CHUBB FINANCIAL SOLUTIONS

Credit risk equivalent exposure: some new thoughts

Financial institutions, insurance companies and other participants in the structured credit derivatives market are acquiring exposure to corporate credit risk in a complex fashion. We describe an approach, to quantifying credit exposure emanating from structured credit products, which permits comparison to conventional measures

How does one reconcile the credit exposure to individual issuers acquired by managing a portfolio of credit derivatives with the credit exposure acquired by more traditional means? Managers of fixed income portfolios typically characterise individual exposures in terms of the aggregate notional amount of debt held.[1] Risk managers of financial institutions usually include, within their arsenal of risk controls, limits on the notional amount of debt that portfolio managers can acquire for any given issuer.[2] These limits are often increasing in the credit quality of the issuing firm.

For example, a portfolio manager may be permitted to acquire $100 million of notional exposure to a 'Aaa' issuer, but only $5 million to a sub-investment grade issuer. How does the risk manager ensure that her credit derivatives trading group is complying with the same individual exposure limits? In the case of single name credit default swaps the risk manager's task is reduced to aggregating notional amounts. However, when the exposure is acquired by means of portfolio default swaps, collateralised debt obligations (CDO's) and other 'basket' products the task becomes a little more demanding.

In this article, we outline an approach to characterising credit exposure, acquired in this fashion, which is comparable with traditional notions. Along the way, we will point out some counterintuitive results.

A measure of credit exposure
We proceed with the following definition of exposure:

☐ The credit risk equivalent exposure C_i to firm i is the amount of par floating-rate bonds of that issuer, with maturity T same as the credit structure, whose cost of protection against default is the same as the cost of protecting the structure against default by firm i.

By cost of protecting the structure against default by i, we mean the fair market value of a contract that pays, at each point in time and for each state of nature, the difference between what the structure pays and what it would have paid had i not defaulted. Cost here refers to single payment up front.

Note that this avoids the informational issues arising with the mark-to-market approach, since buying protection does not change the actual course of events. It can also be readily calculated if one already has a procedure for valuing the credit structure in the first place. For exam-

Table 1. Credit risk equivalents C_i

| Case		C_1	C_2	Total
(1) $\alpha < s$	$\beta < s$	$p_2(1 - s)$	$p_1(1 - s)$	$(p_1 + p_2)(1 - s)$
(2) $\alpha > s$	$\beta < s$	$\alpha - s + p_2\beta$	$p_1\beta$	$\alpha - s + (p_1 + p_2)\beta$
(3) $\alpha < s$	$\beta > s$	$p_2\alpha$	$\beta - s + p_1\alpha$	$\beta - s + (p_1 + p_2)\alpha$
(4) $\alpha > s$	$\beta > s$	$\alpha - s + p_2 s$	$\beta - s + p_1 s$	$1 - s + (p_1 + p_2 - 1)s$

[1]*This notion of exposure is not without its problems. For example, is the exposure on a $1 million 10-year bond equivalent to the exposure on $1 million of 3-month paper? The measure of exposure we describe here is not, in our opinion, subject to this criticism.*
[2]*These limits are generally intended to maintain an appropriate degree of diversity within the portfolio and are generally not substitutes for value-at-risk or mark-to-market based constraints to manage the risk of adverse changes in credit quality and other market variables.*
[3]*This is not the same as excluding all paths on which i defaults, which would lead to a value conditional on i not defaulting, raising the informational issue again.*
[4]*The correlation between defaults can be written as $[\pi_1\pi_2/(1 - \pi_1)(1 - \pi_2)]^{1/2}(1 - p_0)/p_0$.*

Figure 1. Exposure to firm 1

Figure 2. Total credit exposure
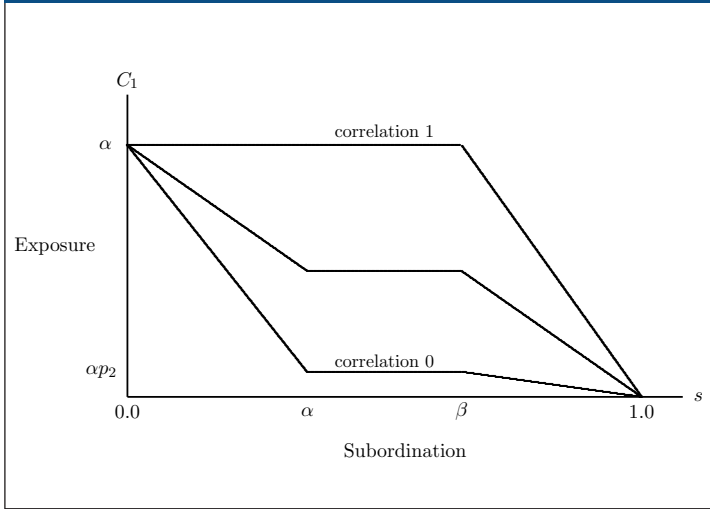
ple, if Monte Carlo simulation of default times was used for valuation, one need only force non-default by i along each path and determine the change in structure value (value of protection).[3]

In what follows, V_0 denotes the market value of the remaining credit structure conditional on events to time 0. T is the contractual maturity of the structure, π_i the risk-neutral probability of default by firm i in $[0, T]$, and δ_i the indicator of default by T for firm i (1 if yes, 0 if no). Let V_0^i denote value of the structure with payouts computed as if i never defaults.

Up-front cost of protecting the structure against default by i is $V_0 - V_0^i$. For expositional simplicity, we assume the riskless interest rate and recovery rates are 0. Hence the up-front cost of principal-only protection on one dollar face amount of floating rate notes equals π_i. Credit risk equivalent to firm i is therefore:

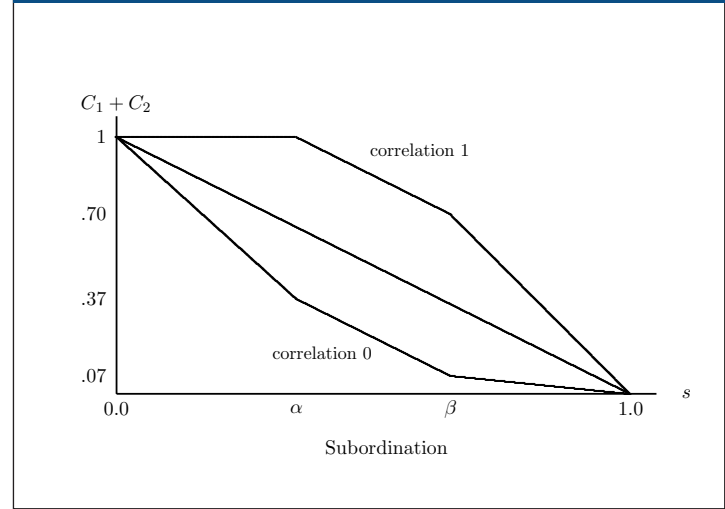$$C_i = \frac{V_0 - V_0^i}{\pi_i} \tag{1}$$

A simple setting

Let us set up a simple environment through which structured credit exposures can be analysed. Consider two firms, $i = 1, 2$, which may default. Let defaults be driven by the exogenous and independent binary random variables $\iota_0, \iota_1, \iota_2$, which take values of 1 with probabilities p_0, p_1, p_2 respectively and 0 otherwise. Default by firm i is given by $\delta_i = \iota_0 \iota_i$. For example, firm i defaults whenever the common factor ι_0 and the idiosyncratic ι_i are both 1. Probability of firm i defaulting is:

$$\mathbf{p}\{\ i \text{ defaults }\} = \mathbf{E}[\delta_i] = p_0 p_i \tag{2}$$

This setup permits (positive) correlation between defaults via the common factor. Conditional and joint default probabilities are readily determined:

$$
\begin{aligned}
\mathbf{p}\{\ i \text{ default}, j \text{ not default }\} &= p_0 p_i (1 - p_j) \\
\mathbf{p}\{\text{ none default}\} &= 1 - p_0 (p_1 + p_2 - p_1 p_2) \\
\mathbf{p}\{\ i \text{ default } | \ j \text{ default }\} &= p_i
\end{aligned}
$$

The case of independent defaults (no correlation) obtains with $p_0 = 1$. Default correlation is positive with $p_0 < 1$ and maximal with $p_0 = \max\{\pi_i\}$.[4] In this setting, we examine credit exposure from a simple portfolio credit derivative to get some sense of how the pieces fit together.

Portfolio credit default swap

Consider a portfolio credit default swap where the portfolio consists of $1 total notional amount split between two names (firms). Notional amount for firm 1 is α; for firm 2 is $\beta = 1 - \alpha$. We hold a senior tranche with subordination $0 \leq s \leq 1$. Assume zero recovery rate in event of default. Assume the riskless discount rate is 0. At time 0 neither firm has defaulted. By time T it will be revealed whether firm 1 has defaulted, firm 2, both or neither. Then the swap pays any excess of total default losses over s. The value V_0 at time 0 of this swap is the expected payout under the appropriate risk-neutral measure. Hence, using $(x)+$ as shorthand to denote $\max(0, x)$,

$$
\begin{aligned}
V_0 &= \mathbf{E}[(\ \alpha \delta_1 + \beta \delta_2 - s\ )^+] \\
&= (1 - s) p_0 p_1 p_2 + p_0 p_1 (1 - p_2)(\alpha - s)^+ + p_0 p_2 (1 - p_1)(\beta - s)^+ \\
&= p_0 p_1 (\alpha - s)^+ + p_0 p_2 (\beta - s)^+ + p_0 p_1 p_2 (1 - s - (\alpha - s)^+ - (\beta - s)^+)
\end{aligned}
\tag{3}
$$

Note that as we raise subordination s, value unambiguously falls. If $s < \min(\alpha, \beta)$, so that any default eats through the subordination, value simplifies to $\pi_1(\alpha - s) + \pi_2(\beta - s) + p_0 p_1 p_2 s$. With zero subordination, only the unconditional default probabilities – not the correlation – matter for swap value. But with positive s, value monotonically increases with correlation (represented as fall in p_0 keeping fixed π_1, π_2).

Turn next to credit risk equivalents. Value of the swap if firm 1 is default-free (setting $p_1 = 0$) reduces to $V_0(0) = p_0 p_2 (\beta - s)^+$. Credit exposure to firm 1 using our measure becomes, after simplification:

$$
\begin{aligned}
C_1 &= \frac{V_0 - V_0^1}{\pi_1} \\
&= p_2 (1 - s) + (1 - p_2)(\alpha - s)^+ - p_2 (\beta - s)^+
\end{aligned}
\tag{4}
$$

Note that the only probability entering the final expression is p_2, which is the conditional probability of default by 2 given default by 1. Neither the probability of default by 1, nor the unconditional probability of default by 2, otherwise matters. One can show that (4) is increasing in p_2. Hence raising correlation keeping default probabilities fixed (by equiproportionately raising p_1, p_2 while shrinking p_0) unambiguously raises credit expo-

With sufficient subordination in the structure, it is possible for exposure to a single issuer to exceed the notional amount associated with that issuer!

sure to firm 1. Changing credit quality through changing probability p_0 of the common factor event has no impact on credit exposure.

However, it is also clear that the relation between credit exposure and contract parameters α, s varies with the direction of various inequalities. There are four regions, according to whether s is greater or less than α and β. Table 1 gives the credit exposure to both firms and the total for each case.

Case (1) can be termed the high subordination case, since s is sufficient to cover losses on any one firm defaulting. Case (4) is the low subordination case. With just two firms, (1) implies $s > .5$ while (4) implies $s < .5$. Observations, which are at times counterintuitive, can be drawn from table 1.

☐ When the portfolio consists of just one bond, exposure equals notional minus subordination (case (2) with $\alpha = 1$).

☐ In the low subordination case, credit exposure to 1 equals its notional amount less a portion of s. The portion is the probability that 2 does not default, given that 1 defaults.

☐ Exposure to firm 1 is always higher, the lower is the credit quality of firm 2 (ie, higher is p_2). Essentially, more of the subordination is being 'used up' by firm 2. If firm 2 is riskless, exposure to 1 is its notional less the subordination.

☐ In the high subordination case, credit exposure is independent of the division of notional amount between firms in the structure. This is because default by firm i only results in payout if both firms default, which leads to payout $1 - s$. Thus the protection of the structure against default by firm 1 involves paying out $1 - s$ with probability p_2, the conditional probability of 2 defaulting given 1 defaults, regardless of the size of α. Thus exposure using this measure can exceed notional amount.

☐ The individual credit exposures add up to the maximum payout $1 - s$ only if $p_1 + p_2 = 1$. For reasonable default probabilities, this -

corresponds to an intermediate level of correlation. For example, half the time that one defaults, the other defaults. For lower correlation levels, total exposure is less than the total notional less subordination. With zero correlation, total exposure equals $1 - 2s + (\pi_1 + \pi_2)$. Hence for very low default probabilities, total exposure equals total notional less twice the subordination amount.

Finally, we explore how the subordination s influences credit exposure. Consider first individual exposure to firm 1. Using measure C_1 and starting from $s = 0$, exposure just equals the notional α. Assume that $\alpha < \beta$ (firm 1 has the smaller notional), so that we progress from case (4) to case (3) to case (1) as s increases. As s rises, exposure initially falls at the less than dollar-for-dollar rate $1 - p_2$ until s just equals α. Further increases in s cause no change in exposure to firm 1 until it just equals β. After that, exposure falls at the rate p_2 until it reaches 0 when subordination equals 1. Figure 1 graphs the relation for the case $\pi_1 = \pi_2 = .10$ with $\alpha = .35$, $\beta = .65$. The 0, mid, and 1 correlation cases correspond to pi values of .1, .5, and 1, respectively (with p_2 respectively 1, .2 and .1).

Now consider total credit exposure to both names. Proceeding again from case (4) to case (3) to case (1), total exposure starts at $1 - s$ and initially falls with s at rate $2 - p_1 - p_2$. This rate varies from just about 2 if defaults are independent and of low probability (each dollar of subordination reduces exposure to each name by approximately one dollar since joint defaults are unlikely), to barely above zero if defaults are almost perfectly correlated ($p_1 = p_2 = 1$, $p_0 = \pi$). Once s exceeds α, total exposure falls one-for-one with s. And once s exceeds both notionals, total exposure falls at rate $p_1 + p_2$, which can range from 0 to 2 depending on correlation. Figure 2 graphs the relation for the parameters as in figure 1. Note that when $p_1 + p_2$ exceeds 1, the total measured exposure exceeds the maximum possible payout under the structure!

The general observation is that increased subordination is distributed across names in a way that reduces their credit exposure from full notional to 0 as s increases to 1. The initial rate of reduction is smaller, the more highly correlated are defaults (higher is p_2). ■