

Analyzing Credit Lines with Fluctuating Credit Quality

Preliminary

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The revolving credit line—an arrangement under which customers may borrow and repay at will subject to a maximum outstanding—has been a standard form of bank lending for decades. The interest rate paid on borrowings, or the *spread* over some reference rate in the case of floating rate loans, is typically fixed for the term of the contract. This grants a valuable option to the borrower: As credit quality, and hence rate that would be paid on alternative borrowings, subsequently fluctuates, he can raise or lower borrowings on the line. The loan spread and likelihood of default from the perspective of *initial* credit quality may thus present a misleading picture of loan profitability to the lender.

This paper examines credit lines using the tools of modern contingent claims analysis. The objective is a broadly applicable, computationally tractable, arbitrage-free framework for defaultable securities when creditworthiness evolves independently of the security being valued. I.e., the security in question does not itself influence solvency of the issuer.¹ Credit exposure to a given customer can take a variety of forms, all considered by him (or the ambitious bank relationship-manager) simultaneously: fixed rate bullet (constant balance) loan; floating rate bullet loan; program of guaranteeing the customer's commercial paper; floating rate loan with spread reset according to credit quality; unsecured fixed/floating interest rate swap with the customer; third party credit default swap based on that customer; revolving credit line; standard credit line (maximum cumulative drawdowns independent of prepayments); option on any of these (loan commitment). Clearly an analytically consistent framework is needed to avoid internal arbitrage and to ensure the same compensation for bearing the same risk regardless of contractual form.

Section I lays out a model of credit quality evolution and the numerical valuation of default-risky securities. Section II describes estimation of risk-neutral process parameters from US traded corporate bond price data. Section III analyses the revolving credit line in particular.

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¹Using the terminology of Duffie and Singleton (1997), we are looking for a reduced-form rather than structural model.

I. The default process and security valuation

1. Credit quality evolution

Let us treat credit quality as a scalar $s(t)$ that follows a calendar time independent jump-diffusion process. As a Markov process, this provides computational tractability, facilitates treatment of securities with American option features, and allows room for a second factor that may influence contractual payouts or values (e.g., riskfree interest rates, variable entering swap or option contract, etc.). Default is associated with the credit state crossing an exogenously specified barrier, here $s = 0$. Thus

$$ds = \alpha dt + \sigma dz + \mu d\pi \quad (1)$$

in which $z(t)$ is a standard Brownian motion, $\pi(t)$ counts the jumps in a Poisson process with instantaneous intensity λ , independent of z , α is the diffusion drift, σ the diffusion volatility, and μ the (random) jump size conditional upon a jump occurring. α , σ , λ are permitted to be ‘nice’ functions of s . Jump sizes are specified by a conditional distribution function

$$F(x, x') \equiv \mathbf{prob}\{s(t^+) \leq x \mid s(t^-) = x', d\pi(t) = 1\} \quad (2)$$

with μ otherwise independent of z and π .² The resulting default time—first t such that $s(t) \leq 0$ —will be denoted by τ . Default states are treated as absorbing: i.e., $ds(t) = 0$ for $t > \tau$.

2. Theoretical security values

Assume there is a continuously functioning, frictionless market in default-free bonds. Let $r(t)$ denote the interest rate on instantaneously maturing bonds at time t , with the process for r being independent of s .

Consider financial securities—or contracts between two parties—of the following form: There is a finite set $\{t_i\}_{i=0,1,\dots}$ of contractual payment dates. Contract maturity T is the last payment date; contract origination $t_0 = 0$ is the first. One party is designated the purchaser or lender, and treated as default free; the other, the seller or borrower, has credit quality evolve as in the previous section. Security valuation is from the perspective of the lender.

The contract specifies net payments to the lender on payment dates of $q(s, r, t_i)$ if no prior default, where s, r are values at t_i . At each t there is also a contractual *recovery balance*

²Appropriate choice of the functions α , σ , λ , F allows representation of a wide range of reduced-form default models, such as ratings-based specifications ($\alpha, \sigma = 0$, F a step function), pure diffusion to barrier ($\lambda = 0$), jumps are always to default ($F(x, x')$ steps up from 0 to 1 at $x = 0 \ \forall x'$), fixed distribution of jump sizes ($F(x + y, x' + y) = F(x, x') \ \forall x, x', y$), and so on.

$B(s, r, t)$. If default occurs at time τ , the lender receives $\rho B(s, r, \tau)$ at that time. ρ is a *recovery rate* assumed constant in the current application.³ Initial loan advance, origination fees or purchase price are subsumed in the (possibly negative) origination date payment $q(s, r, 0)$.⁴

Consistent with equilibrium in the absence of arbitrage opportunities, we assume existence of a (risk-neutral) probability measure Q over realizations such that the time t fair market value of the contract, conditional on no prior default, is the expected discounted cash flows remaining:

$$V(s', r', t) = \mathbf{E}^Q \left[\sum_{t \leq t_i < \tau} R(t_i) q(s, r, t_i) + i_{\tau < T} R(\tau) \rho B(s, r, \tau) \mid s(t) = s', r(t) = r', \tau > t \right] \quad (3)$$

where $i_{\tau < T}$ is an indicator taking value 1 if $\tau < T$, otherwise 0, and

$$R(t_i) \equiv e^{-\int_t^{t_i} r(x) dx} \quad (4)$$

is the riskless rate discount factor along the realized path.

3. Numerical solution for security values

We obtain an approximate numerical solution for $V(s, r, t)$ by recursively working back from contract maturity T in a discretized state space. We will treat r as deterministic in the current paper. Discount factors $R(t_i)$ can thus be calculated directly and consideration of the r state is suppressed. We solve for V on a uniformly spaced rectangular grid on $[0, T] \times [s_{\min}, s_{\max}]$ with mesh size h in the s direction and k in the t direction, chosen so that payment dates lie at gridpoints. Since we also take recoveries to be independent of how negative is s , we let $s_{\min} = 0$ correspond to the default barrier. s_{\max} is chosen to be a large

³Though B will be interpreted in what follows as the contractual loan balance, it could readily be the netted fair market value of a portfolio of derivatives in which the ‘borrower’ is the counterparty, payment received under a credit default swap based on the borrower, etc. Depending on the application, the s referred to might be $s(\tau^+)$, embodying say a dependence of recoveries on how far s jumped into a default region, or it might be $s(\tau^-)$, embodying the influence of prior credit state on borrower actions such as drawdowns on a credit line.

⁴Contractual payments q and recovery balance B will generally depend on *actions* taken by borrower or lender that are permissible under the contract—e.g., drawdown of a credit line or exercise of an option. These plausibly depend on the entire time path of r, s up to the date of such action(s). The current specification assumes that actions influencing contract payments and recovery balance at time t are functions of the then current s, r only, that the contingent actions of both parties are known to the lender, and that all this is already incorporated in functions q, B . That said, some dependence of contractual payments on prior dates’ states will be accommodated, reflecting for example the payment of interest in arrears on loans or swaps.

enough that exceeding it from initial $s(0)$ of interest within time T is sufficiently unlikely.⁵ Let V_{ij} , $i = 0 \dots I$, $j = 0 \dots J$ be the solution at credit quality step i up from the default barrier and time step j away from origination. Vector V_j denotes the solution at time step j . Its length is the number of states considered in the s -direction.

The maturity value $V(s, r, T)$, conditional no prior default, is assumed known from the contract terms and used to fill in values in the terminal vector V_J . We work backward one timestep k by treating the jump and diffusion changes in s as occurring in sequence: I.e., within each interval all changes in s from diffusion over time k occur first, with no jumps; then all jumps in s over time k occur, with no diffusion.

The diffusion part is handled by applying the Crank-Nicholson finite difference method to the vector of s -contingent values, just as if the jump process were not present, with boundary condition at s_{\min} imposed by recovery value ρB . The partial differential equation being solved is what one gets from the Feynman-Kac formula for conditional expected value,

$$\frac{1}{2}\sigma^2 V_{ss} + \alpha V_s + V_t = 0 \quad (5)$$

At the upper boundary we use the (technically incorrect) expedient of assuming V to be quadratic in s . I.e., the value of V_j at s_{\max} is the quadratic extrapolation of its values at the three neighbouring interior gridpoints. In certain cases, if the value of the security were it default-free is known, it works better to suppose that default is effectively inaccessible from s_{\max} and impose that as a known-value condition at that edge. The result of this step is to multiply vector V_j by some matrix C .

The jump part is handled by constructing a transition matrix M over s -states approximating the changes from this source over interval k . This matrix is pre-calculated as follows. Choose a modest sized integer n , and approximate the Poisson process over a subinterval of length $k/2^n$ by supposing at most one jump occurs in that subinterval. From a particular level s_i on the grid, the probability of one jump is $\lambda(s_i)k/2^n$. If that occurs, the probability of moving to level s_l is taken to be $F(s_l + \frac{1}{2}h, s_i) - F(s_l - \frac{1}{2}h, s_i)$. I.e., it is assumed that jumps are exclusively to gridpoints. The subinterval jump transition probabilities are thus⁶

$$\mathbf{prob}\{s_l|s_i\} = \begin{cases} 1 - (1 - F(s_i + \frac{1}{2}h, s_i) + F(s_i - \frac{1}{2}h, s_i))\lambda(s_i)k/2^n & \text{for } l = i \\ (F(s_l + \frac{1}{2}h, s_i) - F(s_l - \frac{1}{2}h, s_i))\lambda(s_i)k/2^n & \text{for } l \neq i \end{cases} \quad (6)$$

The resulting matrix is then squared n times to get the transition matrix M (which allows

⁵This obviously will depend on the particulars of the processes involved and the relative magnitude of cash flows in different regions of the state space.

⁶Jumps to below s_{\min} are treated as being to s_{\min} and to above s_{\max} as being to s_{\max} .

for up to n jumps) over interval k .⁷

Discounting by the default-free rate over time interval $[(j-1)k, jk]$ is accomplished by multiplying the result (for the non-default s levels) by the s -independent quantity

$$d_j \equiv e^{-\int_{(j-1)k}^{jk} r(x) dx}$$

We thus get

$$V_{j-1} = d_j C M V_j \tag{7}$$

In practice, computation costs are reduced by treating as 0 elements of M below some threshold (e.g., 10^{-10}), storing the rescaled result as a sparse matrix, and using a sparse matrix multiplication routine. The ‘multiplication by C ’ is done by solving the tridiagonal equation system that comes out of the Crank-Nicolson algorithm. Computation costs per time step are thus typically much less than a single matrix multiplication of V_j .⁸

One proceeds thus back from time T , pausing at each payment date t_i to add to each element of V_j the s -contingent contractual payment received by the lender in non-default states. Security values for s not at gridpoints is found by (cubic) interpolation on V_0 .

II. Specification and estimation

1. Process chosen

After some experimentation with fit to bond price data (see below), and desiring to keep the number of parameters small, the following specification was tentatively adopted for the processes involved:

$$\begin{aligned} \alpha(s) &= \kappa(\bar{s} - s) \\ \sigma(s) &= \sigma \\ \lambda(s) &= \lambda_0 \max\left\{0, \frac{e^{(10-s)\delta} - 1}{e^{10\delta} - 1}\right\} \\ F(s, s') &= \begin{cases} 0 & s < a \\ (s - a)/(b - a) & s \in [a, b] \\ 1 & s > b \end{cases} \end{aligned}$$

⁷The alternative of allowing for arbitrarily large number of jumps in interval k would require calculating the eigenvalues and eigenvectors of the $I \times I$ matrix of infinitesimal transition probabilities, which would be computationally more costly.

⁸We are losing something by ignoring the interaction of jump and diffusion within the interval when the components of each depends on s . A more ‘centered’ approximation over interval k can be achieved by doing $M^{1/2} C M^{1/2}$ rather than $C M$.

That is, the diffusion volatility of credit quality s is constant; the diffusion drift is mean-reverting; the instantaneous jump intensity is an exponential function with maximum value λ_0 at the default boundary, declining to 0 for s at 10 or above, with curvature parameter δ ; and s -levels after a jump are uniformly distributed over a fixed interval $[a, b]$.⁹

For this specification there are thus eight time-invariant model parameters, including the recovery rate in default: $\theta \equiv \{\kappa, \bar{s}, \sigma, \lambda_0, \delta, a, b, \rho\}$.

2. Parameter estimation from bond prices

Preliminary estimate of model parameters—and some sense of the model’s promise—come from looking at US corporate bond prices. The data set initially available consisted of month-end bond price quotes, obtained from Interactive Data Corporation, for 737 US firms with publicly traded debt. Prices were available for the period May 1993 to December 1997. Even within this period, observations were missing much of the time for a majority of firms. Prices for two months, May 1996 and January 1997, were missing for all bonds. Altogether there were 114614 price observations.

From this, a subset of 105 firms was selected which (i) had price observations every month (except for May 96 and Jan 97 as noted above), and (ii) had multiple bond issues outstanding during the period.¹⁰ Only non-callable, fixed rate bonds were considered. This left 40,955 price observations. The included bonds had agency credit ratings ranging from B to AAA, and maturities ranging from a few months to over 25 years. Average number of bonds outstanding per firm at month end was 8.

Although this subset has many thousands of price observations, certain of its aspects qualify any inference drawn. First, 1993-97 was a period of continuous economic growth. It misses the 1991-92 recession and consequent evidence on the behavior of corporate bond prices in such circumstances. Closely related is the fact that no firms in the sample actually defaulted during this time period. Second, the data set ends before the dramatic mid-1998 rise in credit spreads, which has been characterized by some observers as more a drying-up of liquidity than a widespread deterioration in credit quality. Such phenomena are of obvious importance for both portfolio and single-issuer analysis, and inclusion of that period could quantitatively affect conclusions.

⁹The choice of 10 as the instantaneously default-free credit quality can be viewed as just a scaling parameter given the rest of the specification. Regarding F , experiments were conducted with uniform and normally distributed relative and absolute jumps. $\lambda(s)$ approaches linear as $\delta \rightarrow 0$.

¹⁰Since the default model gives an exact fit to any single market price quote on a given day by construction, firms with single debt issues were excluded as not legitimately testing the model.

Estimation was exclusively of the *risk-neutral process* parameters. I.e., those which, when used to generate a probability measure Q over realizations, make equation (3) a predictor of observed market prices. These are the parameters investors behave as if they believed in, were they actually risk-neutral; they typically differ from those describing the *objective process* of s over time, since investors are risk-averse and not all risks are diversifiable. Additionally, our abstract credit quality is not explicitly observable and must itself be inferred from observed bond prices.

A fixed rate non-callable bond is a simple contract. In valuation equation (3), $\{t_i\}$ are the coupon payment dates remaining, $q(s, r, t_i)$ equal the fixed coupon payments (plus maturity value at T), and B is the par value plus (linearly) accrued interest at default time τ . For given model parameters θ and credit state for firm i at month j of s_{ij} , let V_{ijk} denote the (numerically computed) theoretical value of bond k of that firm.¹¹ Let P_{ijk} denote the corresponding actual market price quote (plus accrued interest to transaction settlement date). Let d_{ijk} denote the Hicks/Macaulay duration of bond ijk (sensitivity of value to its own annualized yield to maturity). The parameter estimation criterion was to find $\theta, \{s_{ij}\}$ minimizing the weighted sum-squared-residuals

$$S \equiv \sum_{i,j,k} \left(\frac{P_{ijk} - V_{ijk}}{d_{ijk}} \right)^2 \quad (8)$$

subject to the constraints

$$\sum_k \frac{P_{ijk} - V_{ijk}}{d_{ijk}} = 0 \quad \forall i, j \quad (9)$$

The deflating of price residuals by duration expresses the residuals (approximately) in terms of difference between quoted yield to maturity and the model's forecast; if not done, the short end of the maturity spectrum would have little influence. Constraints (9) assert that market and model yields, averaged over all bonds of a given firm outstanding on a given day, agree exactly. This identifies the credit state, and renders s_{ij} , conditional on θ , observable without error—one way of handling the latent variable problem were we to bring time series properties of s into the estimation.¹² Minimization of S over θ was done using Marquardt's algorithm, extended to incorporate bounds on parameter values to make the search better behaved.

Notice that we make no use of actual time series properties of bond prices here. Data dates could be shuffled with no effect. The basic question is whether a small set of fixed parameters—common to all firms and all points in time—combined with a *single* firm-and-date-specific variable s can reasonably capture all 'term structures of credit spreads' observed. Equivalently, since it is only the risk-neutral density of time to default that matters for the

¹¹Also needed for this calculation is that date's default-free yield curve, taken from the constant maturity US Treasury yield curve as mentioned earlier.

¹²See Honore (1998) and Jones and Wang (1996) for previous use of this method.

bond values, can the *one-dimensional* family of density functions implied by the specification adequately explain observed prices?

3. Estimation results

Parameter estimation is still in progress. I describe where it now stands. For the data set initially available, within the constraints of the specification chosen, not all parameters were well identified. To cut the story short (and the computational intensity), we imposed the following: recovery rate $\rho = .5$, roughly consistent with Altman and Kishore's (1996) reported average for US corporate bonds;¹³ $\kappa = 0$, so no mean reversion; $\sigma = 1.0$ (per year), viewed as an almost free scaling parameter. The four remaining parameters were estimated with the following results:

$$\begin{array}{rcl}
 \lambda_0 & .48 & \kappa & 0.0 \\
 \delta & .38 & \bar{s} & \text{---} \\
 \bar{s}_a & 1.10 & \sigma & 1.0 \\
 \sigma_a & .80 & \rho & 0.5
 \end{array} \tag{10}$$

Here \bar{s}_a and σ_a are the mean and standard deviation of the uniform distribution on absolute jump destination. I.e., F is uniform on the interval $[\bar{s}_a - 3^{1/2}\sigma_a, \bar{s}_a + 3^{1/2}\sigma_a] = [-.286, 2.49]$.

These parameters give quite a good fit to the observed bond prices. Standard deviation of the *price* residuals is .70 (on par value of 100). Expressed as *yield* to maturity residuals, 53% of observations are below 5 basis points, 77% are below 10 bp, 95% are below 25 bp, 99% are below 50 bp. Model and observed yield curves of a representative firm—US Steel—for a range of observation dates and Treasury yield curves are displayed in the appended figures.

The credit state $s(t)$ can be given more tangible interpretation in terms of the fair credit spread on instantaneously maturing loans. Very short term loans can only default through jump. Expected default losses over an interval of length dt are thus $(1 - \rho)\lambda(s)F(0, s) dt$, the product of the loss rate in default, probability of a jump occurring, and probability that the jump would land in the default region. This implied instantaneous credit spread for $0 < s < 10$ and the above parameter estimates is graphed in Figure 1.

Fair credit spreads for longer maturities must allow for diffusion of s both to default directly and to levels with different jump intensities. They are typically much larger than the instantaneous spread above. The model's predicted par coupon rates for selected s and default-free yield curves are displayed in Figures 2 and 3 (respectively upward sloping Treasury

¹³They report mean recovery rate for senior secured notes of 57.94%, with standard deviation of 23.12%; and mean recovery rate for senior unsecured notes of 47.70%, with standard deviation of 26.60%.

curve, May 1993, and almost flat Treasury curve, May 1997).¹⁴ For currently high quality borrowers, credit spread increases with maturity; for low quality borrowers, after an initial hump occurring somewhere before two years, credit spread declines with maturity.

Time series analysis of the estimated s_{ij} showed them to have small drift (.072/yr), slight negative first order serial correlation (-.067 for monthly observations), and diffusion volatility of σ of 0.57 . The latter was noticeably below the value of 1.0 assumed, so σ was reset to 0.75 and the parameters re-estimated, with better consistency between the assumed and realized σ . A larger data set has since been obtained¹⁵ with the following preliminary parameter estimates:

$$\begin{array}{rcll}
 \lambda_0 & .622 & \kappa & 0.0 \\
 \delta & .244 & \bar{s} & \text{—} \\
 \bar{s}_a & .275 & \sigma & 0.75 \\
 \sigma_a & 1.906 & \rho & 0.5
 \end{array} \tag{11}$$

However the former parameter values are used in the credit line analysis that follows.

III. Revolving credit lines

1. The loan contract

The loan contract is assumed to grant the borrower for a fixed *term* the right—if no default to date—to borrow and repay at will up to fixed *amount*. This is further specialized as follows: There is a fixed *payment interval* associated with the loan (e.g., monthly) during which interest accrues. Drawdowns and repayments can only occur on these ‘payment dates’. Default, though it may be reached between such dates, will only be recognized as of these dates.

Interest accrues between payment dates at a fixed *spread* above the level of a default-free reference rate r_{ref} as of the beginning of the interval.¹⁶ The reference rate is the then prevailing market rate (simple interest) on a default-free zero-coupon instrument of term *reference rate maturity* (e.g., 90 day LIBOR rate). Additionally, the balance due may accrue a *facility fee* payable on the full credit line amount, and/or a *standby fee* payable on the amount undrawn during the interval. The latter are expressed as (simple) interest rates per year. Finally, an up-front *origination fee* expressed as percent of maximum loan amount, may be charged.

¹⁴These are the annual coupon rates for semiannual pay, fixed rate bonds that would give theoretical value of 100. This would generally not coincide with the yield to maturity on an existing bond of the same maturity not trading at par because of its different coupon rate.

¹⁵552,278 observations on 14,617 bonds of 1794 firms over the period Jan 1982 to Aug 1999.

¹⁶Fixed rate rather than fixed spread credit lines, typical for most credit cards, can be easily accommodated but are not done so here.

Contract parameters are thus: the fixed term T , maximum loan amount A , set of repayment/drawdown dates $\{t_i\}$, reference rate maturity T_{ref} , loan spread c , facility c_f , standby c_s and origination c_o fees. These parameters, together with drawdowns and the reference rate at the beginning of a payment interval, determine a contractual *balance owed* at the end of an interval. This is the basis for recoveries if default occurs within the interval; this is the amount assumed paid in full if default does not occur, to be then followed by a new drawdown.¹⁷

2. Borrower drawdown behaviour

The proportion of the credit line drawn down depends on the borrower's current credit quality, as indicated by the fair market rate he would pay elsewhere on new borrowings. This *alternative opportunity rate* r_{opp} is assumed to be the fair (simple) interest rate on a zero-coupon bond of specified maturity T_{opp} . We wish to account for the fact that borrower behaviour may be rational to varying degrees—or influenced by factors other than just immediate borrowing costs—by specifying a flexible functional form that includes full ‘rationality’ as a special case.

In the *interest incentive* model, the proportion of A drawn down is a function of the gap $g \equiv r_{\text{opp}} - (r_{\text{ref}} + c - c_s)$, i.e., the difference between current cost of borrowing elsewhere and (net) cost of incremental borrowing under the credit line. The shape assumed is that of the cumulative normal distribution function:

$$f(g) = d_{\min} + (d_{\max} - d_{\min}) N((g - d_{\text{shft}})d_{\text{sens}}) \quad (12)$$

Here $N(\)$ is the standard normal distribution function, with behavioural parameters d_{\min} , d_{\max} , d_{sens} , d_{shft} representing minimum and maximum drawdowns, sensitivity of drawdowns to interest incentives, and gap level where maximum sensitivity occurs.¹⁸ ¹⁹ The amount borrowed for interval $[t_i, t_{i+1}]$ is thus $Af(g(t_i))$. Fully rational behaviour in this model is characterized by $d_{\min} = 0$, $d_{\max} = 1$, $d_{\text{shft}} = 0$, $d_{\text{sens}} = \infty$, and T_{opp} equal to the payment interval. Constant borrowing at the average of d_{\min} and d_{\max} is characterized by $d_{\text{sens}} = 0$.

An alternative model, more suggestive that *liquidity constraints* rather than interest incentives are the dominant influence on borrower behaviour, is to specify the gap g as $r_{\text{opp}} - r_{\text{ref}}$. This means that the closer the borrower is to default, as measured by the fair market credit spread he would pay on new borrowings, the higher will be his utilization of the credit line,

¹⁷Realistically, of course, these two transactions occur simultaneously and show up as just a net change in drawdowns.

¹⁸Actually, we multiply the argument of N by the scale factor $\sqrt{2\pi}$ so that d_{sens} may be interpreted as the change in drawdown, as a proportion of $d_{\max} - d_{\min}$, induced by a 1%/yr. change in g at the most sensitive point.

¹⁹A non-zero value for d_{shft} can either reflect real transaction or issuance costs associated with alternative borrowing, or reflect a perception threshold, banking relationship value, etc.

regardless of its contractual terms. This g would be used in (12) for valuation purposes. Adverse selection, in the sense that line utilization will be higher when credit quality is lower, will still be present.²⁰ However the implications for loan contract design change since spread c and standby fee c_s no longer affect behaviour.²¹

3. Treatment of risk-free rates

The current implementation treats the default-free instantaneous interest rate $r(t)$ as evolving deterministically. The current term structure is input as an instantaneous-maturity forward rate curve consistent with current Treasury rates. The latter is taken as the Federal Reserve Bank H-15 constant-maturity rates (3 mo. - 30 yr.). The forward rate curve chosen is the minimally-varying step function consistent with the rates reported. This is the path assumed for $r(t)$. Note that this implies no discontinuity in rates (spot or forward) that are for other than instantaneous maturities.

It is tempting to excuse this for the moment by assuming that credit quality and default-free rates evolve stochastically but independently. If loan cash flows depended solely on s , then risklessly-discounted cash flows could be factored into r and s contingent components, with the current term structure used to represent the expected discount factor, and all would be okay. However line utilization is assumed to depend on the credit spread $r_{\text{opp}} - r_{\text{ref}}$. Since that relation is non-linear, and each of r_{opp} and r_{ref} are non-linearly related to $r(t)$, we are clearly missing something by ignoring uncertainty about r . Ascertaining whether it is quantitatively important in the r - s -independent case awaits further work.

More realistically, of course, there are good reasons to believe r and s do not evolve independently. Both are plausibly influenced by common business cycle and macroeconomic factors; and high interest rates by themselves can impact negatively the financial condition of borrowers.

4. Numerical solution for line value

Valuation of the credit line proceeds as in section 3. with some modification. We must accommodate the fact that the size of drawdown is determined by s at the beginning of a payment

²⁰Note that the presence of idiosyncratic additional influences, or noise, affecting drawdowns would not alter valuation of the credit line if such influences have 0 mean effect on f and evolve statistically independently of credit quality, interest and recovery rates. This follows from cash flows both in and out of default being linear in f . Interpret f as expected drawdowns conditional on s, r . Note that this rules out recoveries depending on credit state immediately prior to default.

²¹A mixture of the two models, adding one further parameter, is also possible if empirical evidence on borrower behaviour is available and suggests it is appropriate.

interval, but whether one gets paid is determined by s at the end. Moreover the interest rate charged over the interval is based on r_{ref} prevailing at the start. Thus cash flows at t_i are determined by the r, s state at both t_i and t_{i-1} jointly.²² This is handled as follows. Between payment dates, the vector V_j recursively solved for is the value of the credit line's cash flows from the *next* payment date onwards, as a function of current s , conditional on no default prior to that time. If $s = 0$ currently, then all future cash flows will be zero, captured by imposing boundary condition $V(0, r, t) = 0$ for t not in $\{t_i\}$. Let $\Delta \equiv t_{i+1} - t_i$ be the payment interval. If t is a payment date t_i , then for each s in our discretized state space we calculate $r_{\text{ref}}, r_{\text{opp}}$, gap g , and the fair value p of a unit discount bond of maturity Δ issued by the borrower with the same recovery rate as the credit line. We add to $V_j(s)$ the amount

$$-Af(g) + pA(f(g)(1 + (r_{\text{ref}} + c - c_s)\Delta) + (c_s + c_f)\Delta) \quad (13)$$

First term is the cash paid out as current drawdown; second term is expected^Q discounted value of the contractual payments now due at t_{i+1} . At $s = 0$ this treats the borrower as drawing down, defaulting right away, and providing immediate recoveries a proportion of the balance due at the *next* payment date. For equal length payment intervals the vector of risky bond values $p(s)$, up to a riskless rate discount factor independent of s , need only be computed once.

5. Numerical experiments

This section explores numerically how credit line value varies with contract and model parameters. At times we ask what the 'fair level' of some parameter would be. By this we mean the level which makes initial value of the credit line 0 to the lender. Of primary interest is how the optionality of drawdowns affects the fair contract terms.

Our base case is a 3 year credit line with contractual maximum balance $A = 100$; payment interval, reference and opportunity cost rate maturities of one month ($T_{\text{opp}} = T_{\text{ref}} = .0833$ yr); contractual spread over reference rate $c = 2\%/yr$; origination, facility and standby fees of 0. Drawdown behavior parameters are $d_{\text{min}} = 0, d_{\text{max}} = 1, d_{\text{shft}} = 0$ in all cases. Recovery rate in default is $\rho = .5$. The default-free term structure is assumed flat at $5\%/yr$, continuously compounded.

Consider first the case of $d_{\text{sens}} = 0$: Drawdown is thus fixed at 50 regardless of credit state or contractual spread, so there is no optionality. An initial credit state of $s(0) = 4.72$ makes this contract have 0 value to the lender. For this credit state, the fair contractual spread for alternative maturities (the equilibrium term structure of credit spreads for fixed

²²The payment of interest in arrears is not a problem in the current deterministic interest rate setting, but would become so if the default-free rate were treated as stochastic.

Table 1: Term structure of credit spreads

	1 mo.	6 mo.	1 yr.	2 yr.	3 yr.	5 yr.	10 yr.	20 yr.
spread c	.77	1.04	1.29	1.67	2.00	2.48	2.91	2.94
prob. default	.0013	.0098	.0235	.0590	.1029	.2015	.4144	.6423

Table 2: Line values with high rationality

c	0%	2%	4%	10%	20%	50%	fair c_o	fair c_s
1 yr.	-1.24	-0.45	-0.42	-0.37	-0.34	-0.24	0.45	0.48%
3 yr.	-5.32	-2.95	-2.75	-2.40	-2.18	-1.52	2.95	1.81%

Initial credit quality $s(0) = 4.72$. Fair c_o, c_s are for contractual spread of 2%.

rate, monthly coupon bonds for that credit quality) is given in Table 1. Also given is the probability of default (under the risk-neutral distribution) within the term of the contract.²³

Consider next the case of $d_{\text{sens}} = 1000$. Here drawdown (almost) jumps between 0 and 100 as s crosses the level for which the contract spread just equals the fair credit spread on one month borrowings elsewhere (at approximately $s = 2.4$ for $c = 2\%$). This is a no-win situation for the lender: Borrowing only occurs, and the loan spread is only collected, in states where it is insufficient to compensate for the risk of default. Table 2 displays the negative net value of the credit line for initial $s = 4.72$, various contractual spreads up to 50%, and line maturities of 1 and 3 years. Value is less negative the higher is c since that shrinks the set of credit states for which any borrowing occurs. However loan origination and/or loan standby fees can make the loan break even. Displayed are the fair values of one or the other of these contract terms when c is 2%. Notice that they rise rapidly with maturity. The standby fee (paid on loan amount undrawn) also alters the states in which the line is drawn down since the marginal cost of borrowing is reduced by this amount.

Finally consider intermediate cases of d_{sens} . These represent noisy or quasi-rational behaviour by borrowers, from the perspective of pure interest rate incentives. Line value declines

²³This is computed numerically using the previously described procedure but ‘valuing’ a security that pays 1 when default occurs, 0 otherwise, and whose value is not discounted.

Table 3: Line value as function of contractual spread

c	0%	2%	4%	6%	8%	12%	16%	20%
1 yr.	-0.88	0.06	0.63	0.87	0.87	0.50	0.07	-0.18
3 yr.	-4.26	-1.58	0.08	0.86	0.95	0.04	-1.06	-1.72

$$d_{\text{sens}} = 5$$

Table 4: Fair contractual loan spreads

d_{sens}	0	1	2	3	4	5	6	7	10	20
1 yr.	1.28	1.72	1.75	1.78	1.81	1.84	1.88	1.91	2.06	—
3 yr.	2.00	3.05	3.20	3.36	3.57	3.87	4.39	—	—	—

For initial credit state $s = 4.72$

monotonically with d_{sens} as long as drawdowns are centered around the full rationality point, i.e., $d_{\text{shft}} = 0$ and the rate maturities involved coincide with the payment interval. We will not display these. But unlike the extremes considered so far, line value is no longer monotonic in contractual spread c . Table 3 gives line values for c ranging from 0 to 20% for $d_{\text{sens}} = 5$. Origination and standby fees are 0.

The 1 yr. line reaches a maximum value of .90 at c of approximately 6.9%; the 3 yr. line reaches a maximum of .98 at $c = 7.3\%$. Above these levels, the reduction in balances on which the spread is received more than outweighs the increase in spread and reduction in default losses. The reverse happens as spreads are reduced.

A different perspective comes from looking at the lowest contractual spread at which the line value is 0. Table 4 displays this fair line spread for a range of d_{sens} values and initial credit quality 4.72. For each maturity, however, there is a size of d_{sens} beyond which there is no spread that can make the line break even for the lender. In other words, without origination, facility, standby fees, or a shortening of maturity, there are no terms under which the lender can rationally grant the borrower a revolving credit line.

Table 5 gives some sense of how much drawdowns are varying with credit state for the specification being used. It imposes the contractual spread that would be fair if borrower

Table 5: Line drawdowns as function of credit state

s	0	1	2	3	4	5	6	7	10	20
1 yr.	1.00	0.85	0.53	0.49	0.46	0.44	0.43	0.42	0.41	0.41
3 yr.	1.00	0.78	0.42	0.39	0.36	0.34	0.33	0.32	0.31	0.31

Fair contractual spread for $d_{\text{sens}} = 5$ and $s(0) = 4.72$: 1.843% on 1 yr., 3.868% on 3 yr.

sensitivity were $d_{\text{sens}} = 5$ and initial credit quality $s = 4.72$ of our benchmark case.

6. Conclusion

This paper is not finished. Hopefully, the value of the options implicit in credit lines—and the importance of accounting for them in bank pricing policy—has been made clear. But there is clearly much left to do and more that could be done. Of highest priority is parametrizing an objective process for credit quality consistent with our risk-neutral specification so far. This is needed to make use of the time series properties of bond prices, and historical data on actual defaults, in parameter estimation. Indeed, in the context of bank portfolios, it more likely that only the latter information will be available. This raises econometric issues of estimating jump-diffusions for latent state variables from discrete observations.

Further, there is work to be done applying whatever specification is adopted for default to the wide array of credit-related products mentioned in the introduction. It is unlikely that current practice prices these instruments consistently. If it does not, then these tools can improve management practice and market efficiency; if it does, then that is of academic interest in itself as a positive theory of observed contractual arrangements.

And finally, the question of joint evolution of credit quality across a portfolio of borrowers must be addressed. Many are now working on such problems. But advances in securitization, the advent of portfolio credit derivatives, and progress of regulators toward viewing financial entities as a whole raises the pressure for workable solutions.

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