

# The Timing of Investment

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## 1 Introduction

The decision to invest in fixed capital is central to the theory of the firm, whether for the prescriptive purpose of capital budgeting or for the descriptive purpose of macroeconomic modelling. Yet some aspects of actual investment behaviour appear difficult to reconcile with conventional theory. These include the short payback periods frequently required of risky projects; the notion that decreased ‘confidence’ (increased uncertainty) depresses capital spending, even by those in the business of taking calculated risks; and the relative instability of short run investment relations in econometric models. This essay portrays such phenomena as natural when viewed in terms of the optimal timing of investment under uncertainty.

Shifting focus from ‘whether to invest’ to ‘when to invest’ forces a different perspective on the way capital expenditure is determined. Rather than reflecting the isolated evaluation of a once and for all opportunity, capital expenditure reflects the interaction of a rational investment *policy* with a stochastic environment. Investment results from a process, in which new information arrives and is expected to continue to arrive, rather than from a single static decision.

The investment problem is conventionally approached by assuming that the current decision has no impact on other opportunities, current or future. But for timing to be an issue current investment must impact future options. Investment today and similar investment tomorrow must, to some degree, be mutually exclusive. There are several reasons to think that this is more likely to be the rule than the exception. Investing now may use up some scarce, though perhaps not explicitly purchased, resource: Oil reserves

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developed today cannot be developed again next year; a field planted in corn today cannot also be planted in oats tomorrow. Managerial or technological decreasing returns to scale may cause investment today to reduce the return from investment in the future: A unit of capital installed tomorrow could yield less if it is the second such unit than if it were the first. Or market imperfections may link tomorrow's opportunities with today's action: Financial constraints may bind sooner; the output from prior investment may exhaust rents in monopolistic product markets. Thus investment today may render similar investment tomorrow uneconomical. They become, in effect, mutually exclusive to a rational decision maker.<sup>1</sup>

Such considerations are largely irrelevant whenever investment is *reversible*. There are two distinct aspects to reversibility. The first is the return of funds spent to acquire the capital—can you get your money out? The second is the reinstatement of opportunities you would have had if the investment had never taken place—can you get your options back? Irreversibility in the first sense does not create a problem. It simply implies uncertainty about the price that could be realized if the capital is sold. But irreversibility in the second sense means that options continue to be impaired by the original act of investment. This fundamentally alters the criterion for investment.

This paper considers the consequences of irreversibility and exclusivity in an extreme case. A firm must decide how to manage an opportunity to make a one time investment in an indivisible project. The rational policy has the following properties: (a) with uncertainty, a benefit/cost ratio significantly greater than one is needed to induce investment; (b) a rise in uncertainty usually reduces current investment through increasing the value of the option to invest at a future date; and (c) a rise in interest rates can stimulate investment in projects of short to medium duration. The conventional 'net present value' rule, and unambiguous negative impact of higher interest rates on investment, emerge as limiting cases when the level of uncertainty goes to zero.

Many of these ideas are not new. The notion that irreversibility deters fixed investment, or requires premium rates of return even in the absence of risk aversion, can be traced back to Lavington (1921). It is developed by Hirshleifer (1972), Nickell (1978) and Bernanke (1983), among others. The importance of option value when irreversibly altering the environment

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<sup>1</sup>In the above cases, a cost of investing was the elimination, or worsening of the terms, of opportunities for future investment. There are situations where investment today enhances future options, as in the case of research and development expenditures. However we focus here on options given up.

is shown by Fisher et al. (1972), Henry (1974) and Arrow and Fisher (1974). Hart (1942), Epstein (1980) and Jones and Ostroy (1984) focus more generally on the value of flexibility when waiting brings information. Hicks (1974) suggests the importance of such considerations for macroeconomic analysis. Closely related is the literature on optimal sequential investment. This includes the work of Baldwin and Meyer (1979), Prastacos (1983) and Moene (1985), and has much in common with search models of labour markets. Learning need not take place in the sense of changing beliefs about future opportunities, but incentives exist to wait for strictly positive rents (i.e., net present values).

The problem we set up is most similar to that of McDonald and Siegel (1986). McDonald and Siegel obtain the optimal policy in the case when project value follows a lognormal diffusion (our section III.2) and the Intertemporal Capital Asset Pricing Model assumptions hold. The method by which we characterize the optimal policy is most similar to that of Brennan and Schwartz (1985). They use arbitrage pricing methods to establish the operating policy and value of a mine in terms of spot and futures commodity prices. The decision to initially develop a property, though only one aspect of their much larger problem, corresponds to the investment decision in our context. Like Brennan and Schwartz, we use arbitrage methods to handle risk aversion. This extends McDonald and Siegel's results to situations where ICAPM assumptions could not hold. In contrast with Brennan and Schwartz, however, our investment policy is expressed in terms of the project characteristics (initial cash flow, depreciation rate, etc.) were it to be put in place immediately. This obviates the need to identify *a priori* exactly what causes the project's cash flows to be uncertain.

The paper has three somewhat distinct objectives. First, we wish to emphasize that, with appropriately complete and perfect capital markets, there is an optimal investment timing policy that is independent of the decision maker's attitude towards risk and (some aspects of) expectations about the future. In other words, the Fisherian 'Separation Theorem' that proved powerful in developing standard discounted cash flow analysis applies also in a world of uncertainty and irreversibilities. Second, we illustrate how arbitrage pricing principles allow optimal policy to be expressed in terms of variables (prices of bundles of contingent claims) that are potentially observable by, or at least familiar to, decision makers—in this case the initial project characteristics. It thus suggests how these principles might be useful for implementing optimal policy in practical contexts. Third, we examine the implications of this perspective on investment for questions of macroeconomic interest. Namely, what is the relationship between interest rates and

capital spending? Though our analysis is highly preliminary, it does suggest that the presence of uncertainty has the potential to reverse long standing presumptions about the relations between some aggregate economic variables.

## 2 Direct Solution for the Investment Criterion

Consider the choice between undertaking and postponing a one time indivisible investment when the decision maker is risk neutral.<sup>2</sup> The analysis is kept as simple as possible to highlight the essence of the problem. Assume the decision maker knows the expected Net Present Value (NPV) of the project if undertaken immediately, and the distribution of NPV's realizable at future dates if postponed. Change in perceived value can result from project returns being dependent on economic conditions at the time of undertaking, or simply reflect the arrival of information allowing better assessment of its (implicity fixed) prospects. The decision maker compares the NPV realized by proceeding at once with the most favourable expected discounted NPV realizable by undertaking the project at future dates. This comparison forms the basis for his decision.

To be concrete, let  $N(t)$  be the NPV realized as of time  $t$  if the project is undertaken at time  $t$ , and let  $N(t)$  follow the random walk

$$dN(t) = \sigma dz(t) \tag{1}$$

where  $\sigma$  is a constant and  $z(t)$  is the standardized Wiener process. Given  $N(t_0) \equiv N_0$ , the random variable  $N(t)$  is normal with expected value  $N_0$  and variance  $\sigma^2(t - t_0)$ . Moreover, for  $\sigma > 0$ ,  $N(t)$  with probability one eventually exceeds any given  $\bar{N}$ . Clearly, if the decision maker does not discount the future and if the option of proceeding with the project does not expire, then the project is rationally postponed forever – it is costless to wait for assuredly more favourable circumstances. But if the discount rate is positive, then the higher is this rate, the higher is the cost of waiting, and the more reasonable it would seem to settle for a finite  $\bar{N}$ . In fact, for  $\sigma = 0$  any project with positive  $N$  should proceed at once. Thus it appears that the higher the discount on the future, the more likely the project will be undertaken for a given value of  $\sigma$ ; and the higher is  $\sigma$  the more likely the project will be postponed given the discount rate. We demonstrate this to be the case under plausible assumptions.

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<sup>2</sup>This analysis is a simplified version of that of McDonald and Siegel (1986) under risk neutrality.

Let the decision maker's policy take the form of a reservation price,  $\bar{N}$ , for the NPV of the project: if  $N(t) \geq \bar{N}$  at time  $t$  then the project is undertaken at that time; if not it is postponed. Assume the opportunity to undertake the project never expires. Since process (1) is stationary, the optimal reservation price will be constant. Consider the decision at time 0, with  $N(0) = N_0$ . The decision maker must choose  $\bar{N}$ . If  $\bar{N} \leq N_0$  he proceeds immediately and if  $\bar{N} > N_0$  he waits. For  $\bar{N} > N_0$ , there will be a probability distribution of possible times  $\tau$  when  $N$  first equals  $\bar{N}$ . Let  $\bar{\pi}(N_0, \bar{N}, \tau)$  denote the density function of first passage times conditional on  $N_0$  and  $\bar{N}$ . Figure 1 depicts paths of  $N$  that could lead to investment occurring at  $t_1$ , together with the density function for  $\tau$ . Since the project is undertaken at time  $\tau$ , value  $\bar{N}$  is realized as of that time. From the perspective of time 0, the expected NPV from postponing and holding out for  $\bar{N}$  is thus

$$E_0(NPV) = \bar{N} \int_0^{\infty} e^{-r\tau} \bar{\pi}(N_0, \bar{N}, \tau) d\tau \quad (2)$$

where  $r$  is the decision maker's assumed constant rate of discount. The decision maker chooses  $\bar{N}$  to maximize (2). For  $N$  following process (1) it can be shown that<sup>3</sup>

$$\bar{\pi}(N_0, \bar{N}, \tau) = \frac{\bar{N} - N_0}{\sigma(2\pi\tau^3)^{1/2}} \exp\{-(\bar{N} - N_0)^2/2\sigma^2\tau\} \quad (3)$$

Substituting (3) into (2) and integrating yields

$$E_0(NPV) = \bar{N} \exp\{(N_0 - \bar{N})(2r)^{1/2}/\sigma\} \quad (4)$$

It is easily shown that (4) is maximized by setting

$$\bar{N} = \sigma/(2r)^{1/2} \quad (5)$$

Thus the project is initially postponed if  $N_0 < \sigma/(2r)^{1/2}$ . Since the process is stationary this result holds for all times  $t$ .  $\bar{N}$  does indeed have the anticipated properties: increasing with  $\sigma$  and decreasing with  $r$ .

Direct solution for the optimal investment policy was feasible because the process followed by  $N$  was simple and risk neutrality was assumed. Risk aversion on the part of the decision maker complicates the problem. His marginal utility of wealth will vary with his wealth from other sources, the target level of prospective project returns he selects, and when during his life cycle uncertainties are resolved. The solution to the optimal policy problem,

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<sup>3</sup>See Cox and Miller (1970), pp.220-222.

if opportunities to trade in other risky securities are neglected, depends on the particulars of the individual's situation and utility function. At first glance, this would appear to rule out the possibility of an unambiguously optimal policy. As in the Fisherian deterministic consumption– investment problem, however, these difficulties can be resolved if the individual can control risk through trading in other securities. That is the route taken in the next section.

### 3 An Arbitrage-Based Decision Model

Arbitrage pricing methods circumvent the problems created by individual circumstances by viewing the investment opportunity, properly managed, as a bundle of state contingent claims, and assuming that: a) the market is already complete with respect to claims contingent on those factors affecting the investment<sup>4</sup> ; b) the decision maker trades to equilibrium in this larger market, in addition to managing his own investment opportunity; c) the project is small enough that his policy does not alter the prevailing state claim prices. Under these conditions, the value of the investment opportunity combined with a particular management policy is the market price of the bundle of claims it represents. The decision maker rationally chooses a policy that maximizes this market value, using trading in other securities to accommodate his own tastes. As a result, the optimal policy can be expressed solely in terms of the prices of related traded securities.

This section adopts the above framework to express the value of an investment opportunity, and the criterion for exercising it, in terms of the project's market value were it already in place. An explicit solution is displayed for the case where future prospects have a lognormal distribution.

#### 3.1 The Investment Opportunity

Consider an investment opportunity with the following characteristics. The initial capital cost is  $C$  regardless of when it takes place. Revenue subsequent to commitment accrues at the rate

$$X(t, u, s)$$

where  $t$  is the time of the cash flow,  $u$  is the time the project was undertaken, and  $s$  is a scalar describing those aspects of the state of the world as of time

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<sup>4</sup>The state contingent payoffs in which one can implicitly trade include those attainable through feasible intertemporal trading strategies in existing securities.

$t$  relevant for the project's payoffs (e.g., output price).<sup>5</sup> This state variable follows a continuous Markov process described by the stochastic differential equation

$$ds = \mu(s, t) dt + \omega(s, t) dz.$$

$dz$  is a standardized Wiener process.

Something must be said about trading opportunities available to the decision maker. First, it is assumed able to borrow or lend at the riskless instantaneous interest rate  $r$ . Second, although the project itself and the option to invest in it may not be tradeable (they may embody firm-specific resources or expertise, or reflect a "good idea" to which property rights cannot be assigned), assume it is possible for the decision maker to trade in a security whose return is locally perfectly correlated with  $s$ . If the decision maker trades to equilibrium in these other securities, then the value it rationally ascribes to the investment opportunity must be the same as its market price were it divisible and tradeable.<sup>6</sup> This price can be determined by the now standard technique: combine the asset to be valued in a portfolio with traded securities so that all risk is hedged; the resulting portfolio must then have a value such that its rate of return is the riskless interest rate.<sup>7</sup>

If embarked upon at time  $u$ , let the price for which the project can be sold at time  $t$  in state  $s$  be

$$Q(t, u, s)$$

$Q$  is the risk-adjusted present value of the cash flows remaining — the market value of the bundle of contingent claims implicit in  $X$  given  $(t, u, s)$ . It obeys the partial differential equation

$$\frac{1}{2}\omega^2 Q_{ss} + (\mu - \lambda(t, s))Q_s + Q_t(t, u, s) + X(t, u, s) - rQ(t, u, s) = 0 \quad (6)$$

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<sup>5</sup>The cash flow  $X$  may embody the solution to a subsidiary maximization problem at time  $t$ , such as a choice of production level, as in Brennan and Schwartz (1985), or exercise of an option to scrap the project. In fact,  $X$  need not even be actual receipts as of time  $t$ , but can instead be interpreted as the discounted present value of that portion of the project's future cash flows that becomes certain once  $s(t)$  is known.  $Q(t, u, s)$  would then be the market value of revenues to be received from  $t$  onward, less that portion of those flows already discounted into prior  $X$ 's. The purpose of such interpretation would be to allow limited dependencies of current revenues on past values of the state variable within the present analytical framework.

<sup>6</sup>Equivalently, the value may be thought of as either a) the current price of a portfolio of traded securities that could provide a self-financing replica of the cash flows from the investment opportunity and chosen exercise policy, or b) the present amount that could be borrowed so that, with appropriate hedging against all subsequent fluctuation in  $s$ , the loan would be exactly repaid by the project project cash flows when it proceeds.

<sup>7</sup>See for example Cox, Ingersoll and Ross (1985) and Brennan and Schwartz (1985).

where subscripts indicate partial differentiation,  $r$  is the instantaneous risk free interest rate, and  $\lambda(t, s)$  is the excess expected return on a portfolio constructed so that its value is always equal to  $s$ .<sup>8</sup>

Let  $V(s, t)$  denote the value of the not yet exercised *option* to invest in the project when the policy  $\bar{s}$  is to invest whenever  $s(t)$  first hits  $\bar{s}(t)$ . It obeys the differential equation

$$\frac{1}{2}\omega^2 V_{ss} + (\mu - \lambda(t, s))V_s + V_t(t, s) - rV(t, s) = 0 \quad (7)$$

with the boundary condition

$$V(\bar{s}, t) = Q(t, t, \bar{s}) - C \quad (8)$$

Optimality of the chosen policy (i.e., it maximizes  $V(t, s)$ ) requires that the Merton–Samuelson ‘high contact condition’ be satisfied at the time of exercise:<sup>9</sup>

$$V_s(\bar{s}) = Q_s(t, t, \bar{s}) \quad (9)$$

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<sup>8</sup>See Cox, Ingersoll and Ross (1985, p.375). An informal demonstration of (6) is as follows. Suppose there exists a non-dividend paying (for simplicity) tradeable security whose price  $Z(t, s)$  at time  $t$  depends solely on the then current state  $s$ . Construct a portfolio consisting of  $1/Z_s$  units of this security plus  $s - Z/Z_s$  dollars of riskless loans. Adopt a policy of extracting ‘dividends’ from the portfolio at the (possibly negative) rate per unit time of

$$c(t, s) = [\frac{1}{2}\omega^2 Z_{ss} + \mu Z_s + Z_t + (sZ_s - Z)r - \mu Z_s]/Z_s$$

Let  $S(t, s)$  denote the value of the portfolio at time  $t$ . Applying Ito’s lemma to  $Z(t, s)$ , one may verify that  $S$  follows the same stochastic process as  $s$ , the stochastic component of its return is  $\omega dz$ , and hence that  $S = s$  at all times. The excess expected return on this portfolio is the dividend flow plus expected capital appreciation  $\mu$  less the risk free opportunity cost  $rs$ . This is

$$\lambda(t, s) \equiv [\frac{1}{2}\omega^2 Z_{ss} + \mu Z_s + Z_t - rZ]/Z_s$$

By construction, the gross expected return on the portfolio is  $\lambda + rs$  at each point in time.

Now consider the project with value  $Q(t, u, s)$  combined with a hedge of  $-Q_s$  units of the above  $s$ -portfolio. Applying Ito’s lemma to  $Q$ , the expected return on the combination is

$$[\frac{1}{2}\omega^2 Q_{ss} + \mu Q_s + Q_t + X - (\lambda + rs)Q_s] dt$$

However the return on the combination is riskless since the stochastic components  $+Q_s ds$  and  $-Q_s ds$  offset each other. The return that could be earned on the same amount of money invested in riskless loans would be  $(Q - sQ_s)r dt$ . In an arbitrage-free environment these two riskless returns must be equal. Equating them gives equation (6).

<sup>9</sup>See Samuelson (1965) and Merton (1973). The graphical intuition behind this condition is provided in the next section.



Solution to the optimal timing problem involves the simultaneous solution of (6) and (7) subject to (8) and (9). However without specifying the nature of the underlying state variable  $s$ , the solution is without economic content.

Let us define

$$P(t, s) \equiv Q(t, t, s)$$

That is,  $P(t, s)$  is the value of the project at time  $t$  assuming it was just undertaken at time  $t$ . Figure 2 depicts the relationship between the  $X$ ,  $Q$  and  $P$  for a particular time path of  $s(t)$ .  $X(t, u_1, s)$  and  $X(t, u_2, s)$  are the project's cash flows over its life cycle if initiated at times  $u_1$  and  $u_2$  respectively. The lower graph depicts the corresponding paths of the project's market value, generally declining as cash flows are paid out.  $Q(u_2, u_1, s)$  is the market value of the revenues represented by the shaded area.  $P(t, s)$  is the curve joining the initial values of the project begun at various dates.

To this point, no restrictions have been placed on the way  $\mu$  and  $\omega$  may depend on  $s$  and  $t$ . As long as  $P$  is strictly monotonically related to  $s$ , we may without loss of generality suppose  $s$  to be scaled so that  $s \equiv P(t, s)$  at all times. The distinction between  $P$  and  $s$  will be reintroduced in section IV.

Assuming that  $s \equiv P(t, s)$  at all times  $t$ , so that the value of the option to invest is expressed in terms of the value of the underlying project,  $Q_{ss}(t, t, s) = 0$  and  $Q_s(t, t, s) = 1$ . Let

$$Q_t \equiv \lim_{u \rightarrow t^-} Q_t(t, u, s)$$

denote the instantaneous rate of depreciation on the project newly put in place at time  $t$  assuming no change in  $P$ .<sup>10</sup> Letting  $u \rightarrow t^-$ , equation (6) becomes

$$Q_t + (\mu(s, t) - \lambda(s, t)) + X(t, t, s) - rP(t, s) = 0 \quad (10)$$

Using equation (10) we can eliminate the unobservable  $\lambda$  from equation (7). Thus

$$\frac{1}{2}\omega^2 V_{PP} + (-Q_t - X + rP)V_P + V_t - rV = 0 \quad (11)$$

Equation (11) involves as inputs the cash flow and depreciation rate on the project, parameters which presumably can be estimated by the project managers.  $\omega^2$  is the variance per unit time of the market value of the

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<sup>10</sup>If there was no dependence of  $\omega$ ,  $\alpha$ ,  $\lambda$  on  $t$  then this would be the same as the observed price differential per unit increase in age for similar recently initiated projects, and hence could be calculated by managers from a cross-section of market prices.

project newly put in place, a quantity that might be estimated from the market prices of securities issued against similar projects already in place.

Rewriting the boundary conditions to reflect the use of  $P$  as a proxy for  $s$ , the optimal policy  $\bar{P}(t)$  and value of the opportunity to invest  $V(P, t)$  are then jointly determined by

$$\begin{aligned}\omega^2 V_{PP}/2 + (rP - X - Q_t)V_P + V_t &= rV \\ V(\bar{P}, t) &= \bar{P} - C \\ V_P(\bar{P}, t) &= 1\end{aligned}\tag{12}$$

System (12) is noteworthy for what it does not contain rather than for what it does.  $V(P, t)$  and  $\bar{P}(t)$  are completely determined by the three functions  $\omega(t, P)$ ,  $Q_t(t, P)$  and  $X(t, P)$ . The investment policy is independent of the decision maker's preferences, circumstances, and beliefs about the expected path of  $P$ . This is the Separation Theorem at work. What is perhaps surprising is that all relevant aspects of the project's cash flows, their timing and risk, are embodied in the two numbers  $X$  and  $Q_t$  for new projects alone.

This does not mean, of course, that the basic problem of project valuation has been eliminated. To obtain  $P$  and  $Q_t$  the decision maker would have to solve (6). To do this, he would have to identify the relevant state variable, quantify his expectations about its future path, and form an assessment of the appropriate market risk price. Once all that was done, (10) stipulates that  $(\mu - \lambda)$  could be used interchangeably with  $(rP - X - Q_t)$  in system (12).  $\mu$  would be the expected drift in project prospects and  $\lambda$  the excess expected return that the market would require on the new projects just after they were put in place.

What the above analysis does accomplish is to link the solution of the optimal timing problem to the project valuation problem. If we are willing to add structure concerning the solution to the latter problem we can get some insight into the solution of the former.

### 3.2 Investment Policy when Project Value is Lognormally Distributed

Let us examine a case for which  $\bar{P}$  has an explicit solution. In particular, suppose that

$$\omega(t, P) = \sigma P \quad \mu(t, P) = \alpha P \quad \lambda(t, P) = \beta \sigma P \tag{13}$$

where  $\alpha, \beta, \sigma$  are constants. That is, the value of the project if postponed (which is also the state variable influencing the cash flows of projects in

place) is expected to improve or deteriorate at a constant proportional rate with a constant proportional standard deviation.  $\beta$  is the risk premium the market requires per unit standard deviation of this type of risk.

It is unclear a priori which case – that of positive  $\alpha$  or that of negative  $\alpha$  – has greater relevance. Each has plausible application. A positive growth rate would apply, for instance, if the project was extraction of an exhaustible resource that was becoming increasingly scarce, or if technological advance in the capital goods being purchased (e.g., word processing equipment) meant that later investments would be more productive. A negative growth rate, however, would likely be more common. Market situations develop in which a firm can earn short run quasi-rents by acting quickly. The longer it waits, the greater the erosion of economic profit to the activity by competing entrants. Indeed, because it is such rent seeking behaviour that drives market adjustment, the investment timing problem is closely linked to the problem of competitive price dynamics.

Since the process governing  $P$  is stationary and the option never expires, optimality requires that the critical value of  $P$  needed to induce investment must be constant. I.e.,  $\bar{P}(t) \equiv \bar{P}$  for all  $t$ , and  $V_t = 0$ . In addition, since 0 is an absorbing barrier for  $P$ , we have an additional boundary condition  $V(0, t) = 0$ . Thus (12) becomes the linear ordinary differential equation

$$\sigma^2 P^2 V_{PP}/2 + (\alpha - \beta\sigma)PV_P - rV = 0. \quad (14)$$

The solution to (14), subject to the boundary conditions, is

$$\begin{aligned} \bar{P} &= \left(\frac{a}{a-1}\right)C \\ V(P, t) &= \left(\frac{a-1}{C}\right)^{a-1} \left(\frac{P}{a}\right)^a \end{aligned} \quad (15)$$

where<sup>11</sup>

$$\begin{aligned} a &\equiv D + (D^2 + 2r/\sigma^2)^{1/2} \\ D &\equiv 1/2 + (\beta\sigma - \alpha)/\sigma^2 \end{aligned} \quad (16)$$

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<sup>11</sup>The reader may recognize (16) as the value of Samuelson and McKean's perpetual warrant on a security paying a constant proportional dividend. See Samuelson (1965, p.29), McKean (1965, p.36), Samuelson and Merton (1969, pp.41-42). The similarity is appropriate. Our investment opportunity can be viewed as a call option on a portfolio which is perpetually being rolled over into the newest project. The dividend rate that could be paid from this portfolio is  $(X + Q_t)/P = r + \beta\sigma - \alpha$  which is constant under the assumptions of this section. This formula is also obtained by McDonald and Siegel (1982, p.11) using methods similar to those of section two.

This solution exists only for  $r + \beta\sigma > \alpha$  (equivalently,  $a > 1$ ).  $\bar{P}$  and  $V$  both tend to infinity as  $\alpha \rightarrow r + \beta\sigma$ . A project whose value is growing faster than the risk adjusted discount rate should always be postponed.

In the ‘risk neutral’ formulation  $\beta\sigma - \alpha = x + q_t - r$ , where  $x \equiv X/Q$ ,  $q_t \equiv Q_t/Q$  for a project of 0 age, and  $x, q_t$  are assumed constant. The risk neutral form of  $\bar{P}$  and  $V$  can then be obtained by substitution into (16). The necessary condition for  $\bar{P}$  and  $V$  to be finite then becomes  $x > -q_t$  — the initial rate of cash flow must exceed the rate of depreciation in market value due to aging.

Figure 3 shows how the parts of (12) interact to determine the optimal policy. The family of curves through the origin are the solutions to (14) that satisfy the boundary condition  $V(0) = 0$ , the relations between  $P$  and the value of the unexercised option that are consistent with lack of arbitrage opportunities. Given any  $\bar{P}$ , the value of the option must be on the line  $P - C$  when  $P = \bar{P}$  since the option is exercised immediately. Finally, the optimizing choice of  $\bar{P}$  is that which maximizes the value of the option subject to the above requirements: i.e., which puts one on the highest solution curve that touches line  $P - C$ . Hence the tangency condition  $V_P(\bar{P}) = 1$ .

The solution makes explicit that substantially *positive* differences between market value and cost are required to induce investment. Investment proceeds only when  $P/C$  exceeds the reservation ‘benefit/cost ratio’,  $a/(a - 1)$ . The effect is noticeable for moderate values of the parameters. For example, with  $\alpha - \beta\sigma = 0$  (zero risk adjusted growth in  $P$ ),  $r = .03$  and  $\sigma^2 = .01$ , the required ratio is  $\bar{P}/C = 1.5$ . With  $\sigma = .20$ , the ratio rises to 2.22. For projects with short lifespans, these translate into exorbitant rates of return. Does this mean that projects with positive NPV’s that do not satisfy this requirement are bad investments? Not at all. It simply means that given the choice between investing now or later, there is a strong bias toward waiting to see how the future develops.

The optimal investment policy does reduce to the conventional net present value rule in the limit.  $\bar{P} \rightarrow C$  if any one of the following occur:

$$\alpha \rightarrow -\infty \quad \beta \rightarrow \infty \quad r \rightarrow \infty$$

The first circumstance indicates a fleeting opportunity. If the rents to proceeding immediately reflect a temporary market advantage in a highly competitive industry, then  $\alpha$  is likely to be very negative. Such opportunities must be seized. The last two imply unacceptably high costs of waiting for potentially larger NPV’s. Unlike call options on stocks, the option here

delivers an asset that varies with the time of exercise. The effect of postponement is to push forward in time the entire cash flow of the project. This is analogous to foregoing a dividend equal to  $r$  times the capitalized rent (NPV) were the project to proceed ahead immediately. In addition, postponing the cash flow makes it riskier from today's perspective. High values of  $r$  imply high opportunity costs of waiting, and high values of  $\beta$  imply high costs of hedging the associated greater risk.

The reservation value under certainty (i.e.,  $\sigma^2 \rightarrow 0$ ) is

$$\bar{P} = \max\left\{C, \frac{r}{r - \alpha}C\right\}$$

This is the familiar rule: Harvest the forest when the rate of growth in its value net of harvest costs, here  $\alpha P/(P - C)$ , equals the interest rate,  $r$ . For negative or zero growth in the project's prospects,  $\bar{P} = C$ . Positive NPV is sufficient for investment.

The relationship between  $\bar{P}$  and the various parameters is straightforward but tedious. It can be shown that  $\bar{P}$  unambiguously increases with  $\alpha$ , and decreases with  $r$  and  $\beta$ . However the effect of risk on  $\bar{P}$  reverses for sufficiently positive values of  $\alpha$ :

$$\frac{\partial \bar{P}}{\partial \sigma} = \frac{(\beta\sigma - 2\alpha)a + 2r}{\sigma^3(D^2 + 2r/\sigma^2)^{1/2}} \frac{C}{(a - 1)^2} > 0 \quad \text{iff} \quad \alpha < \frac{\beta\sigma}{2} + \frac{r}{a} \quad (17)$$

The effect of parameter changes on  $V$  are in the same direction as their effects on  $\bar{P}$ . Two observations can be made here. First, in contrast with usual stock options, increased risk can decrease the value of the option to invest. For any  $\beta > 0$  there exists a range of growth rates  $\alpha$  such that  $\partial \bar{P}/\partial \sigma < 0$ . For  $\beta \leq 0$ , however, it can be shown that higher  $\sigma$  always implies higher option values. Second, the fact that  $\bar{P}$  falls as  $r$  increases raises the possibility that higher interest rates could actually encourage current real investment. This latter possibility we explore in the next section.

## 4 The Effect of Interest Rates on Investment

One of the most fundamental results of capital theory is that higher interest rates discourage investment. Yet we just saw how higher interest rates can reduce required benefit/cost ratios. Could a rise in interest rates stimulate capital spending?

Intuitively, two opposing forces are at work when current and future investment are mutually exclusive. On the one hand, higher interest rates

reduce the present value of project revenues, making them less likely to cover initial costs. On the other hand, higher interest rates decrease the value of the option to invest at a future date, which is an opportunity cost of investing now. Whether a rise in interest rates discourages current investment depends on which force is larger.

To compare the magnitude of these forces, structure must be added that reflects the timing of project cash flows. Since the present value of these flows is determined in part by  $r$ , the use of  $P$  as a proxy state variable must be dropped. The simplicity of the explicit policy of the last section can be preserved if, taking  $s$  to be the exogenous state variable,  $P$  takes the form

$$P(t, s; r) = f(r) s \tag{18}$$

$$ds = \alpha s dt + \sigma s dz$$

where  $\alpha, \sigma$  are constant and  $f(r)$  is a non-stochastic function of the interest rate that determines the market value of the cash flows. The elasticity of the project value with respect to  $r$  will be the same as the elasticity of  $f$ . For any given level of  $r$ , the process followed by  $P$  becomes

$$dP = \alpha s f dt + \sigma s f dz = \alpha P dt + \sigma P dz \tag{19}$$

Assuming that the price of  $s$ -risk is constant and unaffected by  $r$ , the requirements of (13) are met and investment policy (16) applies. It must be emphasized that the effect on investment we obtain is for a change in  $r$  that is completely unanticipated, yet viewed as permanent once it has occurred.

Suppose now that the firm is just indifferent between proceeding and not. Only at this point could a small parameter change alter current behaviour. This requires that  $P = \bar{P}$  where  $P = sf$  and  $\bar{P} = aC/(a - 1)$ . Equating these two expressions implies the current state is

$$s = \frac{C}{f} \left( \frac{a}{a - 1} \right)$$

The effect of an increase in  $r$  hinges on the sign of  $\partial(P - \bar{P})/\partial r$ . If positive, the project proceeds; if negative it is postponed. Let  $f'$  and  $a'$  denote derivatives with respect to  $r$ . Then

$$\frac{\partial}{\partial r}(P - \bar{P}) = sf' + \frac{Ca'}{(a - 1)^2}$$

which, upon substituting for  $s$ , becomes

$$\frac{\partial}{\partial r}(P - \bar{P}) = \frac{Ca}{a - 1} \left( \frac{f'}{f} + \frac{a'}{a(a - 1)} \right) \tag{20}$$

Investment falls when  $r$  rises if the expression in parentheses is negative. The derivative of  $a$  (defined in (16)) with respect to  $r$  is

$$a' = \frac{1}{\sigma^2(D^2 + 2r/\sigma^2)^{1/2}} \quad (21)$$

Substituting (21) into (20) and rearranging, investment falls if and only if

$$-\frac{rf'}{f} > \frac{1}{2\sqrt{D^2 + 2r/\sigma^2}} \left( \frac{2r/\sigma^2}{a(a-1)} \right) \quad (22)$$

The left side is the project value's elasticity with respect to interest rates.

To illustrate the plausibility of perverse interest rate effects, assume risk neutrality and zero expected growth in  $s$ . The critical interest elasticity below which higher interest rates *encourage* investing becomes

$$\eta^* = \frac{1}{\sqrt{1 + 8r/\sigma^2}} \quad (23)$$

For  $r = .05$  and  $\sigma = .20$ , one gets  $\eta^* = .30$ . This corresponds to a 'duration', or average time to receipt of revenues, of about 6 years. Shorter duration projects are encouraged, longer duration projects are discouraged, by increases in  $r$ . At an extreme, if  $\sigma$  is infinite then  $\eta^*$  is 1 — the elasticity of a consol. Only if  $\sigma$  equals 0 do we have  $\eta^*$  equals 0, and have uniform dampening of capital spending by interest rates.

One must conclude that for long duration projects the usual present value considerations dominate, while for short duration projects, such as the decision to 'invest' in cutting down trees, the option aspect dominates. How long is long depends on the degree of uncertainty about the value of the project if postponed. But for reasonable parameter values the cutoff elasticity falls in the range of empirical relevance.

The apparent contradiction with capital theory can be resolved by recognizing that the option to invest, even if intangible and nontradeable, is properly part of the capital stock. Going ahead with the project entails giving up the real option at the same time. What appears to be net investment is really a shift of capital from one sector to another. This approach creates its own problems for aggregative analysis, however, because investment opportunities are more like manna than produced capital goods in the usual sense. How one should then model the dynamics of the capital stock becomes an awkward question.

Lest the implications of this section be misinterpreted, it must be pointed out that only the effect of a change in  $r$  on current investment has been

examined. A fuller model would incorporate the stock of available options, and recognize that investment today, by depleting this stock, means fewer opportunities and less investment in the future. A positive initial impact of changes in  $r$  does not rule out a negative correlation between levels of interest rates and investment in some sort of stochastic steady state.

Nevertheless, if one is concerned with spending on produced capital goods, say for macroeconomic forecasting, one must recognize that decisions of whether to invest and when to invest are always to some extent intermingled. Perverse initial impacts of  $r$  imply destabilizing interactions between interest rates and investment, exacerbating business cycles. The option perspective provides some clues about why uncertainty has a negative impact on capital spending, and why aggregate investment is so difficult to predict.



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