

Credit Risk and Credit Rationing

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Abstract

Rational lender behaviour is examined in the market for secured loans with no moral hazard and with symmetric information between contracting parties. The analysis presumes collateral value follows a diffusion process. The focus is on the option of the borrower to control the *timing* of default and prepayment. Results are obtained analytically and by numerical methods to illustrate optimal default policies and lender response.

Qualitative results include: (1) An upper limit on what a rational bank would lend that is a modest fraction of the current collateral value, regardless of the interest rate the borrower offers. (2) A loan supply curve to a particular borrower can bend backward, with the bank preferring a lower loan rate over a higher one. (3) Collateral promising low cash flows is preferable to collateral producing high cash flows. (4) The amount lendable is highly sensitive to the collateral's perceived volatility, even under risk neutrality, suggesting cyclical fluctuation in the availability of credit.

If such credit rationing-like phenomena naturally occur *without* information asymmetry or moral hazard, then there is less reason to suspect market failure requiring government action. If action is called for, it suggests policies of *removing* regulatory restrictions on the enforceable forms loan contracts can take.

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I. Introduction

Why can agents not borrow as much as they wish, even though willing to pay more than the going interest rate? Answers to this question over the last decade are dominated by explanations involving information asymmetries and moral hazard.¹ This paper re-examines lending when borrower and lender have identical information at the time a loan is negotiated and when the borrower cannot influence the subsequent riskiness of his activities. The object is to provide a benchmark analysis of credit relations based on option value alone. These are present in all loans, whether information asymmetries reasonably exist or not. The premise is that many phenomena thought of as ‘credit rationing’ can be best understood in such terms.²

This paper does not address the question of why the standard, non-contingent loan contract is so prevalent. What will be displayed is simply the limit imposed on agents when contracts are restricted to this form. More elaborate contingent contracts would permit funds to be advanced if economically efficient to do so. But alternative arrangements may be infeasible because of regulation (restrictions on banks taking equity positions), uncertain legal enforceability, or high monitoring costs. Why these impediments exist *may* rest on information asymmetries. But, rather than reflecting differential information between contracting parties, it is more likely to reflect the cost of verifying the state to unconnected third parties—e.g., to a court—for contract enforcement. Our focus on standard loan contracts is thus in the spirit of Williamson (1986), motivated by an implicit notion of costly state verification. Though far from unique, loans remain the most common contract through which agents exchange purchasing power across time.

Beyond providing an alternative view on credit rationing, the analysis sheds light on loan contract design, on why the credit granting process can matter for the propagation of shocks and policy changes, and on whether there could be substance to the notion of a credit crunch. More importantly, if such phenomena indicate misallocation of resources by credit markets, and if they naturally occur *without* information asymmetry or moral hazard, then the appropriate public policy remedy differs. Action, if any is called for,

¹Stiglitz and Weiss (1981). See Jaffee and Stiglitz (1990) for a comprehensive survey.

²The term credit rationing loses some of its force upon noting it is the lender that is the purchaser in the transaction (handing over money) and the borrower that is the seller (of his bond).

is along the line of removing regulatory restrictions on the form loan contracts can take (permitting banks to take equity positions in risky borrowers, legally upholding substantial prepayment penalties, and so on).

The contract we consider is the multiperiod, specified collateral, non-recourse loan: A lender advances a sum to a borrower, who promises to make a scheduled stream of payments. If the borrower fails to make a payment as agreed, the lender seizes the collateral and sells it, refunding any excess over the outstanding loan balance to the borrower. The lender has no claim on other assets of the borrower.³ The initial market value of the collateral is known to both agents; the future value is uncertain. We seek to determine the maximum *fraction* of initial market value a rational lender will advance against given collateral, with particular interest in situations where that fraction is noticeably less than one, regardless of the interest rate the borrower offers to pay. That is ‘credit rationing’ for our purposes.

The multiperiod nature of the loan contract introduces two considerations not present with single period loans. The first consideration is that the borrower can receive dividends and/or service flows from owning the collateral as long as he makes his loan payments.⁴ The value of past flows is not retrievable by the lender in the event of default. Although the prospect of such flows contributes to the *initial* market value of the collateral, that value is effectively ‘not there’ as security to the lender. The relevant value from the lender’s perspective is, loosely speaking, the market value of the collateral at the (unknown) time of default, discounted to the present. For a given initial value, this will be lower the higher are the intervening dividend flows.

The second consideration is that the option to default belongs to the borrower. He controls the *timing* of its exercise, and will do so in his own best interest. There is thus an adverse incentive problem, though not of the ‘increased risk taking’ variety. A lender would always prefer a borrower to follow a policy of defaulting at the last moment the loan could be fully repaid by the liquidated collateral. But if there is some chance the collateral

³This arrangement is explicit with, say, a non-recourse residential mortgage. It is implicit if legal costs or adverse publicity render it unprofitable for a lender to pursue its claim further, or if bankruptcy laws limit what might be collected. Alternatively, if the loan is to a corporation, the collateral could be all of the corporation’s assets, but there is no further claim on shareholders.

⁴These include the marginal contribution to output of productive capital and the implicit rental value of property, in addition to cash payments in the case of financial assets.

might rise in value, the borrower would generally prefer to wait longer (especially if the loan payments are less than the ‘dividend’ flows). In the absence of monitoring, any promise by the borrower to do otherwise would not be credible. Ironically, the moral hazard is that the borrower will not default soon enough. These two considerations—the irretrievability of past service flows and the control of default timing by the borrower—provide a basis for credit rationing under symmetric information.

The model is set in continuous time, using arbitrage-free valuation principles familiar from modern asset pricing theory. Riskfree interest rates are assumed constant. Fluctuations in the market value of the collateral are exogenous to the agents, but can be hedged, if they choose, at the prevailing market terms for such insurance. Opportunities for renegotiation of the loan contract are ignored. Both parties are assumed to rationally exercise the options available to them and to anticipate the strategy of the other. We focus primarily on option of the borrower to default. But we also consider the effect of options of the borrower to pay off the loan early, if it could be competitively refinanced on more attractive terms, and of the lender to call for immediate repayment if monitoring reveals a ‘technical default’ has occurred.

The advantages of working in a dynamically complete market, symmetric information setting are compelling. It permits one to obtain equilibrium option exercise strategies and contract values that are independent of the risk attitudes, personal circumstances and expectations about future collateral value of the contracting parties. It enforces consistency between collateral characteristics such as cash flows and rate of change in value. And it offers the promise of more tractable analysis of the welfare implications of regulatory policy. It thus provides the natural benchmark from which to see if, or where, information asymmetries are *needed* for the understanding of credit markets.

A considerable literature on the pricing of corporate debt subject to default and/or early prepayment already exists.⁵ The analysis here both varies the assumptions and changes the focus from these works. The original treatment of Merton (1974) examines a situation with no liquidation or bankruptcy costs, so that the Modigliani-Miller theorem holds: The value of the collateral in the hands of the lender and in the hands of the borrower are identical. The existence of a difference between these values plays a critical

⁵See, for example, Merton (1974), Brennan and Schwartz (1977), Stultz and Johnson (1985). Merton (1990, chap. 13) provides a brief survey and pointers to others.

role in our results. Further, Merton's bond indentures force equality between the dividend/service flows generated by the collateral (taken to be the entire value of the firm) and contractual debt payments.⁶ In our analysis, collateral cash flows are distinct from contractual debt payments. The borrower's decision whether or not to provide additional cash to make up any deficiency as it occurs, from resources outside the loan agreement, becomes the central determinant of default.

In terms of focus, the corporate finance literature is primarily concerned the equilibrium value and (promised) yield to maturity of a bond with given characteristics. Our question is whether there exists any coupon rate which could make a bond have a pre-specified value (permit a loan request to be granted). The two problems are obviously mirror images of each other; but the answer to the latter has not been explored and is the more relevant for apparent credit rationing.

Section II sets out the parameters of the loan contract and principles determining the value to each party. Section III provides analytical results on the extent of credit rationing for loans which only default at maturity and for perpetual loans. Section IV uses numerical methods to examine loans in which both parties have options whose rational exercise depends on the term remaining.

II. The lending environment

1. The loan contract

Consider a loan in which the lender advances \$1 now. The borrower promises to make continuous payments at a rate of p per year followed by a lump-sum payment P at maturity date T . The contractual loan rate is c per year (continuously compounded), by which we mean that the present value of the promised payments discounted at rate c equals the loan amount. The loan balance $b(t)$ outstanding as of time $0 \leq t \leq T$ is defined similarly. That is,

$$1 = p \int_0^t e^{-c\tau} d\tau + b(t)e^{-ct} \quad \text{for } 0 \leq t \leq T \quad (1)$$

⁶Since there is no reason why the firm's net income from operations should exactly equal the contractual debt payments, the firm is implicitly either drawing down inventories of liquid assets, which must also part of the 'collateral', or is incurring further debt senior to the debt being valued to make the payments. It is senior in that this value is assumed not later retrievable by the junior bond-holders in question.

and hence

$$b(t) = e^{ct} - (e^{ct} - 1)p/c \quad P = b(T) \quad (2)$$

Let $s(t)$ be the market value of the collateral at time t . Assume its equilibrium value over time follows a continuous Markov process described by

$$ds(t) = \alpha(s, t) dt + \sigma s(t) dz(t) \quad (3)$$

where $z(t)$ is a standard Brownian motion, σ is a constant volatility parameter and $\alpha(s, t)$ is the instantaneous expected drift in s . Possession of the collateral entitles the holder to dividend or service flows accruing at a rate $d(s, t)$. This flow is received by the borrower until such time as he defaults. If he defaults (does not make the contractual payments), the lender seizes the collateral and sells it for $s(t)$. The lender may incur foreclosure/liquidation costs $l(s, t)$.⁷ Any proceeds from the sale beyond the outstanding loan balance $b(t)$ are refunded to the borrower.⁸

Regarding the rest of the environment around the borrower and lender, we assume that they may trade continuously and without transaction costs in markets that are *complete* with respect to s -risk. By this we mean that at each instant there exist securities (or combinations of securities) that are locally perfectly correlated with s , permitting either party to hedge against the random part of the change in s if they so choose.⁹ For example, if s is the value of the assets of a borrowing firm, risk of their fluctuation might be hedged

⁷In addition to explicit legal, administrative and transaction expenses, liquidation costs can include the opportunity cost of receiving neither loan payments nor interest on the value of the collateral during the interval between default and the time the collateral is sold.

⁸For simplicity, in the numerical illustrations we let the flow of dividends and the liquidation costs be linear functions of the prevailing fair market value of the collateral: $d = d_0 + d_1 s(t)$ and $l = l_0 + l_1 s(t)$. This dividend function only applies for $s > 0$. A positive dividend flow when $s = 0$ would usually be inconsistent with s being an equilibrium price for the collateral. The expected drift $\alpha(s, t)$ is assumed to satisfy Lipschitz and growth conditions in s , uniformly in t , to assure existence and uniqueness of the stochastic process described by (3) (see for example Duffie (1992), p.239). But within these requirements s may be characterized as mean reverting, growing at a constant proportional rate, being (almost) confined within fixed upper and lower bounds, or a having a wide variety of other dynamic properties.

⁹More precisely, complete markets in this setting requires that any random variable with finite variance can be obtained as the terminal value of some self-financing trading strategy (Duffie, 1992, p.103f).

by selling short shares of publicly traded firms in the same industry. Additionally, both parties may trade in default free bonds that provide a continuously compounded yield r . This interest rate is constant over time.¹⁰

2. Options of borrower and lender

The principle option is that of the borrower to default. He may at any time cease making the contractual payments p , or at maturity fail to make the terminal payment P , and forfeit the collateral. Upon this occurring, the lender must liquidate the collateral immediately. We permit the lender to abandon the collateral if liquidation costs exceed its market value. The lender thus would receive $\max\{0, s(t) - l(s, t)\}$ if default occurs at time t .

A second option of the borrower is the right to pay off the loan prior to maturity by making a payment equal to the outstanding balance $b(t)$. If he does so, we suppose he incurs transaction costs $f(b)$ of negotiating a new loan to ‘refinance’ the old, in addition to incurring a new obligation whose market value equals the amount $b(t)$ of this second loan (i.e., we assume a competitive loan market). These refinancing fees are not received by the current lender.¹¹

Why would the borrower refinance in a fixed interest rate environment? The reason is that he is paying a premium $c - r$ over the riskfree rate to compensate the lender for default risk. If the value of the collateral subsequently rises the loan becomes less risky, and a new lender should be willing to accept a lower premium. If the benefit of the lower rate over the remaining term of the loan exceeds the refinancing costs, the borrower will exercise this option.

¹⁰Almost equivalently, for loans of moderate duration, one can view the loan as floating rate, with the contract calling for a constant spread $(c - r)$ above the prevailing default free rate at each instant. Contractual payments would adjust as r fluctuates to maintain the same balance schedule $b(t)$ as in the fixed rate situation. It may seem slightly contradictory to suppose the borrower can borrow elsewhere at default free rates. What is required for the arbitrage valuation argument to apply is that the agent in control of the default decision is in that situation. This could be the borrowing firm itself if the collateral was only a portion of its assets and the loan truly without recourse to its other assets; or it could be the shareholders of the firm, who are protected by limited liability.

¹¹For simplicity, in the numerical analysis we only consider refinancing costs of the linear form $f(b) = f_0 + f_1 b(t)$.

This prepayment option is a ‘bad’ from the lender’s perspective. In principle it could be eliminated by explicitly prohibiting prepayment in the original loan contract. However its legal enforceability is less clear. Unlike refinancings of fixed rate contracts that occur because the general level of interest rates has fallen, it would be more difficult for the lender to make the case he has been harmed by the actions of the borrower. He was charging a premium to compensate for the possibility of default; but it turned out that *this* borrower did not default. How was he harmed by this borrower? When we permit this option, we implicitly assume that the cost of pursuing the case is not worthwhile for the lender.¹²

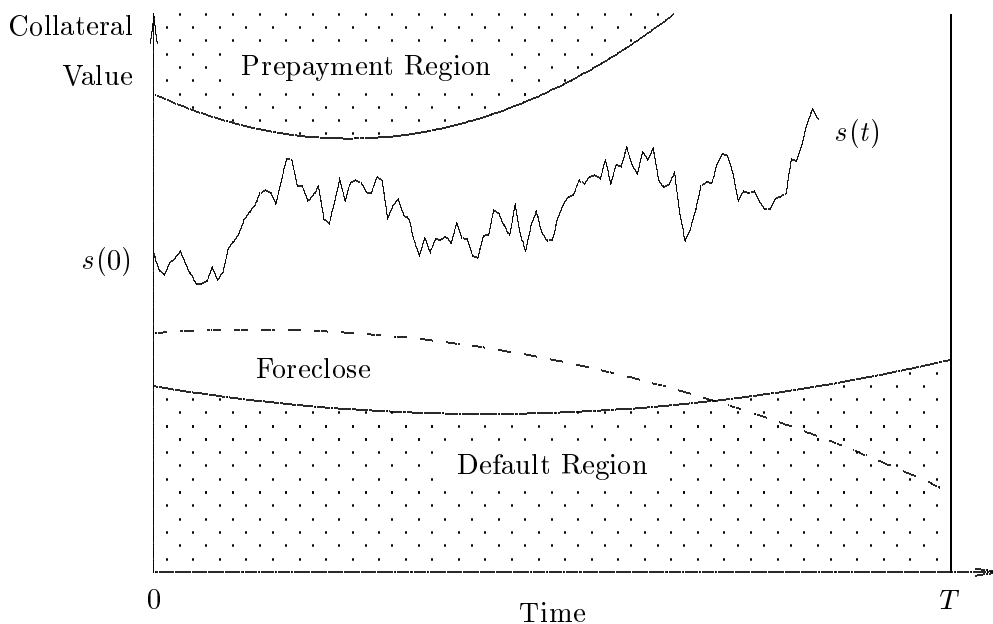
A further option we consider, because it is a common feature in one form or another of many loans, is that of the lender to declare a ‘technical default’. That is, the borrower is viewed as being in default though he has made all agreed upon payments. We suppose the provision takes the following form: If the value of the collateral falls below some fixed fraction δ of the outstanding loan balance, then the lender has the option to force liquidation immediately. This is like the maintenance margin requirement on security loans. Monitoring of the collateral value is costly (higher for some types of collateral than others), however, and not done continuously. We characterize monitoring as a Poisson process with fixed intensity m per unit time.¹³ That is, technical default occurs at time t only if monitoring occurs at that time and $s(t) \leq \delta b(t)$. Varying m alters how frequently the lender is assumed to verify the technical default condition. As one would expect, it can substantially affect how much a lender will rationally advance by counteracting the borrower’s incentive to postpone abandoning the collateral.¹⁴

¹²Of course there are instances of floating rate contracts where prepayment appears more enforceably restricted than in the case of bank loans. An example is publicly traded floating rate notes with specified call prices (above par) for early redemption.

¹³We do not incorporate explicitly the costs of monitoring, or address the question of optimal monitoring intensity.

¹⁴The Poisson process controlling monitoring is assumed independent of s and fully ‘diversifiable’ within the market environment. This means we can treat both agents as risk neutral with respect to this event.

Option Exercise Policies



3. Value to borrower and lender

Borrower and lender choose policies for exercising the options available to them. The loan terminates when either default occurs, prepayment occurs, or T is reached. Given the Markov nature of $s(t)$, and the fact that the loan payments are fixed conditional on the loan not having terminated, we assume the current value of s embodies all relevant information upon which their current actions are based. Letting $S \equiv [0, \infty)$ denote the range of values s can take, and $T \equiv [0, T]$ denote the range of t during the loan, $S \times T$ is the set of possible (information) states for their decision problem. The default policy for the borrower is a closed subset D of $S \times T$. That is, he defaults whenever (s, t) is in D . His prepayment policy is another subset P . When monitoring and technical default are permitted, the lender has a foreclosure policy represented by another subset F of $S \times T$. That is, the lender liquidates the collateral if the monitoring event occurs and (s, t) is in F . Figure 1 depicts what these policies might look like in $S \times T$ space.

The borrower and lender are assumed to choose the policies D, P, F that maximize the value of their own position, given the policy of the other. Each knows the process followed by $s(t)$. The borrower continuously observes s ; the lender gets occasional ‘glimpses’ of s on monitoring dates that occur on average m times per year. Loan payments continue

until the first date that (s, t) hits either D or P, or until the first date on which monitoring occurs and the state is in F, or until time T is reached.

Let the function $L(s, t)$ represent the value to the lender of the remaining cash flows from the loan if the collateral value at time t is s and the loan has not previously terminated. Note that $L(s, 0)$, being the value of the promised cash flows at the loan origination date, is also the maximum amount of cash he would rationally advance against the promise. Let $B(s, t)$ similarly represent the value of the borrower's position, taking into account his options under the contract.

Ignore, for the moment, the possibility of technical default. Using the familiar arbitrage/replication argument of option pricing theory, it can be shown that these functions satisfy the following partial differential equations in the open region where the loan continues:¹⁵

$$\frac{1}{2}\sigma^2 s^2 L_{ss} + (rs - d)L_s + p + L_t = rL \quad (4)$$

$$\frac{1}{2}\sigma^2 s^2 B_{ss} + (rs - d)B_s + d - p + B_t = rB \quad (5)$$

In the above $d = d(s, t)$ is the instantaneous dividend flow of the collateral. The expression $(rs - d)$ is the 'risk-adjusted drift' in the value of the collateral — the expected rate of capital appreciation it would have in equilibrium if agents were risk neutral. The equations state that the expected increase in wealth from holding each position (the left hand sides) must be the same as the increase in wealth from holding riskless bonds of the same value. The first terms on the left hand sides reflect a Jensen's inequality effect arising from the volatility of s . The second terms reflect the (risk adjusted) expected drift in s . The third terms are the net payment flows received by lender and borrower respectively. The last terms capture the shrinking time to maturity.¹⁶

A noteworthy property of these relations is their absence of taste parameters: They apply for any agent, regardless of level of risk aversion and regardless of his personal beliefs about the drift α in s . This is a consequence of assuming that markets are in equilibrium and are effectively complete with respect to s -risk. It implies that all individuals, *at*

¹⁵Subscripts on functions indicate partial derivatives with respect to the subscript variables.

¹⁶An alternative interpretation of these differential equations, termed the Feynman-Kac solution, is that the value of each agent's position equals the expected discounted value of his future receipts, where the probabilities used in calculating the expected value are those that would prevail if s followed the process $ds = (rs - d)dt + \sigma s dz$.

the margin, have the same marginal rate of substitution between riskless assets and the positions represented by L and B . Trading in other securities mediates differences in risk tolerance, just as trading in default-free bonds mediates differences in personal rates of time preference in the standard Fisherian model of consumption and investment.

Equations (4) and (5) both have many solutions. The functions L and B are pinned down by their values on the loan termination boundaries. At maturity T , outside the default region,

$$B(s, T) = s(T) - P \quad L(s, T) = P \quad (6)$$

Whenever $(s, t) \in D$, default occurs and

$$B(s, t) = 0 \quad L(s, t) = \max\{0, s(t) - l(s, t)\} \quad (7)$$

Receipts by the lender reflect his option to abandon the collateral if that makes sense.¹⁷ Whenever $(s, t) \in P$, prepayment occurs, the lender receives the current loan balance, while the borrower incurs refinancing costs plus a new loan obligation with value equal to the amount refinanced:

$$B(s, t) = s(t) - b(t) - f(b(t)) \quad L(s, t) = b(t) \quad (8)$$

Introduce now the possibility of monitoring and technical default. If the monitoring event occurs and $(s, t) \notin F$, nothing happens. If the event occurs and $(s, t) \in F$, then foreclosure occurs. The values to the borrower and lender jump to

$$B'(s, t) = \max\{0, s - l(s, t) - b(t)\} \quad L'(s, t) = \max\{0, \min\{b(t), s - l(s, t)\}\} \quad (9)$$

These expressions reflect the fact that the lender has the option of abandoning the collateral if liquidation costs exceed the market price, and that any excess over the outstanding loan balance must be returned to the borrower. Thus B and L follow mixed jump-diffusion process in the region F . Assuming risk neutrality of both parties toward the Poisson process, the appropriate modification to the valuation pde's is (see Merton, 1976) to add terms $m\phi^L(s, t)$ and $m\phi^B(s, t)$ to the left sides of (4) and (5) respectively, where

$$\begin{aligned} \phi^L(s, t) &= L'(s, t) - L(s, t) && \text{in } F, 0 \text{ elsewhere} \\ \phi^B(s, t) &= B'(s, t) - B(s, t) && \text{in } F, 0 \text{ elsewhere} \end{aligned} \quad (10)$$

¹⁷When a prepayment option is present, a borrower never rationally defaults when $s(t) > b(t)$. For notational simplicity we thus suppress here the fact that the lender must return to the borrower any excess of liquidated value over the loan balance.

These additions are the actuarially fair continuous premium payments that would have to be made on an ‘insurance policy’ that offsets the jump in value in the event of monitoring.

The above pde’s and boundary conditions determine the values L and B for arbitrary policies D , P , F . The conditions which characterize *optimal* policies, and thus determine these sets, are termed ‘high contact’ or ‘smooth pasting’ conditions. They state that the first derivative in the s direction of the value function of the option exerciser is continuous as one crosses each boundary. I.e., there is no kink. These conditions are met if one treats each agent as solving a dynamic programming problem for the policy under his control (cf. van Moerbeke, 1976, and Myneni, 1992). That is, starting from the maturity boundary T , each works backward in time, simultaneously solving his valuation pde and determining whether the value for each s is higher with the contract terminated or not by his actions. This calculation takes as fixed the policy that will be followed at subsequent times both by himself and the other agent. It thus determines a subgame perfect Nash equilibrium in the policies.

III. The supply of credit

Let us examine in this environment a competitive lender’s supply curve for credit to a particular borrower. We are interested in whether and where this supply curve becomes inelastic, or even bends backward. The reasons for this interest are three. First, if the supply of credit to a particular borrower becomes inelastic at a level below the minimum needed to fund a given project, then (otherwise) economically viable investments may fail to proceed.¹⁸ This has the appearance of ‘credit rationing’ in that no adjustment of the contractual loan rate by itself could clear this loan market. Second, the effect of non-price components of the contract (e.g., amortization rate, technical default provisions, prepayment provisions) on the supply curve provides suggestions for contract design and insight into why loan contracts take the forms they do.¹⁹ Finally, if factors influencing the

¹⁸For instance, if an indivisible project requires \$1.00 of real investment, the agent with (non-transferrable) property rights to the investment opportunity has \$.20 of financial capital, but the supply curve for credit to this borrower becomes inelastic at \$.70, then there is no loan contract within the class we consider that could be acceptable to both borrower and lender and allow the project to proceed.

¹⁹Gorton and Kahn (1992) provide an interesting analysis of the role of contract provisions and renegotiation options in the extension of bank credit.

supply curve can be identified with cyclical fluctuations or policy changes, we may learn something further about the role of credit markets in transmitting shocks, whether there is substance to the notion of a ‘credit crunch’, etc.

1. Analytical results

Linked partial differential equations like (4) and (5), subject to free boundary conditions of the type characterizing optimal policies, can seldom be solved analytically. There are however two special situations that are tractable. First, if options can only, or rationally would only, be exercised at loan maturity, then optimal exercise policies are described by single critical values for $s(T)$ and loan values can be readily determined. Second, if the loan is perpetual, so that the decision problems faced by borrower and lender are stationary, then optimal exercise policies are described by constant critical values for s , and again some results can be obtained. These two situations are examined below.

But first we must point out²⁰

Proposition 1 *If the collateral has 0 liquidation costs and 0 dividend flows, then the supply of credit approaches the collateral value as the contractual loan rate approaches ∞ . That is,*

$$\lim_{c-r \rightarrow \infty} L(s, 0) = s$$

This proposition states that a rational lender *will* lend the full market value of the collateral, if offered a sufficiently high interest rate, when there are no dividend flows prior to loan maturity or liquidation costs. The reason is that with a sufficiently high loan rate default will occur with certainty, and the lender is effectively just buying the collateral from the start. Zero liquidation costs imply no value is lost through the default process; zero dividend flows imply no value is received by the borrower. The value to the lender thus equals the value to the borrower and lender combined, which is simply the current market value of the collateral. Looking at the proposition the other way around, there *must* be either liquidation costs or dividend flows for credit to be rationed to less than full market value of the collateral in this symmetric information world.

²⁰This observation is put in proposition form simply for emphasis. It restates in another form Merton’s (1990, eqn. 12.31ff) observation that the value of a bond must approach the value of the collateral (firm) as the risklessly discounted value of the promised payments, relative to the value of the firm, becomes infinite.

2. Default only at maturity

Consider the situation where the loan continues to maturity with certainty. The only relevant option is that of default.

Proposition 2 *If the collateral has 0 liquidation costs, pays constant proportional dividends at rate d , contractual payment flows are constant at p , and default cannot occur prior to maturity T , then the loan value is*

$$L(s, 0) = s + (1 - e^{-rT})\frac{p}{r} - (1 - e^{-dT})s - C(s, T; P)$$

in which $C(s, T; P)$ is the value of a European call option on the collateral with expiry date T and strike price equal to the terminal payment P .²¹

The proposition states that the value of the loan is the value of the collateral plus contractual loan payments, less market value of the dividend flows received by the borrower and the value of the borrower's terminal call option to retrieve the collateral by making the balloon payment P . Rational default would never occur prior to T if, for instance, the loan is a pure discount note ($p = 0$). This leads to

Proposition 3 *If the collateral pays a constant proportional dividend d , the supply of credit under pure discount loans is limited to*

$$L(s, 0) \leq se^{-dT}$$

This follows from the proposition 2, the non-negativity of the value of the call option C , and the fact that positive liquidation costs could only diminish further the value to the lender. We finally see an upper bound on the amount a rational lender could advance which is strictly less than the collateral value s .²² To illustrate, if the collateral pays

²¹The value of this option can be given by the Black-Scholes formula, modified to allow for the positive dividend rate on the collateral. See Merton (1973), fn.62. But beware: He neglects to state that $r - d$ must replace r in his equation (21).

²²If the collateral provides a constant *absolute* flow of benefits d to the borrower while in his possession, the value of the loan to the lender becomes $L(s, 0) = s + (1 - e^{-rT})\frac{p-d}{r} - C(s, T; P)$, and the limit on the supply of credit under pure discount discount loans becomes $L(s, 0) \leq s - (1 - e^{-rT})\frac{d}{r}$. For a constant absolute dividend flow to be consistent with s being non-negative, the proportional volatility σ could not be constant. Thus the value C of the call option would

dividends at an instantaneous rate of 15%/year of its value, the maximum amount that could be rationally advanced through a 5 year pure discount note would be .47 of the initial market value of the collateral, no matter how high a contractual loan rate is charged, and regardless of the *objective* expected rate of appreciation of the collateral.

3. Perpetual loans

Consider a perpetual loan with constant contractual payment flow p . The loan can only terminate with the exercise of one of the borrower/lender options. If the dividend flow d is time-independent, and the Markov process followed by s is time-independent, then the decision problems faced by borrower and lender are stationary, and the optimal exercise policies and (current) value functions L and B are independent of time. This simplifies the situation immensely: Optimal policies can be characterized as constant critical values of s at which default, prepayment or technical foreclosure, respectively, will occur; and the independence of L and B from t means that the partial derivatives with respect to t in equations (4-5) vanish, leaving more readily solved ordinary differential equations.

The perpetual loan may be viewed as approximating long term loans, or loans for which there is high probability of termination before contractual maturity. It highlights the value of timing options to the borrower and their implication for rational lending. Because of its tractability, we focus on the case where default is the only option considered:

Proposition 4 *An interest-only loan with infinite maturity, constant payments p , and collateral paying constant proportional dividends d , will be rationally defaulted at time t if*

$$s(t) \leq \underline{s} = \frac{\lambda}{\lambda - 1} \frac{p}{r}$$

and have value to the lender

$$L(s) = \frac{p}{r} + (\underline{s} - l - \frac{c}{r})(\frac{s}{\underline{s}})^\lambda$$

where

$$\lambda = \frac{\alpha - \sqrt{\alpha^2 + 8r/\sigma^2}}{2}$$

$$\alpha = 1 - \frac{2(r - d)}{\sigma^2}$$

not be as given by the Black-Scholes formula. Note that it would never be rational to default before maturity if $d \geq p$, since the borrower receives a positive net cash flow as long as the loan is outstanding.

Proof of the proposition is in the appendix.²³ Note that $\lambda < 0$ in the above expressions and hence $\underline{s} < p/r$. $l \equiv l(\underline{s})$ are liquidation costs to the lender evaluated at the rational default point. The above offers a simple, operational procedure for valuing risky long term loans.²⁴

Several things can be learned from this result. First, note the default point \underline{s} is proportional to p . Since the principal will never be repaid, it is only the level of payments relative to collateral that concerns the borrower. Whether it was a large loan at a low interest rate or a small loan at a high interest rate that resulted in the given p is irrelevant, as is any expected rate of appreciation in s . Second, the default collateral value is strictly less than p/r , which is the riskless rate capitalized value of the remaining promised loan payments. A loan must thus be considerably ‘under water’ for a borrower to rationally default: simple comparison of current collateral value to contractual obligations is *not* a valid indicator of the imminence of default. Third, totally differentiating L with respect to p and evaluating at \underline{s} results in

$$\frac{dL(\underline{s})}{dp} = \frac{\lambda l}{p} < 0 \quad (11)$$

Lender value is thus decreasing in p in the vicinity of default if l is positive. There is thus a positive incentive for the lender to offer *permanently reduced* payments if default is imminent, either by reducing the contractual loan rate or by forgiving some of the notional principal.²⁵ This reflects the fact that the collateral is more valuable in the hands of the borrower, unliquidated, than in the hands of the lender.

²³Merton (1974) also works out the value of a risky perpetual bond, obtaining a rather different expression involving the confluent hypergeometric function (eq. 12.42 as reprinted in Merton, 1990). The difference is accounted for by Merton’s constraining the ‘cash flow’ off the collateral to equal the bond payments at all times. Since such a coincidence is unlikely to occur in practice, the borrower is implicitly permitted to issue further debt, more senior to that of the proposition, to make up any shortfall (though this is at variance with his stated bond indentures). Default occurs when that is no longer possible, at which point the junior bond would have 0 value. Here, dividends vary with the value s of the collateral, but the borrower has the option to inject further cash that is not included in the original collateral to defer default.

²⁴The situation with both default and prepayment options is qualitatively similar and briefly described in the Appendix. However the boundary conditions result in a pair of simultaneous quadratic equations in the optimal default and prepayment points, \underline{s} and \bar{s} , that would require numerical solution for particular parameter values, and hence are not presented.

²⁵The value of the derivative here presumes this is an irrevocable, one-time alteration of the

Let us go a step further and ask, what is the optimal payment size from the perspective of the lender and what is the associated maximum value the loan can have? The results are clearer if we assume proportional liquidation costs for the collateral: $l(s) = l_1 s$.

Proposition 5 *With constant proportional liquidation costs and the assumptions of Proposition 4, value to lender is maximized at*

$$\begin{aligned} p^* &= sr \left(\frac{\lambda - 1}{\lambda} \right) (1 - \lambda l_1)^{1/\lambda} \\ L^*(s) &= s(1 - \lambda l_1)^{1/\lambda} \end{aligned}$$

where s is the initial market value of the collateral.

To illustrate, suppose that the riskless interest rate and the collateral dividend rate are both 5%/year, that the collateral volatility is 20%/year, and that the lender obtains *no* value from the collateral in the event of default. With $r = d = .05$, $\sigma = .20$ and $l_1 = 1.0$, we get, per dollar of current collateral value $s(0)$,

$$\underline{s} = .5147 \quad p^* = .04795 \quad L^* = .5147$$

Thus the maximum loan amount is for only about half the initial value of the collateral, and involves payment expressed as a fraction of the amount lent of $p^*/L^* = .09316$ per year—the contractual loan rate is the riskless rate plus 4.316%.

This coincidence of $L^* = \underline{s}$ holds for all values of l_1 , as does the optimality of $p^*/L^* = r(\lambda - 1)/\lambda$ for the contractual loan rate. As the marketability of the collateral changes, the monopoly loan rate stays put; only the size of the loan (and drop in collateral value required to induce to default) changes.

IV. Finite maturity loans

For loans with finite maturity, the level of s at which the borrower rationally defaults or prepays varies with the term remaining. Analytical solution for the optimal exercise regions and borrower/lender values is not feasible. Numerical solution permits us to consider the entire range of options simultaneously and provides some insight.

contract with no further changes anticipated. It is tempting to explore how the anticipation of optimal ‘forgiveness’ might alter the granting of credit, since it unambiguously raises value to borrower and lender combined by delaying the payment of deadweight liquidation costs.

We use a finite difference procedure to solve the problem for representative cases. $S \times T$ is represented by a discrete rectangular grid of s and t values. A solution is a set of L and B values for these gridpoints, together with an indication whether each point is or is not in D , P and F . Working backwards from time T , the pde's are 'solved' for each time step using a Crank-Nicholson discrete approximation for the partial derivatives.²⁶ After each step, one checks for each agent and for each s level whether they could raise the value by exercising the options available to him. If so, that is noted and the value functions of *both* agents are appropriately adjusted to reflect this. One continues moving backwards in this fashion until the origination date of the loan is reached.

The examples assume that dividend flows, liquidation costs and refinancing costs are independent of time and are homogenous of degree one in their other arguments. I.e., $d(s, t) = d_1 s$, $l(s, t) = l_1 s$, $f(b) = f_1 b$. The solutions obtained can thus be viewed as values per dollar initially lent, and the policies as independent of loan scale with $s(t)$ interpreted as collateral value at time t per dollar initially lent.

1. Effect of the loan parameters

The base case used as starting point is summarized at the top of Table 1. The riskfree interest rate is $r = .05$ per year; annualized proportional volatility of the collateral's value is $\sigma = .2$; term of loan is $T = 5$ years; no payments are due before maturity—i.e., $p = 0$; the contractual loan spread over the riskless rate is $.02$ per year; there is no cash flow from the collateral, refinancing cost, liquidation cost or monitoring. The amount due at maturity is \$1.419 per \$1 lent.²⁷

Table 1 shows how the value to the lender changes as we alter various loan parameters, keeping other parameters fixed at their base case levels. The column headed %Loan gives

²⁶See for example Press et al, *Numerical Recipes*, 1986. The procedure approximates partial derivatives by the average of difference expressions in the gridpoint values of B and L at time t , for which one has the last solution, and time $t + k$, for which one does not. Requiring the pde to hold at all gridpoints results in a tridiagonal system of simultaneous linear equations in the unknown values at the current time slice. This is then solved for the values at $t + k$. And so one proceeds. The method has the virtue of being numerically stable for all sizes of k , and second order accurate in both the t and s directions.

²⁷The annualized volatility chosen is of the same order of magnitude, for example, as major North American stock market indices.

the ratio $L(s, 0)/s$. This ratio is independent of loan scale and represents the proportion of fair market value of the collateral a competitive lender would be willing to advance.

Twelve experiments were conducted: (a) demonstrates how higher dividend rates on the collateral (without commensurately higher loan payment flows) *reduces* willingness to lend against given collateral. (b) indicates that larger payment flows (faster amortization) significantly increases willingness to lend. (c) shows the negative impact of loan term. (d) the negative impact of increased uncertainty about the future value of the collateral, enhancing the default option. (e) demonstrates that the credit supply curve—required loan spread as a function of loan amount—does slope upwards, approaching 1 when the dividend flows are 0. (f) illustrates that credit supply is bounded below 1 when dividend flows are positive, as foretold by Proposition 3. (g) documents the negative effect of increasing collateral liquidation costs. At the extreme of $l_1 = 1.0$ no value can be retrieved by the lender, though the threat of loss of collateral value to the the borrower induces loan repayment in most states, and thus a positive fraction of the collateral value can still rationally be lent.²⁸ (h) shows how the ability to fund projects through loans worsens when both liquidation costs and dividend flows are present. (i) makes the important observation that the *proportion* that can be advanced against given collateral is *not* affected by the level of the risk free interest rate r . This is because the risk-adjusted expected rate of return on all assets in equilibrium must equal r . Of course, a rise in r may reduce the market value s of given collateral, reducing the absolute *amount* that can be advanced. (j) demonstrates that, with liquidation costs, the credit supply curve can actually bend backwards, increasing then decreasing as the loan spread $c - r$ rises. (k) is the only case presented with prepayment available as an option to the borrower. The amount rationally lendable is an increasing function of the proportional refinancing costs f_1 faced by the borrower. (l) shows the effect of introducing technical default (defined here to be $s(t) \leq b(t)$), and varying the monitoring intensity from 0 to on average 12 times/year.

2. Default/Prepayment policies

Tables 2 and 3 display for two cases the numerically obtained value functions, identifying the regions of $S \times T$ in which various options are exercised. In both cases the loans are

²⁸The farmer's aging family donkey may thus usefully serve as collateral, though it has no realizable value to the lender.

of \$1.00 initial amount for 5 years. Collateral volatility of 20% and riskless interest rate of 5% are as in our base case. The contractual loan rate is 8%. Contractual payments are \$.10 per year with a maturity payment of \$.88. Technical default and prepayment are permitted. Liquidation costs are 10% of collateral value. The lender monitors the collateral on average 4 times per year. The loan is in technical default if it is found to be worth less than the remaining loan balance (i.e., $\delta = 1$). In Table 2, the collateral provides no dividends or service flows, and the refinancing fee is 2% of the remaining loan balance. In Table 3, the collateral provides service flows at a rate of 10% per year of the collateral's value, and the refinancing fee is 4%. Parameter values are repeated at the top of each table.

The upper panel of each table displays the value to the lender $L(s, t)$ for a range of times remaining to maturity and collateral values. The lower panel displays the corresponding value to the borrower $B(s, t)$. The option exercise regions are indicated by a symbol after the dollar value: + indicates prepayment would rationally occur, * indicates default would rationally occur, - indicates that the loan is technically in default but the lender would not rationally exercise his option to call the loan, and : indicates that the lender would rationally foreclose. When the regions overlap, default takes precedence over foreclosure over unforced technical default in the display.

Several observations can be made that apply to both of these cases. First, the prepay region disappears well before maturity at T . This reflects the fact that if little time is left, then the cost of paying a now unwarranted high interest rate over the remaining term must still fall short of any refinancing cost. Second, the default region lies well below the region of technical default. A borrower must have *substantial* negative equity before rationally defaulting if considerable time is left during which things might improve. Third, as a check, one may verify that the combined value to borrower and lender, $B + L$, approaches $(1 - l_1)s$ as $s \rightarrow 0$, approaches $s - bf_1$ as $s \rightarrow \infty$ when prepayment is a viable option, and approaches s when it is not. That is, value to all parties (including the third party bankruptcy trustees in the event of default or new lender in the event of prepayment) is conserved and adds up to s . The maximum value of $B + L$ lies in the interior of the continuation region, where the third parties have the least prospects. Finally, in both cases the default region has a jump up at T when the lump-sum terminal payment is due.

Turning now to the differences, notice that in Table 2 the lender does not rationally

foreclose in the technical default region. It is better for him to carry on receiving the 3% premium over the riskless rate, with the collateral expected to appreciate at 5%/year (risk adjusted), and postpone the loss in value that accompanies liquidation as long as possible.

In contrast, in Table 3, foreclosure before maturity now is rational. The borrower is irretrievably extracting value at the dividend rate. But note that if s falls too far before the technical default is caught then it is better not to foreclose! In the region between default and foreclose, the borrower will shortly default of his own volition if the value falls further, the lender will have an option to foreclose on more advantageous terms if the value rises, and he continues to receive his premium in the meantime. Second, in comparison with Table 2, the default region is noticeably lower, by about 10% in s value. The income received by the borrower reduces the net carrying cost of keeping his options alive and thus raises the expected time to default. Although at first glance this might seem attractive to the lender, it is in fact not. The loan value L in the vicinity of s equals 1 is about 5% lower in Table 3 than in Table 2. This reflects the reality that much less value is left in the collateral if and when default does occur. Third, around a value of $t = 2$ at the prepayment boundary, this case shows that L can actually decrease as s increases. A rise in collateral value makes the loan safer; but for that very reason it increases the likelihood of early prepayment and an end to the now excessive loan spread relative to the true risk of default.

Finally it must be pointed out that neither of these seemingly reasonable loans would ever be made in the first place: Neither has the requisite L value of \$1 at time 0, except for s levels inducing immediate prepayment. Did someone mention loan points?

V. Conclusion

Such illustrations indicate the rich possibilities and subtle interactions that can occur in even simple loans under symmetric information. To summarize, we have examined rational lender behaviour in the market for secured, non-recourse loans in a benchmark complete (with respect to risk) market environment. Major qualitative results include: (1) The upper limit on what a rational bank would lend can be a small fraction of the current fair market value of the collateral, regardless of the interest rate the borrower offers. (2) The loan supply curve to a particular borrower can ‘bend backward’, with the bank preferring a lower loan rate over a higher one. (3) Collateral promising low cash

flows is preferable to collateral of the same current value producing high cash flows. (4) Unilateral loan rate reduction or principal forgiveness by a lender can be rational. (5) The maximum amount lendable increases with the intensity of monitoring collateral value. (6) The amount lendable is highly sensitive to the collateral's perceived volatility, even under risk neutrality, suggesting cyclical fluctuation in the availability of credit. This last point is consistent with empirical evidence that there might indeed be a 'credit crunch' (especially for small firms) during business downturns (Gertler, 1988; Bernanke and Gertler, 1989).

The significance of all this is that, if such credit rationing-like phenomena naturally occur in situations *without* information asymmetry or moral hazard, then there is less reason to suspect any market failure requiring regulatory intervention or government action. At the least, if action is called for, it suggests remedies in the direction of *removing* restrictions on the enforceable contractual forms through which banks can advance funds to risky borrowers.

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Appendix

Proof of proposition 4

In the stationary perpetual case, let \underline{s} denote the collateral value below which the borrower defaults. Equation (4) for the value of the borrower's position becomes

$$\frac{1}{2}\sigma^2 s^2 B_{ss} + (r-d)sB_s + sd - p = rB \quad (\text{A.1})$$

with boundary value $B(\underline{s}) = 0$, high contact condition $B_s(\underline{s}) = 0$, and boundary condition at infinity $\lim_{s \rightarrow \infty} B(s) = s - p/r$. The latter comes from recognizing that as $s \rightarrow \infty$ the default within any given time frame becomes increasingly improbable, and the value to the borrower is simply the value of the collateral less the present value of the unavoidable loan payments. The solution to the second order linear ordinary differential equation (A.1) is

$$\begin{aligned} B(s) &= s - \frac{p}{r} + k_1 s^{\lambda_1} + k_2 s^{\lambda_2} \quad \text{where} \quad (\text{A.2}) \\ \lambda_i &= \frac{\alpha \pm \sqrt{\alpha^2 + 8r/\sigma^2}}{2} \\ \alpha &= 1 - \frac{2(r-d)}{\sigma^2} \end{aligned}$$

The boundary condition at $s = \infty$ requires the coefficient associated with the positive root be zero: $k_1 = 0$. Substituting the boundary value at \underline{s} into (A.2) then gives

$$k_2 = \frac{p - r\underline{s}}{r\underline{s}^\lambda} \quad (\text{A.3})$$

and hence

$$B(s) = s - \frac{p}{r} + \frac{p - r\underline{s}}{r} \left(\frac{s}{\underline{s}}\right)^\lambda \quad (\text{A.4})$$

This holds for arbitrary \underline{s} . To determine the *optimal* \underline{s} , use the high contact condition by differentiating (A.4) with respect to s , evaluating at $s = \underline{s}$, and equating to 0 to give

$$\underline{s} = \frac{p}{r} \left(\frac{\lambda}{\lambda - 1}\right) \quad (\text{A.5})$$

Notice that the optimal \underline{s} is linear in p . Only the size of payments relative to collateral value is relevant for the default decision.

With borrower default behaviour thus established, turn now to the value to the lender. Equation (5) becomes

$$\frac{1}{2}\sigma^2 s^2 L_{ss} + (r-d)sL_s + p = rB \quad (\text{A.6})$$

with boundary value at default $L(\underline{s}) = \underline{s} - l(\underline{s})$ and at infinity $\lim_{s \rightarrow \infty} L(s) = p/r$. The solution takes the same form as (A.2) and, as in that case, only the negative value for λ is relevant. Thus

$$L(s) = \frac{p}{r} + k_2 s^{\lambda_2} \quad (\text{A.7})$$

Substituting the boundary condition at \underline{s} determines k_2 , resulting in

$$L(s) = s - \frac{p}{r} + (\underline{s} - l(\underline{s}) - \frac{p}{r}) \left(\frac{s}{\underline{s}}\right)^\lambda \quad (\text{A.8})$$

Extension to include prepayment

If, in addition to having the option to default, the borrower has the option to cancel the loan by making lump-sum payment b (time independent), assumed raised by incurring a new liability of market value $b + f(b)$, the solution for $B(s)$ still takes the form (A.2). Denoting by \bar{s} the collateral value above which prepayment rationally occurs, the boundary condition at infinity is replaced by the dual conditions $B(\bar{s}) = \bar{s} - b - f(b)$ and $B_s(\bar{s}) = 1$. Substitution of these together with the conditions at \underline{s} already noted above into (A.2) gives a set of simultaneous nonlinear equations to solve for the four unknowns $k_1, k_2, \underline{s}, \bar{s}$.

Table 1:

BASE CASE: no prepayment permitted or monitoring

r	sigma	T	p	c-r	d_1	f_1	l_1	m	s(0)
.05	.2	5.	0	.02	.0	.0	.0	.0	1.1

P = payment at maturity = 1.419

p = payments/yr. = .00

All parameters are as in Base Case unless explicitly indicated otherwise.

%Loan is fraction of s(0) lent such that c-r is breakeven loan spread.

(a)		(b)		(c)		(d)	
d_1	%Loan	p	%Loan	T	%Loan	sigma	%Loan
.00	.717	.00	.717	1	.809	.00	.999
.02	.649	.02	.781	2	.772	.05	.983
.04	.587	.04	.840	3	.748	.10	.917
.06	.531	.06	.890	4	.731	.15	.822
.08	.481	.08	.928	5	.717	.20	.717
.10	.435	.10	.953	6	.707	.25	.612
.12	.394	.20	.995	8	.690	.30	.512
.14	.356	.30	.999	10	.676	.35	.409

(e)		(f)		(g)		(h)	
With d_1 = .0		With d_1 = .1		With d_1 = .0		With d_1 = .1	
c-r	%Loan	c-r	%Loan	l_1	%Loan	l_1	%Loan
.00	.000	.00	.000	.0	.717	.0	.435
.02	.717	.02	.435	.1	.655	.1	.397
.04	.827	.04	.502	.2	.608	.2	.369
.06	.890	.06	.540	.3	.572	.3	.347
.08	.930	.08	.564	.4	.545	.4	.330
.10	.955	.10	.579	.5	.522	.5	.317
.12	.972	.12	.590	.6	.504	.6	.306
.14	.983	.14	.596	.7	.489	.7	.296
.16	.990	.16	.600	.8	.476	.8	.288
.18	.994	.18	.603	.9	.464	.9	.281
.20	.997	.20	.605	1.0	.454	1.0	.275

(i)		(j)		(k)		(l)	
r	%Loan	d_1=.1	l_1=.5	Prepayment permit		No prepayment	
		c-r	%Loan	f_1	%Loan	m	%Loan
.00	.718	.00	.000	.00	.588	0.	.717
.01	.718	.02	.317	.02	.604	2.	.820
.02	.718	.04	.352	.04	.647	4.	.863
.03	.718	.06	.370	.06	.699	6.	.889
.04	.718	.08	.373	.08	.717	8.	.907
.05	.718	.10	.373	.10	.717	10.	.925
.06	.718	.12	.372	.12	.717	12.	.925
.07	.718	.14	.365				
.08	.718	.16	.356				
.09	.718	.18	.348				
.10	.718	.20	.340				

Table 2:

```

r   sigma  T     p     c-r   d_1   f_1   l_1   m     s(0)
.05  .2     5.   .1    .03   0     .02   .1    4     1.1

```

Prepayments ON

%L: 0.8 Sfair: 1.25 Vlmax: 1.00 Lval: 0.97 Bval: 0.10

Loan status: *=default +=prepay :=foreclose -=tech.default

Loan value to lender

```

1.50 | 1.00+0.99+0.98+0.97+0.96+0.94+0.93+0.92+0.91+0.90 0.88
1.45 | 1.00+0.99+0.98+0.97+0.96+0.94+0.93+0.92+0.91+0.90 0.88
1.40 | 1.00+0.99+0.98+0.97+0.96+0.94+0.93+0.92+0.91+0.90 0.88
1.35 | 1.00+0.99+0.98+0.97+0.96+0.94+0.93+0.92+0.91+0.90 0.88
1.30 | 1.00+0.99+0.98+0.97+0.96+0.94+0.93+0.92+0.91+0.90 0.88
1.25 | 1.00+0.99+0.98+0.97+0.96+0.94+0.93+0.92+0.91+0.90 0.88
1.20 | 0.99 0.98 0.98+0.97+0.96+0.94+0.93+0.92+0.91+0.90 0.88
1.15 | 0.98 0.97 0.97 0.96 0.95 0.94+0.93+0.92+0.91+0.90 0.88
1.10 | 0.97 0.96 0.96 0.95 0.94 0.93 0.92 0.92 0.91 0.90 0.88
1.05 | 0.94 0.94 0.94 0.93 0.93 0.92 0.91 0.91 0.90 0.89 0.88
1.00 | 0.91 0.91 0.91 0.91 0.90 0.90 0.89 0.89 0.89 0.88 0.88
0.95 | 0.87-0.87-0.87-0.88-0.87-0.87 0.87 0.87 0.87 0.87 0.87 0.88
0.90 | 0.81*0.81*0.81*0.83-0.83-0.83-0.83-0.83-0.84-0.84 0.88
0.85 | 0.76*0.76*0.76*0.76*0.76*0.76*0.76*0.76*0.79-0.79-0.76*
0.80 | 0.72*0.72*0.72*0.72*0.72*0.72*0.72*0.72*0.72*0.72*
0.75 | 0.68*0.68*0.68*0.68*0.68*0.68*0.68*0.68*0.68*0.68*
0.70 | 0.63*0.63*0.63*0.63*0.63*0.63*0.63*0.63*0.63*0.63*
0.65 | 0.58*0.58*0.58*0.58*0.58*0.58*0.58*0.58*0.58*0.58*
0.60 | 0.54*0.54*0.54*0.54*0.54*0.54*0.54*0.54*0.54*0.54*
S: | -----
T: 5.00 4.50 4.00 3.50 3.00 2.50 2.00 1.50 1.00 0.50 0.00

```

Value of borrower equity

```

1.50 | 0.48+0.49+0.50+0.51+0.52+0.54+0.55+0.56+0.58+0.60 0.62
1.45 | 0.43+0.44+0.45+0.46+0.47+0.49+0.50+0.51+0.53+0.55 0.57
1.40 | 0.38+0.39+0.40+0.41+0.42+0.44+0.45+0.46+0.48+0.50 0.52
1.35 | 0.33+0.34+0.35+0.36+0.37+0.39+0.40+0.41+0.43+0.45 0.47
1.30 | 0.28+0.29+0.30+0.31+0.32+0.34+0.35+0.36+0.38+0.40 0.42
1.25 | 0.23+0.24+0.25+0.26+0.27+0.29+0.30+0.31+0.33+0.35 0.37
1.20 | 0.18 0.19 0.20+0.21+0.22+0.24+0.25+0.26+0.28+0.30 0.32
1.15 | 0.14 0.15 0.15 0.16 0.18 0.19+0.20+0.21+0.23+0.25 0.27
1.10 | 0.10 0.10 0.11 0.12 0.13 0.14 0.15 0.16 0.18 0.20 0.22
1.05 | 0.06 0.07 0.07 0.08 0.09 0.10 0.11 0.12 0.13 0.15 0.17
1.00 | 0.03 0.03 0.04 0.05 0.05 0.06 0.07 0.08 0.09 0.10 0.12
0.95 | 0.01-0.01-0.01-0.02-0.02-0.03 0.04 0.04 0.05 0.06 0.07
0.90 | 0.00*0.00*0.00*0.00-0.00-0.01-0.01-0.02-0.02-0.03 0.02
0.85 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00-0.01-0.00*
0.80 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*
0.75 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*
0.70 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*
0.65 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*
0.60 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*
S: | -----
T: 5.00 4.50 4.00 3.50 3.00 2.50 2.00 1.50 1.00 0.50 0.00

```

Table 3:

```

r   sigma  T    p    c-r   d_1   f_1   l_1   m    s(0)
.05  .2     5.   .1   .03   .1    .04   .1    4    1.1

```

Prepayments ON

```
%L: 0.69  Sfair: 1.45  Vlmax: 1.00  Lval: 0.92  Bval: 0.11
```

```
Loan status: *=default +=prepay :=foreclose -=tech.default
```

Loan value to lender

```

1.50 | 1.00+0.99+0.98+0.97+0.96+0.94+0.93+0.95 0.93 0.90 0.88
1.45 | 1.00+0.99+0.98+0.97+0.96+0.94+0.93+0.95 0.93 0.90 0.88
1.40 | 0.99 0.99 0.98+0.97+0.96+0.94+0.93+0.95 0.93 0.90 0.88
1.35 | 0.99 0.98 0.97 0.96 0.95 0.95 0.94 0.94 0.93 0.90 0.88
1.30 | 0.98 0.97 0.96 0.96 0.95 0.94 0.94 0.94 0.92 0.90 0.88
1.25 | 0.97 0.96 0.95 0.95 0.94 0.94 0.94 0.93 0.92 0.90 0.88
1.20 | 0.95 0.95 0.94 0.94 0.93 0.93 0.93 0.92 0.91 0.90 0.88
1.15 | 0.94 0.93 0.93 0.93 0.92 0.92 0.92 0.91 0.91 0.90 0.88
1.10 | 0.92 0.92 0.91 0.91 0.91 0.90 0.90 0.90 0.90 0.89 0.88
1.05 | 0.90 0.90 0.89 0.89 0.89 0.88 0.88 0.88 0.88 0.88 0.88
1.00 | 0.87 0.87 0.87 0.87 0.87 0.86 0.86 0.86 0.86 0.87 0.88
0.95 | 0.85:0.85:0.85:0.85:0.85:0.84 0.84 0.84 0.84 0.85 0.88
0.90 | 0.81:0.81:0.81:0.81:0.81:0.81:0.81:0.81:0.81:0.82 0.88
0.85 | 0.77-0.77-0.77-0.77-0.77-0.77-0.77-0.77-0.77-0.78-0.76*
0.80 | 0.72*0.72*0.72*0.72*0.73-0.73-0.73-0.73-0.73-0.73-0.72*
0.75 | 0.68*0.68*0.68*0.68*0.68*0.68*0.68*0.68*0.68-0.68*0.68*
0.70 | 0.63*0.63*0.63*0.63*0.63*0.63*0.63*0.63*0.63*0.63*0.63*
0.65 | 0.58*0.58*0.58*0.58*0.58*0.58*0.58*0.58*0.58*0.58*0.58*
0.60 | 0.54*0.54*0.54*0.54*0.54*0.54*0.54*0.54*0.54*0.54*0.54*
S: | -----
T: 5.00 4.50 4.00 3.50 3.00 2.50 2.00 1.50 1.00 0.50 0.00

```

Value of borrower equity

```

1.50 | 0.46+0.47+0.48+0.49+0.51+0.52+0.53+0.54 0.57 0.60 0.62
1.45 | 0.41+0.42+0.43+0.44+0.46+0.47+0.48+0.50 0.52 0.55 0.57
1.40 | 0.36 0.37 0.38+0.39+0.41+0.42+0.43+0.45 0.47 0.50 0.52
1.35 | 0.31 0.32 0.33 0.34 0.36 0.37 0.38 0.40 0.42 0.45 0.47
1.30 | 0.27 0.28 0.29 0.30 0.31 0.32 0.33 0.35 0.37 0.40 0.42
1.25 | 0.23 0.23 0.24 0.25 0.26 0.28 0.29 0.31 0.32 0.35 0.37
1.20 | 0.18 0.19 0.20 0.21 0.22 0.23 0.24 0.26 0.28 0.30 0.32
1.15 | 0.15 0.15 0.16 0.17 0.18 0.19 0.20 0.22 0.23 0.25 0.27
1.10 | 0.11 0.11 0.12 0.13 0.14 0.15 0.16 0.17 0.19 0.20 0.22
1.05 | 0.08 0.08 0.08 0.09 0.10 0.11 0.12 0.13 0.15 0.15 0.17
1.00 | 0.04 0.05 0.05 0.05 0.06 0.08 0.08 0.09 0.11 0.11 0.12
0.95 | 0.01:0.01:0.01:0.02:0.02:0.05 0.05 0.06 0.07 0.07 0.07
0.90 | 0.00:0.00:0.01:0.01:0.01:0.01:0.01:0.02:0.02:0.02:0.04 0.02
0.85 | 0.00-0.00-0.00-0.00-0.01-0.01-0.01-0.01-0.01-0.02-0.00*
0.80 | 0.00*0.00*0.00*0.00*0.00*0.00-0.00-0.00-0.00-0.01-0.01-0.00*
0.75 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00-0.00*0.00*
0.70 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*
0.65 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*
0.60 | 0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*0.00*
S: | -----
T: 5.00 4.50 4.00 3.50 3.00 2.50 2.00 1.50 1.00 0.50 0.00

```