Pricing GNMA Mortgage Certificates

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This paper sets out and estimates a joint model of mortgage prepayment behaviour and GNMA certificate valuation. Prepayment behaviour by households is assumed to be quasi-rational. GNMA certificate values are taken to be those that would prevail in an arbitrage–free asset market equilibrium. The interest rate environment is the two factor continuous time model of Jacobs and Jones (1986).

GNMA certificates are complex securities. They are claims to a fixed share of the principal and interest payments on a pool of U.S. government guaranteed residential mortgages. The mortgages in each pool have identical coupon rates and maturity dates (to within a year). The mortgages are assumable and prepayable at any time without penalty. They are usually of 30 year original maturity, though some of 15 year maturity are issued. The mortgage rate charged to the borrower is 0.5% higher than the passthrough rate assigned to the certificate.[1] The originating lender retains 0.5% per year of the remaining balance each month as a servicing fee before passing on the borrowers' payments to the certificate holder.[2]

A GNMA certificate is an interest rate contingent claim — in this case a type of callable bond. Its value depends on the course of interest rates, features of the contract determining cash flows in the absence of call, and the call policy followed by the borrower. A first cut solution might focus on the absence of stated prepayment penalties and assume rational exercise on the basis of interest rate factors alone. However significant complications arise in the case of residential mortgages.

First, prepayments related to interest rate changes are typically refinancings. Refinancing involves significant up front fees, intermediation costs paid to a third party. Therefore the cash flows perceived by the borrower, whose value he would rationally try to minimize by choice of call policy, differ from those received by the lender. The fact that these cash flows are not identical means

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[1]Thus an 8% GNMA certificate is in fact a claim on a pool of 8.5% mortgages.

[2]Further details on these and other mortgage passthrough securities can be found in Sullivan, Collins and Smilow (1985).

that the nature of equilibrium in the market for financial services (i.e., markups and spreads) has to be addressed in any solution.

Second, prepayment options are clearly exercised suboptimally from the perspective of interest rate factors alone. We do not see all mortgages in a given pool prepaying at the same moment, as should occur with perfectly rational exercise. Observed prepayment rates avoid the extremes of 0 and 100%. This may reflect the presence of queues and time lags in the processing of prepayments, individual specific factors which would rationalize the behaviour, or simply limitations on individual rationality. In any case, it raises the problems of selecting factors in addition to interest rates that might warrant explicit consideration, of the history of a pool being relevant (at the least, the proportion of mortgages remaining), and of parametrizing the call policy in a form for empirical implementation.

It is be desirable to structure a model with fully rational exercise as a benchmark or limiting case, around which an empirical call policy might be centered. This facilitates appraising what might happen if borrowers were to become more responsive to interest rates — a conservative valuation approach for long positions — and allows prepayment predictions to be more confidently extended to situations and interest rate levels not yet experienced. Monte Carlo approaches to the valuation problem, which have become widely used for mortgages (e.g., Schwartz and Torous, 1989), do not have this characteristic since individual behaviour is not forward–looking.

Section I develops the theoretical prepayment model. Section II outlines the underlying model of interest rate movements used for the empirical work. Section III then estimates the parameters of the prepayment model from monthly prepayment rates on GNMA pools from 1982-87. Coupon rates on the mortgages range from 6.5 – 16%. Various hypotheses regarding rationality, stability of the model parameters across coupon rates and time, seasonality and 'seasoning' effects are examined. The relationship between equilibrium mortgage rates and Treasury bond rates is discussed.

I. Theoretical Framework

This section lays out the theoretical prepayment and valuation model. It is developed in four stages. First, we review the equilibrium valuation of interest rate contingent claims in arbitrage–free capital markets without intermediation costs. Next, we incorporate refinancing costs into the problem, creating a distinction between the cash flows paid by the borrower and those received by the holder of the mortgage certificate. Third, a parametric prepayment policy suitable for empirical implementation is described. This family of policies includes 'fully rational' prepayment as a special case. Finally, the method of handling seasonality, seasoning, non-homogenous pools and servicing fees is detailed. Discussion of the particular stochastic process assumed for interest rates is deferred to the next section.

1. Valuing interest rate dependent claims

We adopt the arbitrage–based valuation model for interest rate dependent claims of Brennan and Schwartz (1977), Cox, Ingersoll and Ross (1978), Vasicek (1977) and others.

Assumption 1 *Trading takes place continuously without taxes or transaction costs in perfectly competitive markets. The price at time t of any interest rate dependent claim is a function of some vector x of state variables. The state x follows an exogenous stochastic process*

$$dx = \alpha\, dt + \sigma\, dz \tag{1}$$

In the above, x and α are n dimensional column vectors; σ is a $n \times n$ matrix. The column vector dz is the increment of a Weiner process with 0 mean, variance 1 per unit time for each element, and local correlation ρ_{ij} between elements i and j. The process parameters α and σ can be functions of x, t.

Now consider any security whose value is contingent solely on the state variables in x and time. Let $P(x,t)$ denote its value with time t left to maturity in state x. Suppose the security offers continuous coupon payments at a rate $q(x,t)$ and has maturity value $P(x,0)$. Applying the principle that equilibrium requires that no riskless arbitrage opportunities exist implies that P must satisfy the following differential equation (cf. Brennan and Schwartz, 1977, 1985; Cox, Ingersoll and Ross, 1985):

$$\frac{1}{2}\sigma^2 P_{xx} + (\alpha - \lambda)P_x + q - P_t = rP \tag{2}$$

In the above r is the instantaneous riskless interest rate, λ is the market price of x-risk (common to all securities), and subscripts indicate partial differentiation.[3] $\alpha, \sigma, \lambda, q, r$ may all be functions of x, t. The particulars of a given security imply boundary conditions that determine which of the many functions satisfying (2) is the appropriate one.

A first cut solution to the GNMA valuation problem would be to assume no intermediation costs, fully rational exercise of the prepayment option and a uniform mortgage pool. Let c denote the constant payment rate that amortizes the mortgage to 0, and $B(t)$ denote the balance remaining at time t per dollar of initial mortgage balance. The payment rate c and contractual mortgage rate are related in a one–to–one fashion for given original term T. A borrower would rationally prepay whenever the market value of his remaining liability exceeds

[3] For notational simplicity we will write all relationships as if the state vector x is a scalar for the rest of this section. When x is a vector, equation (2) takes the same form, with $\sigma^2 P_{xx}$ denoting $\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}P_{x_i x_j}$ and the second term of (2) interpreted as an inner product.

the option exercise price $B(t)$. The cash flows received by the certificate holder are identical to those paid by the borrower. The value of the certificate would be given by the solution to (2) that satisfies

$$\begin{aligned} q(x,t) &= c \\ P(x,0) &= 0 \\ P(x,t) &= \min\{B(t), P(x,t)\} \end{aligned}$$

In such a case all mortgages would prepay at the same instant, so there would be no issue of the fraction of mortgages remaining in the pool.

2. Suboptimal call policies

We do not observe all mortgages in a given pool prepaying simultaneously. Prepayment behaviour is evidently suboptimal from the perspective of interest rate considerations alone. Following Brennan and Schwartz (1985), we suppose

Assumption 2 *The fraction of remaining mortgages prepaying at each point in time is independent of the fraction that have prepaid to date.*

Suppose the borrowers in a given pool follow a given *prepayment policy*. Let the policy be denoted by $\pi(x,t;c)$, where π is the proportional redemption rate in state x on mortgages with payment rate c and t years remaining.[4] Let f denote the fraction of original mortgages that remain at time t. Assumption 2 implies

$$df = -\pi(x,t;c) f \, dt \tag{3}$$

The value of the average borrower's liability and of the GNMA certificate depend on f, which summarizes the relevant aspects of past interest rates, in addition to x and t. However, as is shown in the Appendix, the resulting proportionality of all cash flows to f implies that the value of the security is also proportional to f. The value when $f = 1$ is the solution to

$$\frac{1}{2}\sigma^2 P_{xx} + (\alpha - \lambda)P_x + c + (B - P)\pi - P_t = rP \tag{4}$$

subject to the maturity value $P(x,0) = 0$. The value $\tilde{P}(x,t,f)$ when f is less than 1 is simply $fP(x,t)$. This homogeneity assumption is useful because it permits some aspects of the history of the pool to be considered without expanding the number of state variables that have to treated when numerically solving the valuation equation. Note that the above relations include the case of optimal prepayment behaviour, for which the policy is

$$\pi(x,t;c) = \begin{cases} \infty & \text{for} \quad x,t \text{ such that } P \geq B \\ 0 & \text{for} \quad x,t \text{ such that } P < B \end{cases}$$

[4] At the level of the individual mortgage in the pool, π can be viewed either as the fractional prepayment rate or as the intensity of a Poisson process whose event is total prepayment of the loan.

3. Refinancing costs

Mortgage prepayments related to changes in interest rates are primarily refinancings. That is, the existing mortgage balance is paid off with the proceeds of a new mortgage issued at currently prevailing rates. However this is not costless to the borrower. Although there are no penalties for prepaying the existing mortgage, there are significant up-front fees associated with arranging the new financing. Such fees will influence the optimal and actual prepayment policies households choose. A rational borrower would select a policy that minimizes the value of his total mortgage liability, inclusive of current and future refinancing costs.

Refinancing fees are not received by the holder of the existing mortgage. They are received by a third party, or financial intermediary. Thus the cash flows from the perspective of the borrower differ from the cash flows from the perspective of the GNMA certificate holder. It is the former that guide the borrower's prepayment policy, while it is the latter determines the market value of the GNMA certificate. To fit this in requires hypotheses about the nature of equilibrium in the market for financial services—specifically, mortgage origination.

The market value of the borrower's cash flows necessarily exceed the value of the certificate holder's. Denote the former by $Q(x, t; c, \pi)$, and the latter by $P(x, t, ; c, \pi)$ as before. The roles played by the payment rate c (rate on the mortgage in place) and prepayment policy π are emphasized by writing them as explicit arguments in Q and P.

For simplicity assume that refinancing costs are directly proportional to the loan size, and independent of the characteristics of the mortgage being prepaid (i.e., non-discriminatory).

Assumption 3 *Refinancing fees are a fraction $\eta(x, \pi)$ of the amount of a loan.*

Notice that refinancing costs can depend in principle on the prepayment strategy followed by borrowers and the interest rate state x. More will be said about these costs below.

Rational prepayment behaviour involves looking forward to the distribution of future interest rates, weighing any gain from currently refinancing against both the fees involved and the possibility that even more attractive terms might be obtained by waiting. Were it not for refinancing fees the latter factor could be ignored: If more attractive terms became available, one could costlessly refinance again. But in the presence of such fees a significant further drop in rates may be required. Refinancing now and refinancing again in the near future become, in a sense, mutually exclusive.

An interest rate whose distribution is particularly relevant to a borrower is the rate on new mortgages of the term T for which he would renew. Let this rate be denoted by $\bar{c}(x, \pi)$. We assume this rate is determined by equilibrium in the market for GNMA certificates:

Assumption 4 *The current coupon rate $\bar{c}$ for mortgages is that which makes GNMA certificates backed by such mortgages worth par. i.e.,*

$$P(x, T; \bar{c}, \pi) = 1$$

In our context, T is 30 years. The current mortgage rate is thus endogenous to the model. It is the rate that an investor acquainted with actual prepayment behaviour would require to pay par for new GNMA certificates. This captures the notion that the GNMA certificates are readily traded substitutes for other Treasury securities and that their coupon rate must offer investors fair compensation for the prepayment option in the mortgage.

It follows from the above that $Q(x, T; \bar{c}, \pi) > P(x, T; \bar{c}, \pi)$. Since Q and P are the values of flows for mortgages already in place, the difference between the two is the market value of the *future* refinancing costs associated with a new mortgage. The realized costs will depend on the path of future interest rates, the fee schedule η, the equilibrium mortgage rate schedule $\bar{c}$ and the exercise policy π followed by households. Let the proportionate difference be denoted by $\psi(x, \pi)$. I.e.,

$$\psi(x, \pi) \equiv \frac{Q(x, T; \bar{c}, \pi) - P(x, T; \bar{c}, \pi)}{P(x, T; \bar{c}, \pi)} = Q(x, T; \bar{c}, \pi) - 1$$

The last equality above follows from the previous assumption.

The situation is clearly getting messy. The refinancing fee schedule η may depend on prepayment behaviour, and we would expect prepayment behaviour to reasonably depend on refinancing fees. The difficulties in simultaneously determining these elements is minimized by the following:

Assumption 5 *The value of current plus future refinancing costs per dollar of loan is independent of the current state. i.e.,*

$$(1 + \eta(x, \pi))(1 + \psi(x, \pi)) = 1 + m \tag{5}$$

where m is a constant.

Borrowers thus anticipate total fees over the life of a mortgage equal to some constant fraction m of the amount borrowed, a 'constant proportional markup'. We offer no reason beyond technical convenience why equilibrium in the mortgage origination market should take this form rather than some other.

We do suppose, however, that the *marginal* investment opportunities available to the borrower during the life of the mortgage are the same as those for purchasers of the GNMA securities. That is, we envisage homeowners as holding positive balances of Treasury securities or other securities which, after adjustment for convenience or liquidity, have comparable yields. It will thus be appropriate to value the borrower's liability in the same manner, and with the same assumption about the interest rate process, used for valuing the GNMA certificates. The valuation equation for the borrower's liability differs slightly from (4) to reflect the payment of refinancing fees. Q is the solution to

$$\frac{1}{2}\sigma^2 Q_{xx} + (\alpha - \lambda)Q_x + c + ((1+m)B - Q)\pi - Q_t = rQ \qquad (6)$$

subject to the maturity value $Q(x,0) = 0$. This embodies the notion that not only will the borrower make contractual payments at a rate c on his loan, but also with probability π (or for a fraction π of his remaining balance), he will incur a liability worth $(1+m)B$ but be relieved of one worth Q through refinancing.

An implication of this assumption is that borrowers and lenders have identical beliefs about the course of future interest rates. This rules out any incentive to postpone refinancing simply because households 'expect interest rates to fall'. GNMA certificate buyers should have the same expectation, and it would be reflected in the current mortgage rate. However it does not rule out a more complex interaction between equilibrium mortgage rates and interest rate expectations in which *new* borrowers do not have an incentive to postpone borrowing, yet existing borrowers do have one because of refinancing costs.

Let us see how the above pieces interact in the case of an optimal prepayment policy. A borrower rationally pays off his existing loan balance $B(t)$ with a new loan if it reduces the value of his overall mortgage liability after allowing for current refinancing fees. That is, if [5]

$$(1+\eta)B(t)Q(x,T;\bar{c},\pi) \leq Q(x,t;c,\pi) \qquad (7)$$

This supposes that the individual has an existing mortgage at rate c and would be constrained in the future to follow prepayment policy π. Substituting Assumptions 4 and 5 in the above yields

$$\frac{Q(x,t;c,\pi)}{B(t)} \geq (1+m) \qquad (8)$$

The desirability of prepaying thus hinges on whether the value of the existing mortgage liability relative to remaining balance is greater or less than some critical ratio that is state independent. This ratio Q/B will be used in the next section as an indicator of the prepayment incentive.

Let us summarize what can now be determined. Given a stochastic process for x and a prepayment policy π, we can determine the equilibrium market value $P(x,t;c,\pi)$ of existing certificates and through this the equilibrium rates $\bar{c}(x,\pi)$ on new mortgages. Given the markup rate m for new loans, we can also determine the value $Q(x,t;c,\pi)$ of an existing borrower's liability, inclusive of future refinancing costs, and the strength of his current prepayment incentive

[5] We assume that the refinancing costs are financed by increasing the balance of the new mortgage.

Q/B. The exogenous elements so far are the state variable process parameters α, σ, the 'price of risk' parameters λ, the markup rate m and the prepayment policy $\pi(x, t; c)$.

4. A parametric prepayment policy

As pointed out earlier, observed prepayment behaviour differs substantially from optimal behaviour on the basis of interest rate considerations alone. This section proposes a family of prepayment policies whose parameters can estimated from available data.

The first element of structure, and link to rational behaviour, is to suppose that all borrowers respond to the prepayment incentive in similar fashion.

Assumption 6 *The prepayment rate on each class of mortgage depends only on Q/B. i.e.,*

$$\pi(x, t; c) = \pi(Q/B) \tag{9}$$

The important feature here is that the prepayment rate does *not* depend on the term remaining, balance remaining or coupon rate of the mortgage being refinanced, except to the extent that these factors influence Q/B. Notice that since Q depends on π, borrowers assess the prepayment incentive at a given moment recognizing that they will be constrained to follow policy π for any *future* refinancings. If π is not optimal, due either to non-interest factors or to bounded rationality, households are aware that such considerations will continue to apply in the future.

Second, we impose a particular parametric family of prepayment functions. Since we will be using monthly data, and the prepayment option is only exercisable in conjunction with the regular monthly mortgage payment, we henceforth take π to be the proportion of remaining mortgages that prepay in the current month. Specifically, let π take the form

$$\pi(y) = k_1 + (k_2 - k_1)\, N((y - k_0)k_3) \tag{10}$$

in which $N(.)$ denotes the standard normal cumulative distribution function and $y \equiv Q/B$. In addition, denote by k_4 the value of $1 + m$ in Assumption 5. The parameters k_0–k_3 determine the prepayment function's shape: π varies between a minimum value k_1 and maximum value k_2. Between these bounds it looks like the normal cdf centered around $Q/B = k_0$. The slope at the inflection point is proportional to k_3.

Although this form was chosen primarily because of its flexibility and potential for 'fitting' the data, its parameters do have interpretations. Parameter k_1 is the minimum monthly prepayment rate that would prevail when interest rate incentives are strongly against prepayment. The prepayments could be associated with defaults, sale of house with the purchaser unwilling or unable to assume the remaining balance (e.g., if it were small compared to the purchase price), lack of investment opportunities for accumulated wealth or simply irrational behaviour. Parameter k_2, a cap on the maximum monthly prepayment rate, represents the probability that a given household in the pool bothers to contemplate refinancing in a given month, or the probability that their refinancing could be processed by lenders (i.e., queues form in periods of high refinancing demand). Parameter k_3 characterizes how sensitive prepayments are to interest rates. High values are associated with a very rapid rise in repayments as Q/B passes the critical level k_0. One interpretation of k_3 is one over the standard deviation of some household specific random component to refinancing costs, ϵ_h for household h (or incentive to refinance if negative).[6] Parameter k_4 is the value of current plus expected future refinancing costs that households behave as if they face. It enters into the calculation of Q. Rational exercise of the *current* prepayment option is associated with $k_4 = k_0$, regardless of their common level. A higher common level of these parameters implies myopic behaviour in the sense that *future* refinancings are considered progressively less likely.

Fully rational prepayment behaviour is a special case of the above policy. It corresponds to $k_1 = 0$, $k_2 = 1$, $k_3 = \infty$, and $k_0 = k_4$ being at most a small multiple of objective refinancing costs. At the opposite extreme is completely interest insensitive behaviour. It would be characterized by $k_1 = k_2$ or by $k_3 = 0$.

5. Seasoning, seasonality and inhomogenous pools

The data we use to estimate the prepayment policy parameters consists of monthly observations of the prepayment rates on various mortgage pools. Casual inspection of the data, and studies by others, suggest the following patterns. First, there is a noticeable seasonal pattern in the prepayment rates of discount issues (pools whose coupon rate is below the current mortgage rate), peaking in late summer and bottoming out after the turn of the year. Second, recent issues (time left to maturity close to 30 years) have very low prepayment rates compared to issues that are a few years older, suggesting a 'seasoning' effect. We should point out that this could be as much a consequence of the fact that mortgage rates are not likely to have changed as much for recent issues as for older ones. Thus the casual observation may simply reflect the workings of rational prepayment incentives combined with a diffusion process for interest rate changes. Finally, pools with large fractions of the original mortgages already

[6] This additional component would have to be randomly redistributed across households in the pool each period to be consistent with Assumption 2. Otherwise pools which had high prepayment rates in the past would be depleted of those with above average propensity to prepay, which would alter the average prepayment behaviour of those remaining. Randomly redrawing the component each period has its own consistency problems however. If households are rational, and anticipate fluctuation in the idiosyncratic portion of refinancing costs, then they will rationally wait (search) for an abnormally good draw before refinancing. As a result, households would appear on average to postpone refinancing beyond the optimal time based on interest rate considerations alone: that is, behaviour would involve $k_0 > k_4$.

prepaid — older high coupon pools — often have lower proportional prepayments than many lower coupon pools. This has been termed 'burnout' by some. In this section we embellish the prepayment model somewhat to incorporate these possibilities and allow their presence to be tested.

Consider first the seasonality and seasoning effects.

Assumption 7 *Seasoning and seasonality are 'proportional hazards' applied to the base prepayment rates k_1 in each pool.*

More specifically, we introduce one additional parameter to allow for each effect. The seasonality factor is assumed to be a sine wave with amplitude k_6, peaking in August of each year. The seasoning factor takes the form of a negative exponential function varying between 0 at mortgage inception and 1 with decay rate k_5. Thus the prepayment rate is actually taken to be

$$\pi(\frac{Q}{B},t) = k_1[1-e^{-k_5(T-t)}][1+k_6\cos(2\pi(t-t_{\mathrm{Aug}}))] + (k_2-k_1)\,N((\frac{Q}{B}-k_0)k_3)$$

where π is the function defined in the preceding section, less k_1. In the calculation of Q by the borrower these effects are assumed rationally anticipated.

The 'burnout' effect is treated by postulating two classes of borrowers.

Assumption 8 *All pools initially consist of some fraction k_7 of borrowers whose prepayments are completely insensitive to interest rates, and have a fixed prepayment rate k_8. The remaining fraction $(1-k_7)$ follow policy π.*

The seasoning and seasonality factors described above are also to be applied to the interest insensitive borrowers. Since the behaviour of this latter group is completely deterministic, the proportion of their original loan balance remaining is a deterministic function of the mortgage rate and age. Details are provided in the appendix. The monthly data includes the fraction of original loan balances left. Thus, given k_7, k_8 and c, the proportion of borrowers that remain who fall in this group can be inferred. Once this fraction is known, the values, expected average maturities, and prepayment rates on loans to the two types of borrowers can simply be linearly combined to get the corresponding quantities for the pool as a whole.

Some comment should be made on the handling of the .5%/year on the remaining balance servicing fee retained by the mortgage intermediary. We treat this as a fee for services currently rendered in the eyes of the borrower. It is not avoidable by refinancing. Consequently its only impact is to alter the balance schedule $B(t,c)$ slightly, since the amortization schedule for the mortgage is based on the pool rate plus .5%. For the certificate holder, monthly receipts from non-prepaid mortgages rise slightly over time as the remaining balance and hence servicing fees shrink. This is taken into account in the calculation of the value of the GNMA certificate.

6. So how does one value the GNMA certificate?

Determining the theoretical value of a GNMA certificate with this setup is a multistep procedure. Given a specification of the interest rate process α, σ, a coupon rate on the mortgage, and the first seven parameters k_0–k_6 determining prepayment behaviour, one first solves the differential equation

$$\frac{1}{2}\sigma^2 Q_{xx} + (\alpha - \lambda) Q_x + c + ((1+m)B - Q)\pi - Q_t = rQ \tag{11}$$

for the state contingent value of the hypothetical borrower's liability $Q(x,t)$. In this π is the function of Q described previously. Substituting the resulting $Q(x,t)$ back into this function gives the theoretical state contingent prepayment rates $\pi(x,t)$. This prepayment rate function is then inserted into the valuation equation for claims on interest sensitive mortgages:

$$\frac{1}{2}\sigma^2 P_{xx} + (\alpha - \lambda) P_x + q(t) + (B - P)\pi - P_t = rP \tag{12}$$

This is almost identical to the borrower's equation, except that the cash flows $q(t)$ to the certificate holder differ from those c of the borrower by the adjustment for servicing fees, and the net receipt by the holder when prepayment occurs does not include refinancing costs m.

Next, one solves the certificate holder's equation again with π replaced by the constant autonomous prepayment rate k_8 for interest insensitive borrowers (and similar allowance for seasoning) to get the value per dollar of remaining balance on those mortgages.

Finally, one uses the known fraction of original balance remaining for the pool to infer the current fraction of the pool that is interest sensitive, and forms a weighted average of the two solutions for P. The value of this average in the current state x is then the theoretical value of the certificate per dollar of remaining loan balance.

As a byproduct, the framework can be used as a model of mortgage rate determination. For a given state x, let time to maturity T be 30 years and vary the coupon rate on the mortgage until the theoretical value of the associated certificate equals par. This specifies what the theoretical 'current coupon' GNMA mortgage rate would be as a function of x.

If all this seems inordinately cumbersome, it is. The objective, however, is simply to capture the interest incentives for prepayment with refinancing costs in a way that is logically tight, yet sufficiently flexible to allow for many of the patterns that seem to be in the data. Full rationality of behaviour is a parameter restriction that is testable; the endogenous nature of mortgage rates is accounted for; borrowers and lenders have the same perception of future interest rate movements and their own behaviour.

To this point we have not said anything about movements in the term structure of interest rates, except that it is described by some vector of state variables

x. The next section briefly describes the interest rate model used for the empirical work.

II. The Interest Rate Model

We assume that the instantaneous risk free rate $r(t)$ follows a two factor continuous time stochastic process derived and estimated in Jacobs and Jones (1986):

$$\begin{aligned} dr &= \kappa_1 r(\ln\mu - \ln r)\,dt + \sigma_1 r\,dz_1 \\ d\mu &= \kappa_2 \mu(\ln\gamma - \ln\mu)\,dt + \sigma_2\mu\,dz_2 \end{aligned} \tag{13}$$

In the above, dz_1 and dz_2 are the increments of a standard Weiner process with zero mean, variance one per unit time and instantaneous correlation coefficient ρ. The parameters κ_1, κ_2, γ, σ_1, σ_2 and ρ are constant. Short term interest rates are treated as moving toward some current target $\mu(t)$, while μ in turn regresses toward some long run fixed level γ. Both are subject to stochastic shocks whose magnitudes are proportional to their current levels. A two factor representation of interest rate movements is the minimum needed to allow both shifts and twists in the term structure.

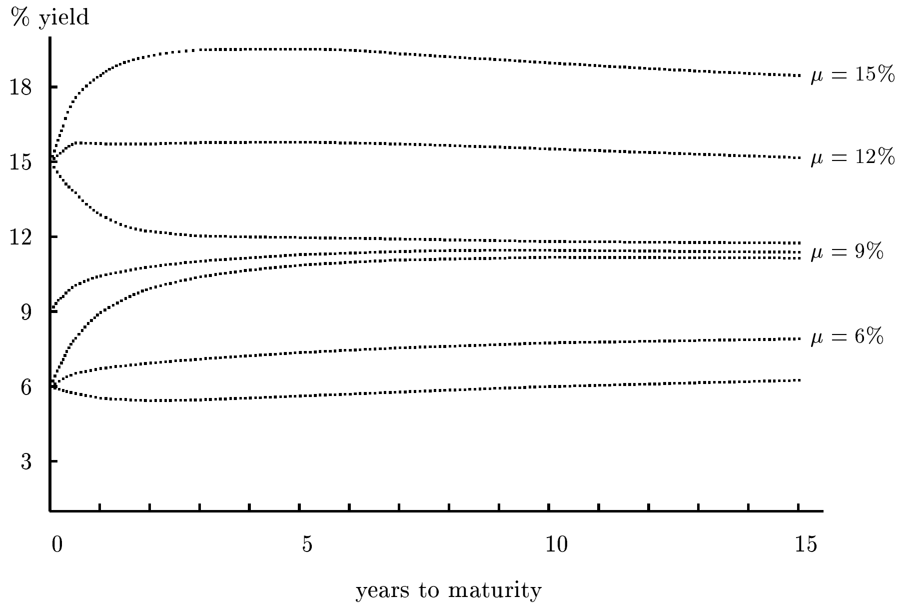
The parameters of the interest rate model were estimated from U.S. Government Treasury Bill and Note data following the procedure described in Jacobs and Jones. Weekly observations of the prices of 4, 13, 26 and 52 week Bills and yields on 3, 5 and 7 year Notes were used for the period January 1978 to April 1986. The yield on 1 week Treasury Bills was used for the instantaneous interest rate $r(t)$. The unobservable $\mu(t)$ was treated as a latent variable, its time series estimated along with the fixed parameters. The results were as follows:[7]

$$\begin{array}{lcllcllcl} \kappa_1 & = & .04461 & \sigma_1 & = & .1042 & \lambda_1 & = & -.2032 \\ \kappa_2 & = & .00134 & \sigma_2 & = & .0271 & \lambda_2 & = & -.0059 \\ \gamma & = & .00175 & \rho & = & -.221 & & & \end{array}$$

The above estimates take the time unit to be one week. Figure 1 depicts the theoretical yield curves for a variety of (μ, r) states. These parameter estimates were taken as fixed for the purpose of estimating the prepayment model parameters. Their most notable feature is the very large value of κ_1 relative to κ_2. The value of κ_1 implies of half-life of deviations of r from μ of about fifteen weeks; the value of κ_2 implies a half-life of deviations of μ from γ (which is equivalent to 9.09%/year) of about ten years.

[7] The parameter estimates differ from those of JJ (1986) for several reasons. First, the data set is extended by almost two years. Second, the one week bill rate was used instead of the Federal Funds rate as the instantaneous interest rate. Third, serial correlation in the pricing residuals was handled in a slightly different fashion. The result is somewhat lower estimated volatility of r and a lower value of κ_1.

Figure 1: Theoretical yield curves



III. Estimated Prepayment Behaviour

The nine parameters $k_0 - k_8$ describing prepayment behaviour were estimated by fitting the model to observed monthly prepayment rates on GNMA pools published by Salomon Brothers. The data consists of 56 monthly observations from July, 1982, to February, 1987, on certificates with 10 different coupon rates ranging from 6.5% to 16.0%. The information on various pools was aggregated by Salomon Brothers by coupon rate, with balance weighted prepayment rates, average remaining life, and proportion of original balance remaining provided. Annualized prepayment rates ranged from 0.7%/year to 48.8%/year. Some information was lost in the aggregation since mortgages with the same rate but different ages were aggregated. However for the time period used the origination date for mortgages of each rate lay in a relatively narrow window, leading us to expect the loans in each group to be of fairly similar age.

The estimation criterion was to minimize the weighted sum of squared prepayment rate residuals

$$\epsilon_{it} \equiv \hat{\pi}_{it} - \pi(x, t, c_i) \tag{14}$$

where $\hat{\pi}_{it}$ is the observed prepayment rate at time t on certificates with coupon rate c_i and π is the prepayment rate predicted by the model. The vector x is the interest rate state (r, μ) at the time the prepayment decision is made. Allowance was made for the vector of residuals at each point in time to have a covariance structure Ω and first order serial correlation coefficients ρ_i. The actual objective was thus to minimize

$$\mathrm{SSR} \equiv \tilde{\epsilon}_t^T \Omega \tilde{\epsilon}_t$$

in which $\tilde{\epsilon}_{it} = \epsilon_{it} - \rho_i \epsilon_{i\,t-1}$. The results reported here, however, assume $\rho_i = 0$ and $\Omega = I$. As will be seen shortly, the bulk of the total SSR comes from the higher coupon, higher prepayment rate issues. Correcting for covariance by, say, iteratively estimating Ω, would put less weight on these issues in determining the prepayment model parameters. Yet it is the behaviour in conditions of high prepayment that has the greatest financial consequence for the value of certificates. It is not obvious that such correction would lead to more efficient estimates of what we interests us most.

Estimation was accomplished by an iterative procedure (Marquardt's algorithm) of adjusting the nine parameters until SSR was as small as possible. The resulting parameter values are

$$\begin{array}{lll lll lll}
k_0 & = & .1.278 & k_3 & = & 16.32 & k_6 & = & .3945 \\
k_1 & = & .00147 & k_4 & = & 1.183 & k_7 & = & .2327 \\
k_2 & = & .20546 & k_5 & = & .090 & k_8 & = & .0113
\end{array}$$

The time unit is one month. In the context of the model, the apparent proportion of borrowers who are interest insensitive (k_7) is about 23%, prepaying at an annualized rate (from k_8) of 12.8%/year. The annualized prepayment rates of the interest sensitive group have a theoretical minimum of 1.75%/year (from k_1) and maximum of 93%/year (from k_2).

The prepayment function is symmetric about, and exhibits maximum sensitivity to interest rates at, a value of Q/B of 1.278 (k_0). That is, when the value of the borrower's liability, inclusive of likely future refinancing costs and with cash flows valued relative to the *Treasury* yield curve, reaches \$1.278 per \$1.00 of remaining loan balance. At this point a 1% rise in Q induces a rise in the prepayment rate of 1.32% per month. Notice that this level of Q/B is larger than the implicit value of the liability incurred per dollar of new mortgage (k_4) of \$1.183 . This suggests that borrowers wait somewhat too long after a rate decline before refinancing to be rational 'on average'. We have not addressed the question of whether these numbers are both too large (or small) given the objective costs of mortgage refinancing and premiums investors require on GNMA certificates relative to Treasury securities.

The total SSR for the 560 observations is 7270. Roughly 80% of this total comes from 10% of the data. Observed annualized prepayment rates range from 0.7% to 48.8%. The standard prediction error is larger for the high coupon, high prepayment rate issues. It goes from 1.5% for the 8.5% coupon rate certificates to 5% for the 16% coupon rate GNMA's, with an average of about 3% over all certificates. This does, of course, mask the considerable variation occuring across certificates on different pools with the same coupon rate and age.

Once a borrower decides to prepay, some time elapses before new financing is in place and the payments are in the hands of certificate holders. Prepayment decisions are made in an interest rate state occurring prior to the time when prepayments are received. We experimented with different lags on the interest rate state variables (r, μ). A two month lag gives a noticably lower SSR than either one month or three month, and is used for the remaining tests:

	8297	one month lag
SSR	7270	two month lag
	9012	three month lag

In each case the interest rate state was an average of the weekly states occuring during the month.

Allowance for seasonality and seasoning gave the most improvement in predictive capability when applied solely to the interest insensitive group. The parameter values reported should be so interpreted (or equivalently viewed as a simple additive component in the overall observed prepayment rate). The seasonal component (k_6) was 39% of the 12.8%/year prepayment rate of this group, which comprised just 23% of the average pool. The seasoning coefficient (k_5) of .09, with time measured in months, resulting in prepayments reaching .66 of their ultimate level by the end of one year, .89 by the end of two, and .96 by the end of three. The effect on prepayments is slight compared to interest rate movements.

Various restrictions were imposed (or relaxed) on the parameters to get some sense of their importance. The effects on total SSR were as follows:

Restriction	Separate SSR	Total SSR
base case		7270
fit only 5 lowest coupons	775	
fit only 5 highest coupons	5909	6684
fit only first half data	1877	
fit only second half data	4970	6847
no seasoning or seasonality $k_5 = k_6 = 0$		7552
reasonable refinancing cost $k_4 = 1.03$		7790
fully rational exercise with $k_0 = k_4 = 1.03$		37023

A formal test of the hypotheses based on an asymptotic F-statistic rejects the null hypothesis in all cases at the .95 significance level. However the test is not meaningful since the presumed 0 serial correlation and identity covariance matrix used above are not supported by the data.

One can however get some quantitative sense of how important particular assumptions are to the description of prepayment behaviour. On the one hand, the less than ten percent reduction in total SSR obtainable by splitting the data into halves by either time period or coupon rate suggests reasonable stability of the model parameters over time and across classes of certificates. This is fortunate, since one objective of the exercise is to find a unifying approach to the behaviour of borrowers with quite different contracts. Similarly, the less than five percent rise in SSR that accompanies removal of 'seasoning' and 'seasonality' from the analysis indicates they are not likely to be significant factors in determining certificate value. On the other hand, if we suppose that a reasonable objective level of refinancing costs are 3% of the amount refinanced ($k_4 = 1.03$), and require fully rational exercise of prepayment options on the basis of interest rate considerations alone by all borrowers, the SSR increases by a factor of five! The hypothesis of fully rational option exercise fares poorly as a description of observed behaviour during this period. The implication for the value of GNMA certificates is explored in the next section.

IV. Implications for Mortgage Rates and Certificate Values

Although an assumption of fully rational (or, alternatively, completely interest insensitive) prepayment behaviour may fail to describe actual behaviour, it might serve to adequately describe observed mortgage pricing. That is, the theoretical difference between certificate values under full rationality and under the estimated level of rationality might be slight. Similarly, the actual prices at which GNMA certificates are observed to trade may reflect a presumption of rational behaviour in the future, or the somewhat less rational behaviour historically observed.

We first examine the implication for equilibrium mortgage rates of full rationality versus observed levels. We then look at the theoretical values and characteristics of a range of currently traded certificates under these two hypotheses.

1. Equilibrium Mortgage Rates

The equilibrium mortgage rate for a given interest rate state is that coupon rate $\bar{c}$ on newly issued GNMA certificates required for them to trade at par. The actual rate paid by borrowers would be .5% higher to cover the previously mentioned servicing fees. The reference set of alternative securities (opportunity cost of funds) available to investors are presumed here to be U.S. Treasury bills and bonds. We assume investors will pay the full theoretical value for GNMA certificates as given by the solution to equation (12). I.e., no additional yield is required to compensate for model uncertainty, liquidity differences, etc.

For a given coupon rate, numerical solution of (12) results in a grid of theoretical certificate values for a range of (r, μ) interest states. These grids were constructed for coupon rates of 5-20% in half percent increments, then interpolated to find the coupon rate giving a theoretical certificate value of 100 for each rate state.

Table 1 gives the theoretical par coupon rate for r and μ values ranging from 3% to 16% under the assumption that prepayment behaviour will continue to follow the historical parameters as estimated in the preceeding section. (Note that μ is referred to as L in these tables.) For example, in a interest rate state or $(8, 8)$ – corresponding to a 'normally shaped' yield curve for Treasuries with 8% short rates – investors should be indifferent to holding Treasuries or new GNMA certificates with a 9.93% coupon rate purchased at par.

Table 2 gives the theoretical par coupon rates that would be required if borrowers were fully rational in the exercise of their prepayment options and refinancing charges were 3% of loan balance. In a $(8, 8)$ rate state, investors should now require a 11.69% coupon rate – an 1.76% higher rate. Comparison of the two tables gives differentials of between 1% and 5%. Differentials are higher at higher levels of rates (for equal r and μ); and they are higher the higher is r relative to μ. The former effect results from the fact that there is some mean reversion in the interest rate model (thus high rates are likely to fall over time inducing refinancings), and from the fact that higher rates are associated with higher absolute volatility of rates. The other effect, that the rationality premium is higher the higher are short rates relative to long, is consistent with the idea that a downward sloping yield curve indicates a rational expectation that short to medium rates will soon decline, again inducing mortgage refinancings. These refinancings would come faster under fully rational than under historical behaviour.

For comparison, Table 3 gives the theoretical par coupon rates appropriate if prepayments were completely insensitive to interest rates and occurred at a constant rate of 1% per month (seasoning is presumed with the parameter as estimated in the previous section). For the $(8, 8)$ rate state, investors should require a 9.28% coupon – .65% lower than under historical behaviour.

Table 4 provides a short series of theoretical par coupon rates under these three prepayment assumptions for the (r, μ) states prevailing monthly during 1982 and 1983. Actual par coupon rates based on GNMA price quotes tended to lie 30 to 90 basis points above the historical behaviour column (Sorry. I appear to have temporarily mislaid the series of actual rates for comparison – rather embarassing).

2. Actual Certificate Values

We finish with a more detailed examination of three specific GNMA issues as of November 8, 1990. They are a 7.5% GNMA with 322 months remaining, a 10.0% certificate with 356 months remaining, and a 12.0% issue with 295 months remaining. As of the close of business that day their bid prices were 88.50, 101.34 and 109.59 respectively (cf. "Mortgage Securities Research", November 9, 1990, Goldman Sachs). The Treasury yield curve for that day was modestly upward sloping with 1 year yields of 7.35%, 5 year of 8.16% and 10 year of 8.57%. A best least-squares fit of the theoretical yield curve of our interest rate model to the observed yield curve implies an interest rate state of $r = 6.80$ and $\mu = 6.91$.

Tables 5 and 6 provide (r, μ) contingent theoretical prepayment rates and market values for a 356 month 10% certificate under historical prepayment behaviour assumptions. Contingent prepayment rates and values are also provided under the alternative extreme assumptions for comparison.

Table 7 gives details the characteristics of all three certificates in the particular interest rate state prevailing on the day of the price quotes under the three prepayment scenarios.

From the first column, note that the market quotes are always below the theoretical values implied by historical behaviour in the context of the current model. A number of institutional factors may contribute to this. First, certificate trades usually settle only once per month, with some notice required for transfer of ownership. Thus settlement is often several weeks distant, and the price quotes in fact forward prices. In an upward sloping yield curve environment, forward prices for longer term instruments will generally be below spot prices. Second, price quotes are for 'generic' issues of the prescribed coupon rate (and degree of 'seasoning'). The seller has the option of which particular issue to deliver. To the extent that other pool specific factors influencing certificate values can be identified, the seller would presumably deliver certificates from the mortgage pool believed to have the lowest value. The possibility of a valuable delivery option on the part of the seller should further depress price quotes. We have not explored either of these avenues empirically.

Alternatively, investors may expect prepayment behaviour in the future to be more rational than in the past. One could, say, extract 'implicit' levels of future interest sensitivity (our parameter k_3) that would reconcile theoretical values and market prices on a given day.

Finally, investors may simply require some sort of excess return to be induced to hold these certificates over regular Treasury securities. There is considerable uncertainty about the parameters of any prepayment model that purports to apply for the next thirty years, considerable noise in even the current predictions of such models, possible doubt over the quality of the default guarantee, and less liquidity than in Treasuries.

Taking this approach, one can ask what excess return is implicit in the market pricing of the certificates, assuming the prepayment model and its parameter estimates are valid? A direct comparison if internal rates of return on certificates against 'comparable Treasuries' makes little sense. The certificates do not have fixed cash flows, giving in effect a random maturity date, and the option value is likely to be missed. However a notion of 'yield spread' relative to the entire Treasury yield curve can be readily developed from arbitrage notions. The theoretical value of the certificate is the (estimated) cost of replicating its cash flows through trading over time solely in Treasury securities.

Reversing the positions that accomplish this replication, and purchasing the certificate itself to make the cash flow payments on the Treasury position, in principle permits one to 'withdraw' from the portfolio an initial amount equal to the difference between the theoretical certificate value and its market price. This excess may instead be withdrawn according to a set interest rate and time contingent schedule. If this excess cash flow at each point in time is closely related to what one has invested in the certificate, the size of the flow may be interpreted as an excess return, or spread, on the amount invested.

The implied yield spread relative to Treasury securities reported in Table 7 assumes this excess is withdrawn in proportion to the remaining (unprepaid) balance owing on the underlying mortgages. I.e., it is that percentage per year of remaining loan balances that may be taken as 'profit' on holding the certificate, assuming the purchase was funded with a loan at Treasury rates and all remaining interest rate risk was fully hedged through positions in Treasury securities. This is closely related to the 'option adjusted spread' notion used by some investment banks. The latter, however, implicitly presumes that 'profits' are withdrawn in proportion to the remaining theoretical value of the security rather than the loan balance. Our spread notion blends more easily with book value accounting and can be compared directly with IRR based yield spreads on non-contingent cash flow securities.

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A. Solution details with homogeneous call policy

Filling in the details of Brennan and Schwartz (1985, p.214), the cash flow from the security with fraction f of the original mortgages left, a remaining contractual balance of $B(t)$ per initial dollar on those not prepaid, and a proportional prepayment rate $\pi(x,t;c)$ will be

$$q(x,t,f) = cf + \pi B f \tag{A.1}$$

Let $\tilde{P}(x,t,f)$ denote the value of the certificate in such a case. As f is a locally nonstochastic additional state variable, $\tilde{P}$ will satisfy a valuation equation

$$\frac{1}{2}\sigma^2 \tilde{P}_{xx} + (\alpha - \lambda)\tilde{P}_x + q - \pi f \tilde{P}_f - \tilde{P}_t = r\tilde{P} \tag{A.2}$$

subject to the maturity value condition $\tilde{P}(x,t,f) = 0$. The claim is that

$$\tilde{P} = f\tilde{P}(x,t,1) \equiv fP(x,t) \tag{A.3}$$

where $P(x,t)$ is the solution to equation (4). If this is the case, then the following relations hold:

$$\begin{array}{rclcrcl} \tilde{P}_x & = & fP_x & \quad & \tilde{P}_t & = & fP_t \\ \tilde{P}_{xx} & = & fP_{xx} & \quad & \tilde{P}_f & = & P \end{array}$$

Making these substitutions into (A..2) and dividing through by f gives equation (4) as claimed.

B. Mortgage Arithmetic Relationships

Listed below are relationships between the rate on a mortgage, periodic payments, prepayment rates and average maturity used in various calculations in the text. The time unit is the interval between payments (i.e., one month). The initial loan balance is \$1. The following notation is used for this appendix:

r	=	coupon rate per period
T	=	term to maturity of mortgage
c	=	periodic payment made by borrower (constant)
x	=	fraction of mortgages remaining prepaying per period
B_n	=	mortgage balance after n^{th} payment made
p_n	=	payment received by GNMA holder (includes prepayments)
δ	=	per period fraction of B_n charged by institution servicing mortgage (paid at time $n+1$)
WAM	=	weighted average maturity of loan

The prepayment rate x is assumed constant. Weighted average maturity is the average time to maturity of *principal* repayments only, weighted by the fraction of the principal repaid at each time, i.e., $\sum_{n=1}^{T} n(B_{n-1} - B_n)$. It is of descriptive significance only.

1. With no prepayments or servicing fees:

$$\begin{aligned} c &= \frac{r(1+r)^T}{(1+r)^T - 1} \\ B_n &= \frac{(1+r)^T - (1+r)^n}{(1+r)^T - 1} \\ p_n &= c \\ \text{WAM} &= \frac{T(1+r)^T}{r[(1+r)^T - 1]} - \frac{1}{r} \end{aligned}$$

2. With prepayments but no servicing fees:

$$\begin{aligned} c &= \frac{r(1+r)^T}{(1+r)^T - 1} \\ B_n &= \left[\frac{(1+r)^T - (1+r)^n}{(1+r)^T - 1}\right](1-x)^n \\ p_n &= c(1-x)^{n-1} + xB_n \end{aligned}$$

3. With servicing fees but no prepayments:

Let $\tilde{r} \equiv r + \delta$ be the mortgage rate paid by the borrower.

$$\begin{aligned} c &= \frac{\tilde{r}(1+\tilde{r})^T}{(1+\tilde{r})^T - 1} \\ B_n &= \frac{(1+\tilde{r})^T - (1+\tilde{r})^n}{(1+\tilde{r})^T - 1} \\ p_n &= c - \delta B_{n-1} \end{aligned}$$

4. With prepayments and servicing fees:

$$
\begin{aligned}
c &= \frac{\tilde{r}(1+\tilde{r})^T}{(1+\tilde{r})^T - 1} \\
B_n &= \left[\frac{(1+\tilde{r})^T - (1+\tilde{r})^n}{(1+\tilde{r})^T - 1}\right](1-x)^n \\
p_n &= c(1-x)^n - \delta B_{n-1} + x B_n
\end{aligned}
$$

C. Expected Average Maturity a GNMA Certificate

This section describes how the expected average maturity of a GNMA certificate can be obtained. Specifically, we determine the function

$M(r, \mu, t) =$ the expected average maturity of \$1 of unpaid balance in state r, μ at time t yrs. from contract maturity

This is accomplished as follows. M is that function of the state which is expected to decline one for one with the passage of time over intervals where no payment of principal can occur. This is the case between contractual payment dates. Applying Ito's lemma to M to get the expected rate of change in M, equating the resulting expression to 1 and rearranging gives

$$
\begin{aligned}
M_t \;=\;& \frac{1}{2}r^2\sigma_1^2 M_{rr} + \rho r\mu\sigma_1\sigma_2 M_{r\mu} + \frac{1}{2}\mu^2\sigma_2^2 M_{\mu\mu} + \kappa_1 r\ln(\mu/r)M_r \\
&+ \kappa_2\mu\ln(\gamma/\mu)M_\mu + 1
\end{aligned} \tag{C.1}
$$

The initial condition for the problem is $M(r, \mu, 0) = 0$. That is, the expected average maturity of remaining balances is 0 at contractual maturity.

On payment dates principal is repaid either through the regular contractual payments or through prepayment of the entire remaining balance. Let $\pi(r, \mu, t_i)$ denote the probability of prepayment on the i^{th} payment date, and $a(t_i)$ be the fraction of the principal remaining immediately prior to t_i that is amortized by the contractual payment on that date.

Immediately prior to t_i, the expected average maturity of \$1 of remaining principal is 0 times the fraction of that amount expected to be repaid at time t_i, plus M immediately after t_i times the fraction expected to remain. I.e.,

$$
\lim_{t\to t_i^+} M(r,\mu,t) = (1-\pi(r,\mu,t_i))(1-a(t_i)) \lim_{t\to t_i^-} M(r,\mu,t) \tag{C.2}
$$

This sequence of boundary conditions is easily incorporated in a numerical solution for M if the state contingent prepayment probabilities π are available from solving the borrower's problem. The numerical solution works backwards from

$t = 0$ to $t = T$, calculating the value of M on a discrete grid of r, μ values. When a payment date t_i is encountered, the M values in the current grid are multiplied by the factor $(1 - \pi(r, \mu, t_i))(1 - a(t_i))$. The value of $1 - a(t_i)$ is B_i/B_{i-1} from the contractual balance schedule. The value of $\pi(r, \mu, t_i)$ is the state dependent prepayment rate established in solving the borrower's prepayment problem — another partial differential equation. Computationally, the two pde's are solved together, with the prepayment rates from the borrower's problem generated as required for use in the above boundary conditions for M.

D. Inferring composition of a seasoned pool

This section sets out the details of how the composition of a pool can be inferred from its age, fraction of original balance remaining and model parameters. Suppose the pool initially consists of some fraction α with a constant autonomous prepayment rate of x per payment period. The fraction $1 - \alpha$ have a nonconstant prepayment rate. Let

r	$=$	coupon rate per period
T	$=$	term to maturity of mortgage
β	$=$	fraction of pool remaining after t payments
$B(t)$	$=$	principal balance remaining after t payments on an interest insensitive of \$1 initial balance
$A(t, \beta)$	$=$	fraction of pool remaining after t payments that is interest insensitive

The remaining interest insensitive loan balance in the pool at time t equals $\beta A(t, \beta)$. But it must also equal $\alpha B(t)$. Equating these two values and solving for A gives

$$A(t, \beta) = \frac{\alpha}{\beta} B(t) = \frac{\alpha}{\beta} \left[\frac{(1 + r)^T - (1 + r)^t}{r(1 + r)^T - 1} \right] (1 - x)^t \tag{D.1}$$

The last equality comes from substituting for $B(t)$ the expression from appendix B.4 .[8]

[8] Actually, this expression for A would not be correct if a 'seasoning' effect is introduced for the interest insensitive group. Specifically, suppose that the prepayment rate for this group is $x(1 - e^{-\delta t})$ with t measuring time from mortgage inception. Let $F(t)$ denote the fraction of these mortgages that have not yet prepaid. F will satisfy $dF/dt = -x(1 - e^{-\delta t})F$. Solving for F subject to $F(0) = 1$ gives

$$F(t) = e^{-x(t - \frac{1 - e^{-\delta t}}{\delta})}$$

This expression replaces $(1 - x)^t$ in the above equation for calculation of $A(t, \beta)$ in the empirical implementation.

The theoretical prepayment rate on a pool with a fraction of original balance remaining β in state interest rate state s would then be

$$\Pi(s,t,\beta) = A(t,\beta)\,x + (1 - A(t,\beta))\,\pi(s,t) \tag{D.2}$$

in which π is the theoretical prepayment rate on the interest sensitive mortgages in the pool. Similarly, the theoretical value per dollar of remaining balance of claims to the pool would be

$$P(s,t,\beta) = A(t,\beta)\,P_1(s,t;x,r) + (1 - A(t,\beta))\,P_2(s,t) \tag{D.3}$$

in which P_1 denotes the value per dollar of remaining balance of claims to interest insensitive mortgages and P_2 denotes the value of claims to interest sensitive mortgages. The expected average maturity of the GNMA certificate is a similarly weighted average of the average maturities of the two mortgage types.