumed fixed by the nature of other business activities (e.g., writing insurance or regulatory constraints). The objective is to choose the flexible asset portfolio to minimize the variance of total return inclusive of the fixed assets/liabilities subject to a target expected return on the flexible asset portfolio alone. This target continues to be denoted by g.

Note that the risk accepted by writing a insurance contract can be viewed as an asset with certain cash inflows equal to the stream of premiums, and uncertain cash outflows equal to claims. I.e., its risk primarily follows from the uncertain claims liability. Consequently we will refer to these fixed positions as liabilities, and the flexible positions as assets, in what follows.

Let their be m types of fixed liabilities, with amount of liability j denoted by β_j. These are expressed as proportions of total *flexible assets* being invested.[2] Let the random component of the proportional return on liability j be ϵ_j, just as for the risky assets, and the expected excess return component be q_j. Variance of total return on assets plus liabilities can be expressed as

$$\mathbf{Var} = (\alpha', \beta') \begin{bmatrix} \Omega_{\alpha\alpha} & \Omega_{\alpha\beta} \\ \Omega_{\beta\alpha} & \Omega_{\beta\beta} \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha'\Omega_{\alpha\alpha}\alpha + 2\alpha'\Omega_{\alpha\beta}\beta + \beta'\Omega_{\beta\beta}\beta \quad (12)$$

In the above partitioning of Ω, the $n \times m$ matrix $\Omega_{\alpha\beta}$ is the matrix of covariances of each asset type with each liability type. Ω is now of size $n + m$.

Minimization of this expression with respect to α, subject to the constraint $g = r + \alpha'p + \beta'q$, results in optimal portfolio shares

$$\alpha^* = \frac{\lambda}{2} \, \Omega_{\alpha\alpha}^{-1} p \, - \, \Omega_{\alpha\alpha}^{-1} \Omega_{\alpha\beta}\beta \quad (13)$$

where

$$\frac{\lambda}{2} = \frac{g - r - \beta'q + p'\Omega_{\alpha\alpha}^{-1}\Omega_{\alpha\beta}\beta}{p'\Omega_{\alpha\alpha}^{-1}p} \quad (14)$$

The last term in equation (13) is the portfolio of assets which, when combined with the fixed liabilities without constraint, gives the minimum variance of overall return. It is now this portfolio (rather than one consisting solely of the riskless asset) which gives minimum overall variance. The Lagrange multiplier λ has interpretation of being the increment to overall risk per unit of additional asset portfolio expected return. Note that the optimal proportions in which to *add* assets to increase expected return is still $\Omega_{\alpha\alpha}^{-1}p$ — the same as in the previous sections.

7. Hurdle rates of return on incremental liabilities

In the above, the size of positions referred to as liabilities were taken as a constraint. However one can ask what is the effect on the minimized risk of an increase in the level of these constraints. I.e., what is the effect on σ^* of a unit increase in β_j? Once that is established,

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[2]Since many of these 'liabilities' involved zero cash outlay or receipt at inception, it is not entirely clear what we mean here. Further thought is needed to clarify how such liabilities are being measured.

one can then further ask, what is the expected excess return on liability j that leaves one just indifferent to a unit increase in β_j? This can be interpreted as the minimum *hurdle rate of return* that liability j would have to offer in the sense that taking it on would neither increase overall risk or reduce overall expected return.

The Langrangian for the minimization problem of the previous section is

$$L(\alpha, \lambda; \beta) = \alpha'\Omega_{\alpha\alpha}\alpha + 2\alpha'\Omega_{\alpha\beta}\beta + \beta'\Omega_{\beta\beta}\beta + \lambda(g - r - \alpha'p - \beta'q) \qquad (15)$$

Using the Envelope Theorem of constrained optimization theory, the total derivative of the optimized objective function with respect to any fixed parameter such as β is simply the partial derivative of L with respect to that parameter. Hence the vector of derivatives of the minimized variance with respect to the various fixed positions is

$$\frac{\partial \mathbf{Var}^*}{\partial \beta} = 2\Omega_{\beta\alpha}\alpha + 2\Omega_{\beta\beta}\beta - \lambda q \qquad (16)$$

where λ is as given in equation (14). Generally, these derivatives are non-zero: a positive value indicates one would prefer to hold less of liability j, a negative value indicates one would prefer to hold more. Let us define the hurdle rate for liability j as the q_j needed for $\partial L/\partial \beta_j$ to be 0. That is, one is indifferent to small increases or decreases in β_j. The vector of hurdle rates is found by setting (16) equal to the 0 vector and solving for q.

$$q = \frac{\Omega_{\beta\alpha}\alpha + \Omega_{\beta\beta}\beta}{\lambda/2} \qquad (17)$$