ADAPTIVE HEDONIC UTILITY

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ABSTRACT

Recent research in neuroscience provides a foundation for a von Neumann Morgenstern utility function that is both hedonic and adaptive. We model such adaptation as arising from a limited capacity to make fine distinctions, where the utility functions adapt in real time. For minimizing the probability of error, an optimal mechanism is particularly simple. For maximizing expected fitness, a still simple mechanism is approximately optimal. The model predicts the S-shaped utility characteristic of prospect theory. It also predicts that risk aversion or risk preference will remain evident over low stakes, resolving a vexing puzzle concerning experiments. JEL Codes A12, D11.

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1. Introduction

Jeremy Bentham is famous for, among other things, the dictum “the greatest happiness for the greatest number”, which, as Paul Samuelson was fond of observing, involved one too many “greatests” to be operationally meaningful. The happiness that Bentham described was cardinal, and so capable of being summed across individuals to obtain a basic welfare criterion. Conventional welfare economics, even allowing for non-additivity, remains needful of some degree of cardinality. However, in the context of individual decisions, of consumer theory, in particular, economics has completely repudiated any need for cardinality, on the basis of “Occam’s Razor,” a theme that culminates in the theory of revealed preference.

On the other hand, there is persuasive neurological evidence that economic decisions are actually orchestrated within the brain by a mechanism that relies on hedonic signals, which signals are measurable and therefore cardinal. In particular, there is evidence that economic decisions are mediated by means of neurons that produce dopamine, a neurotransmitter that is associated with pleasure. (See, for example, a key paper for the present purpose, Stauffer, Lak and Schultz, 2014.)

Although there is no logical need for the utility used in consumer theory to be hedonic and cardinal, it is so, as a brute fact.

Further, there is neurological evidence that this hedonic utility is adaptive, so that dopamine-producing neurons adapt rapidly to changes in the distribution of physical rewards (see Tobler, Fiorillo and Schultz, 2005). For example, if the variance of rewards increases, the sensitivity of such a neuron to a given increase in reward is reduced.

A primary motive of the present paper is then to harmonize the neurological view of hedonic utility with economics. We develop a model of hedonic adaptive utility that draws directly on neuroscience. The model is not fundamentally at odds with conventional economic theory.
in that the only reason for a divergence from economics is the inability to make arbitrarily fine distinctions.

The key paper by Stauffer, Lak and Schultz (2014) is described in some detail in the next section. It provides evidence linking increments in von Neumann Morgenstern utility, in a cardinal sense, to the activity of the dopamine neurons. These neurons evaluate economic options, in an adaptive fashion. Our model also concerns how these evaluations feed into a decision rule that is noisy. The rule involves “just noticeable differences”—JND’s—in the activity of dopamine neurons. (Matlin, 1988, is a textbook account of JND’s.) Adaptation involves shifting the thresholds at which a just noticeable jump in dopamine neuron activity occurs. This formulation was used by Laughlin (1981) to introduce the efficient coding hypothesis to capture maximal informational transfer by neurons.

It pays to shift the capacity to discriminate to the thick of the action, so hedonic utility needs to adapt, and it needs to adapt rapidly. We show simple neural adjustment mechanisms exist by demonstrating the existence of a particular simple automatic mechanism, increasing the empirical plausibility of the present basic approach.²

The present paper presents then a mechanism that generates rapid adaptation to a entirely novel distribution. When the objective is to minimize the probability of error, a particularly simple rule of thumb yields optimal adaptation for an arbitrary number of thresholds.³ When the objective is the more plausible one of maximizing the expected outcome chosen, a different rule of thumb yields adaptation that is approximately optimal for a large number of thresholds.

We demonstrate the empirical power of this approach by sketching an application to prospect theory, readily predicting the S-shaped utility

²We abstract from the interesting and complex question of how conscious inputs influence automatic processing.
³This rule of thumb generates efficient coding, as in Laughlin (1981). His criterion is the formal one of informational transfer; ours is to minimize the probability of error in a concrete binary choice problem.
that is one of the key features (Kahneman and Tversky, 1979). Of equal and independent interest, the model predicts substantial attitudes to risk will remain over the small stakes gambles that arise in experiments, as is well-known to contradict standard expected utility theory (Rabin, 2000).

2. A Framework from Neuroscience

A remarkable paper that grounds the current work in neuroscientific fact is Stauffer, Lak, and Schultz (2014). They argue that von Neumann Morgenstern utility is realized in the brain, in an hedonic fashion, by the activity of dopamine-producing neurons. These neurons number about a million in humans and are located in the midbrain, between the ears and behind the mouth. Dopamine is a neurotransmitter, a chemical that relays a signal from one neuron to the next, and it has a number of functions in the brain, a key one of which is to generate hedonic motivation. These dopamine producing neurons have forward connections—"projections"—to all of the sites in the brain that are known to implement decisions.

Most basically, perhaps, a burst of activity of the dopamine neurons is associated with the arrival of an unanticipated physical reward. Furthermore, a larger reward generates a greater intensity of the burst of activity in the neuron (as measured by the number of impulses per second).

One of the most firmly established results in the neuroscience literature that bears on decisions is the "reward prediction error", which is as follows. Suppose the individual is trained to anticipate a particular reward by a cue, perhaps a unique visual signal. The dopamine neurons then shift much of their firing activity back in time from the actual activity back in time from the actual activity. 

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4See Schultz, (2016), for a less formal treatment of the issues.

5Many of these experiments are done on monkeys, and involve implanted electrodes reading the activity of individual dopamine neurons. The rewards that the monkeys obtain are typically food or fruit juice.

6Caplin and Dean, (2008), present a model of this phenomenon, and sketch several applications to economics.
reward to the cue. If the size of reward is as expected, indeed, there is no further response by the neuron. If the reward is larger than expected, however, there is a supplementary burst of activity upon the receipt of the reward, which is larger the larger the discrepancy on the upside; if the reward is smaller than expected, the firing rate of neuron is reduced below the base rate.\footnote{The dopamine neuron system apparently represents unexpected upside shifts in rewards more accurately than unexpected downside shifts. Different systems, sometimes involving another neurotransmitter, serotonin, may help generate appropriate responses to downside surprises. Rogers, (2011), reviews experimental evidence involving drugs concerning the roles of dopamine and serotonin. See also Weller, Levin, Shiv, and Bechara, (2007), for evidence that the neural systems that deal with gains and losses may be partially dissociated.}

Stauffer, Lak, and Schultz argue that von Neumann Morgenstern utility can be related rather convincingly to this reward prediction error, by proceeding as follows. First they estimate von Neumann Morgenstern utility in a precise revealed preference manner, by deriving the certainty equivalents for a variety of binary gambles that are presented to the monkeys. This step makes no use of neural data, that is. The von Neumann Morgenstern utility is convex at low levels of juice rewards, but concave for higher levels, so monkeys adhere to this property of prospect theory.

Next, Stauffer, Lak, and Schultz consider the response by dopamine neurons to several binary gambles, where the absolute difference between the high and the low reward is held constant. Each gamble is signalled by an associated cue. Neural activity then occurs with the arrival of the cue, but there is additional activity if the higher reward from the binary gamble is obtained. The extra neural activity is low for gambles involving low rewards and for those involving high rewards, but is high for gambles involving intermediate rewards. This additional dopamine neuron activity is then in close cardinal agreement with the incremental (“marginal”) utility estimated from revealed preference.\footnote{See their Figure 3, in particular.}

Stauffer, Lak, and Schultz then check the firing rates of dopamine neurons that arise from unanticipated rewards. Being unanticipated should...
be equivalent to generating an expected level of the reward of zero. Indeed, these unanticipated neural firing rates are indistinguishable statistically from the von Neumann Morgenstern utility derived from revealed preference.

Further, Stauffer, Lak and Schultz establish that the dopamine neuron responses to a cue for a binary gamble reflects the expected utility of the gamble. Hence, for gambles over low levels of juice rewards, dopamine neuron activity exceeds that for the mean reward, reflecting risk-preference over these rewards. For gambles over high levels of juice rewards, the reverse is true, reflecting risk-aversion over these levels. Both of these observations then agree with the S-shape of von Neumann Morgenstern utility established by revealed preference.9

Tobler, Fiorillo, and Schultz (2005) establish further adaptive properties of the dopamine neurons’ response to anticipatory cues, properties that are key here.10 That is, these neurons adapt not only to the expected value of the distribution of rewards, but their response is also scaled up or down in response to the variance of the distribution.11 Our theory generates adaptation to the full distribution, not merely to the first two moments. Adaptation is a pervasive property of neurons. Baccus and Meister, (2002), for example, consider the adaptive properties of visual neurons. The full adaptation of dopamine neurons that we hypothesize here is analogous to that under the efficient coding hypothesis of Laughlin (1981), who illustrates the hypothesis with data for visual neurons.

9In a separate paper, Lak, Stauffer, and Schultz (2014) show that dopamine neurons encode utility for multidimensional choice problems. Monkeys chose between rewards that differ in terms of amount, risk, and type. The monkeys’ preferences over these rewards can be found by revealed preference. These preferences agree with the firing activity of dopamine neurons given a cue for the particular reward, buttressing the notion that these dopamine neurons reflect utility in a full economic sense.

10Rangel and Clithero, (2012), is a recent review of adaptation in neural decision-making.

11Burke, Baddeley, Tobler, and Schultz, (2016), further investigate adaptation, in humans. They now find partial rather than complete adaptation. The rationale they advance for the desirability of this is that unlikely signals still need to generate appropriate reactions. One possibility is that adaptation is complete with respect to a distribution extended to allow for such unlikely signals.
For the current paper, the key fact that the foregoing establishes is that rewards are encoded as expected utility by the firing rates of dopamine neurons that arise in anticipation of the actual reward. Furthermore this encoding adapts to the circumstances. We model this adaptive encoding as an optimal response to the noisy choice mechanism that occurs subsequently.

The process by which rewards are encoded is then relatively well understood. Indeed, so are some of the precise ways in which choice is implemented.\footnote{Shadlen and Shohamy (2016), for example, discuss how sequential sampling drives the choice of a physical action. A random walk arises in the premotor neurons of a monkey seeking a reward for predicting the preponderant drift of a moving pattern of dots. (A premotor neuron, as the name suggests, is immediately upstream from the motor neurons that cause the monkey to press one button or another, for example.) The random walk is driven up or down by accumulating evidence favoring one or the other of two choices. When the random walk hits an endogenously determined barrier, the corresponding choice is implemented. These models work remarkably well, with convincing details accounted for, but the issues they raise concerning motor implementation of a choice are not central here.} Less well understood is how the encoded rewards are compared prior to implementing a decision.\footnote{Possibly, for example, there is further processing of value prior to comparison. It could even be that the process of comparing values proceeds in parallel to the encoding of values. See Hare, Schultz, Camerer, O’Doherty, and Rangel (2001), for example. Indeed, the hedonic interpretation is not crucial to the validity of the model here, which could reflect processing by neurons that do not produce dopamine. Nevertheless, the present formulation seems parsimonious in the light of the empirical findings. That is, value is represented hedonically by dopamine neurons and this feeds forward to comparison and decision.} However, there are tantalizing hints that the comparison of value is the comparison of dopamine neuron activity (Jocham, Klein, and Ullsperger, 2011).\footnote{These authors administered a $D_2$-selective antagonist, amisulpride, to humans engaging first in learning values and then in exploiting these. ($D_2$ is a particular type of dopamine receptor.) Amisulpride did not affect reinforcement learning but enhanced some subsequent choices made on the basis of the learned values.}

For the present purpose, we apply a noisy imprecise mechanism that compares dopamine neuron outputs. Such imprecision has a form that is familiar from psychology—involving “just noticeable differences” (Matlin, 1988, for example).
3. The Model

The neuroscientific evidence discussed in the previous section suggests the following model. A physical reward $y \in [0, 1]$ arises, where $y$ is taken to be fitness. After neural processing, the prospect of this reward induces dopamine neurons to evaluate its possible consumption at a rate given by $w = h(y)$, where $h : [0, 1] \to [0, 1]$. This formulation of $h$ abstracts from noise in these dopamine neuron firing rates.

We focus here instead on noise in the choice that is made after the dopamine neuron evaluation is made. The function $h$ will be modified to reflect adaptation, as is a key concern here. If there are two stimuli given by $y^i \in [0, 1]$ for $i = 1, 2$ then let $w_i = h(y^i)$ for $i = 1, 2$ be the associated dopamine neuron activity levels.

Suppose then choice is made according a function $J(w_1 - w_2)$, which is the probability of choosing option 1, for neural outputs $w_i$, $i = 1, 2$. That is, choice is modelled as inherently noisy, which is empirically compelling. It is assumed that

$$J(0) = 1/2; \ J(w_1 - w_2) \to 1, \text{ as } w_1 - w_2 \to 1, \text{ and}$$

$$J(w_1 - w_2) \to 0, \text{ as } w_1 - w_2 \to -1.$$
For tractability, we adopt a function $J$ that is described by one parameter, interpreted as a “just noticeable difference”. That is,

\[ J(w_1 - w_2) = \begin{cases} 1/2, & \text{for } |w_1 - w_2| < \delta; \quad J(w_1 - w_2) = 0, & \text{for } w_1 - w_2 \leq -\delta, \\ J(w_1 - w_2) = 1, & \text{for } w_1 - w_2 \geq \delta, \end{cases} \]

for some “just noticeable difference”—JND—$\delta > 0$.

Choice represented in this way remains noisy. That is, if particular $w_1$ and $w_2$ are drawn repeatedly, then sometimes $w_1$ is chosen and sometimes $w_2$.

Consider now how the function $h$ reflects adaptation. It is analytically attractive to describe $h$ by a finite number of parameters, where the number of parameters will then determine the accuracy of the map. Anticipating also how the map will conveniently feed into the choice function $J$ suggests the following step function, which compresses ranges of rewards into classes assigned common evaluations, where these evaluations are “just noticeably different”.

Suppose then that $h : [0, 1] \rightarrow \{0, \delta, 2\delta, \ldots, N\delta = 1\}$ for some integer $N$, and where $\delta$ is the JND built into $J$. Hence the number of parameters, $N$, in $h$ is inversely related to $\delta$. Since $h$ should also be non-decreasing, it is characterized by $N$ thresholds in $[0, 1]$, $0 \leq x_1 \leq \ldots \leq x_N \leq 1$, say, where we formally set $x_0 = 0$ and $x_{N+1} = 1$. At these thresholds, $h$ jumps up by $\delta$ and so we have $h(y) = n\delta$ for all $y \in [x_n, x_{n+1}), n = 0, \ldots, N$. Such a step function can approximate an arbitrary continuous function, if $\delta$ is small.

These simplifications still capture key elements of choice orchestrated by neurons. There is a capacity to reshape the evaluation of a reward, which is manifested empirically in the firing rate of dopamine neurons. This capacity is advantageous because the choice mechanism that

\[19\] This formulation segregates adaptation, which arises in the $h$ function, from noise, which arises in the $J$ function. This is largely for convenience. That is, adaptation might first arise, with noise arising later, but all within the process summarized by the $h$ function. This would render the $h$ noisy. Whether choice was also noisy or not, this would give a similar rationale for adaptation.
keys on the evaluations is necessarily noisy. Hence reshaping the reward evaluations to push frequent rewards apart will help discriminate between these rewards.

The foregoing motivates the following choice problem.\(^{20}\) The individual must choose one of two outcomes, \(i = 1, 2\). These are realizations \(y^i \in [0, 1]\), for \(i = 1, 2\), that were drawn independently from the cumulative distribution function, \(F\). This has a continuous probability density function, \(f > 0\) on \((0, 1)\). The cdf \(F\) represents the background distribution of rewards to which the individual is accustomed.

As implied by the construction of \(h\) and the JND formulation of \(J\), the only precise information that the individual has prior to choosing one of the arms is the interval \([x_n, x_{n+1}]\) that contains each realization. If the two realizations belong to different intervals, the gamble lying in the interval further to the right is clearly better; if the two realizations lie in the same interval, choice is noisy with an error being made with probability \(1/2\).

We interpret the number of thresholds that an outcome surpasses as utility. In general, we could assign utility \(U_n \in [0, 1]\) say to any outcome lying in the interval \([x_n, x_{n+1}]\), for \(n = 0, \ldots, N\), so that \(0 = U_0 < U_1 < \ldots U_n < U_{n+1} < \ldots U_N = 1\).

Only the ordinal properties of utility are relevant for the basic model here. Later, however, we sketch an application to attitudes to risk, where cardinal properties become relevant. Since the evidence in Stauffer, Lak, and Schultz, (2014), implies this, and for simplicity throughout, we set \(U_n = n/N\), for \(n = 0, \ldots, N\).

What are the optimal \(0 \leq x_1 \leq \ldots \leq x_N \leq 1\)? Robson (2001) shows that the thresholds that minimize the probability of error are equally spaced in terms of probability. If \(N = 1\), for example, the threshold should be at the median of \(F\). At the other extreme, when \(N \to \infty\), it

\(^{20}\)This is now as in Robson (2001).
follows that the limiting density of thresholds matches the pdf, \( f \), and that \( U(y) = F(y) \), where \( U(y) \) is the utility assigned to \( y \) in this limit.

This result is in striking agreement with the efficient coding hypothesis proposed by Laughlin (1981). He considers a function precisely analogous to \( h \), where a continuous input intensity is mapped onto a finite set of “responses”, spaced apart by the just noticeable difference. Laughlin then argues that the response function of a neuron to a single \( y \) should match the cumulative density function in order to maximize the information content of the neural responses. (See Louie and Glimcher, 2012, for a recent review of this efficient coding hypothesis.) We replace the abstract notion of information transfer with a more concrete binary choice problem but arrive at precisely the same conclusion. However, this agreement only holds for the probability of error criterion.\(^{21}\)

If the criterion were instead to maximize the expected value of the \( y^i \) chosen, expected fitness, that is, and \( N = 1 \), the optimal threshold is at the mean of \( F \). With \( N \) thresholds, each threshold should be at the mean of the distribution of outcomes, conditional on the outcome lying between the next threshold to the left and the next threshold to the right. This uniquely characterizes the optimal thresholds in this case. Now, in the limit as \( N \to \infty \), Netzer (2009) shows that the density of thresholds is proportional to \( f^{2/3} \), so that

\[
U(y) = \frac{\int y f(y)^{2/3} dy}{\int f(y)^{2/3} dy}
\]

In either case, the thresholds are optimally concentrated where the action is—where \( f \) is high. That is, if the distribution shifts, the pattern of thresholds must shift to match.

Previous work has not considered the mechanism of adaptation. If the thresholds were chosen by evolution, this would make adjustment painfully slow, too slow, indeed, to fit the stylized facts. How then could the thresholds adjust to a novel distribution, \( F' \)?

In order to study this question, suppose then that the thresholds react to draws, are allowed to move, that is, but are confined to a finite grid

\(^{21}\)The rate of informational transfer is a symmetric concave function of the probabilities of each output. Hence maximizing this entails equalizing these probabilities.
$K_G = \{0, \epsilon, 2\epsilon, \ldots, G\epsilon, 1\}$, for an integer $G$ such that $(G + 1)\epsilon = 1$. This restriction is for technical simplicity since it means that the adjustment process for the thresholds will be a (finite) Markov chain. Define the state space $S_G = (K_G)^N$ and let $S = [0, 1]^N$.

At first we abstract from the choice between the two arms. Instead, we focus on the process by which the thresholds adjust.\footnote{It simplifies matters to suppose that most draws adjust the thresholds, but choices are made only occasionally.} Hence we use $y$ at first to represent either $y^1$ or $y^2$. Eventually, when considering the performance of the limiting rule of thumb, we will again need to consider the outcomes on both arms.

Suppose the thresholds are time dependent, given as $x^t_n \in K_G$, where $0 \leq x^t_1 \leq \ldots \leq x^t_N \leq 1$, at time $t = 1, 2, \ldots$.

3.1. Minimizing the Probability of Error. The first of the two criteria considered here is to minimize the probability of error. This is less basic than maximizing expected fitness, but it leads to simpler results, and is intuitively illuminating. It is also noteworthy that the rule of thumb for the probability of error case generates efficient neural coding.

Consider the rule of thumb for adjusting the thresholds—

$$
(3.1) \quad x^{t+1}_n = \begin{cases} 
  x^t_n + \epsilon & \text{with probability } \xi \text{ if } y \in (x^t_n, x^t_{n+1}] \\
  x^t_n - \epsilon & \text{with probability } \xi \text{ if } y \in [x^t_{n-1}, x^t_n) \\
  x^t_n & \text{otherwise}
\end{cases}
$$

for $n = 1, \ldots, N$.

The parameter $\xi \in (0, 1)$ represents additional idiosyncratic uncertainty about whether each threshold will actually move, even if the outcome lies in a neighboring subinterval.\footnote{This technical device simplifies the argument that the Markov chain is irreducible.}
This is perhaps the simplest possible rule in this context. It moves thresholds towards where the action is, roughly speaking. This seems like a step in the right direction, at least. More than that, we will show that, in the limit of the invariant distribution as the grid size, $\epsilon \to 0$, the thresholds are in exactly the right place.

The rule may not provide the most rapid possible adjustment, but it is sufficient for the present purpose.24

We have—

**Theorem 3.1.** *In the limit as $G \to \infty$, so that $\epsilon \to 0$, the invariant joint distribution of the thresholds $x^t_n$ converges to one with point mass on the vector with components $x^*_n$, where $F(x^*_n) = n/(N + 1)$, for $n = 1, \ldots, N$.*

This theorem is a corollary of a more general result—Theorem 3.2—to follow.

That is, this rule of thumb generates optimal adaptation of the utility function to any unknown distribution, in a non-parametric way, for any number of thresholds, $N$.

An intuition for Theorem 3.1 is as follows. Consider $N = 1$, so there is a single threshold $x_1 \in \{0, \epsilon, 2\epsilon, \ldots, G\epsilon, 1\}$. If $F(x_1) < 1/2$, for example, then the probability of moving to the right, $1 - F(x_1)$, exceeds the probability of moving to the left, $F(x_1)$. As $\epsilon$ becomes small, the

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24 A Bayesian optimal updating rule entails a prior distribution over distributions $F$. Suppose, for example, there is one threshold, and the pdf is either $f_1(y) = 2$ with support $[0, 1/2]$ or $f_2(y) = 2$ with support $[1/2, 1]$, where these pdf’s are equally likely. The optimal initial threshold should then be at $1/2$. If the outcomes are to the left, the pdf must be $f_1$ and the next position of the threshold should be at $1/4$; if the outcomes are to the right, the pdf must be $f_2$ and the threshold should be set next at $3/4$. That is, there is rapid resolution of the uncertainty about the distribution. On the other hand, this procedure would be wildly inappropriate for a different prior. Even with a definite general prior, it is not obvious that placing the threshold at the median of posterior is always fully optimal. Furthermore, if the distribution is subject to occasional change, this will also affect the Bayesian optimal rule. Although the current rule can only be slower than the optimal rule with a specified prior and mechanism for change, it ultimately yields optimal placement of the thresholds, in a robust fashion, without a specified prior, and without a specified mechanism for redrawing the distribution.
speed at which $x_1$ moves decreases in proportion. This can be precisely offset by increasing the frequency with which draws are taken, without affecting the long run limiting distribution. Now, more and more independent draws are packed into each unit of time. The law of large numbers then dictates that $x_1$ moves deterministically to the right at a rate given by $1 - 2F(x_1)$.

Similarly, $x_1$ moves to the left if $F(x_1) > 1/2$. In the limit as the grid size, $\epsilon$, tends to 0, the limiting invariant distribution puts full weight on the median of $F$ where $F(x_1) = 1/2$.

When there are more thresholds, the same intuition applies, since each threshold is situated in the limit such that the probability of its moving to the left equals its probability of moving to the right.

### 3.2. General Case—Maximizing Fitness

The most basic general criterion is to maximize expected fitness. That is, individuals who successfully do this should outperform those who do not.\(^{25}\)

The situation is now more complicated than it was with the criterion of minimizing the probability of error. There are no longer simple rules of thumb that implement the optimum exactly. However, there do exist simple rules of thumb that implement the optimum approximately, for large $N$. These rules of thumb involve conditioning on the arrival of a realization in the adjacent interval, as above, but also modify the probability of moving using the distance to the next threshold, in a symmetric way.

Although it is possible to accurately estimate the median of a distribution from the limited information available to such a rule of thumb, it is not possible to do this for the mean. Hence the result for the

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\(^{25}\)This assumes that the risk is independent across individuals. See Robson (1996) for a treatment of this issue. Another possibility would be that fitness depends on relative payoffs.
probability of error case are sharper than the results for the expected fitness case.\textsuperscript{26}

It is important that this general rule of thumb uses only information that is available—the location of the neighboring thresholds and whether an outcome lies in the subinterval just to the right or just to the left. It would contradict the interpretation of the model here to use detailed information about the precise location of the outcome within a subinterval.

At the same time, the general rule of thumb here makes greater demands on neural processing than does the rule of thumb for the probability of error case. The need to utilize the position of adjacent thresholds must entail a greater complexity cost.

The general rule of thumb considered here is—

\begin{equation}
  x_{n+1}^{t+1} = \begin{cases} 
    x_n^t + \epsilon & \text{with probability } \xi (x_{n+1}^t - x_n^t)^\beta \text{ if } y \in (x_n^t, x_{n+1}^t] \\
    x_n^t - \epsilon & \text{with probability } \xi (x_n^t - x_{n-1}^t)^\beta \text{ if } y \in (x_{n-1}^t, x_n^t) \\
    x_n^t & \text{otherwise}
  \end{cases}
\end{equation}

Again, the parameter \( \xi \in (0, 1) \) and the draws that are made with probability \( \xi (x_{n+1}^t - x_n^t)^\beta \) or \( \xi (x_n^t - x_{n-1}^t)^\beta \) conditional on the outcome lying in the subinterval just to the right or left, respectively, are made independently across thresholds.\textsuperscript{27}

\textsuperscript{26}To see that simple rules of thumb like this cannot implement the optimum exactly, consider first the case that \( N = 1 \). Suppose that \( F \) has median \( \frac{1}{2} \) but a mean that is not \( \frac{1}{2} \). Consider a symmetric rule of thumb based on the arrival of an outcome to the left or the right of the current position of the threshold at \( x \), say, and the distance to the ends—\( x \) or \( 1 - x \). This will then generate a limiting position for the threshold at \( \frac{1}{2} \), thus failing to implement the optimum. This is also an issue for any number of thresholds, since this argument applies to the position of any threshold relative its two neighbors.

\textsuperscript{27}A few technical considerations are as follows. Given the Markov chain described here, it is possible that the order of thresholds is reversed at some stage so that \( x_{n+1}^{t+1} < x_n^{t+1} \), for example. In such a case assume that the thresholds are renumbered so as to preserve the natural order. It is also possible that the process superimpose one threshold on another so that \( x_{n+1}^{t+1} = x_n^{t+1} \), for example. In this case the independence of the draws made conditional on an outcome lying in
If the parameter $\beta = 0$, we have the old rule of thumb. Formally, then, Theorem 3.1 follows from Theorem 3.2.

If $\beta > 0$ this will encourage the closing up of large gaps that arise where $f$ is small, which is useful to maximize expected fitness. Consider, for example, a threshold situated so that the probability of an outcome in the adjacent interval to the left equals the probability of an outcome just to the right. Suppose, however, that the distance to the next threshold on the right exceeds the distance to the left, because the pdf, $f$ is lower to the right. It will then pay to move to right, since the expected fitness stakes on the right exceed those on the left. Indeed, if $\beta = 1/2$, the resulting rule will be shown to be approximately optimal for large $N$.

We have—

**Theorem 3.2.** In the limit as $G \to \infty$ so that $\epsilon \to 0$, the invariant joint distribution of the thresholds $x_n^*$ converges to one that assigns a point mass to the vector with components $x_n^*, n = 1, ..., N$. These are the unique solutions to

$$(F(x_{n+1}^*) - F(x_n^*))(x_{n+1}^* - x_n^*)^\beta = (F(x_n^*) - F(x_{n-1}^*))(x_n^* - x_{n-1}^*)^\beta,$$

for $n = 1, ..., N$.

**Proof.** See the Appendix.

The intuition for Theorem 3.2 straightforwardly extends that given for Theorem 3.1. Again, the limiting position of each threshold is such that the probability of moving to the left is equal to the probability of moving to the right.

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A neighboring subinterval will eventually break such a tie. These possibilities become vanishingly improbable as $G \to \infty$, so these issues are purely details.

The above considerations simplify the proof that the Markov chain defined here is irreducible. That is, there exists a number of repetitions such that, for any initial configuration, $x^0$, say, there is positive probability of being in any final configuration, $x^T$, say. There is therefore a unique invariant distribution for this chain. See Footnote 49 in the Appendix.
3.3. Approximate Optimality of the Rule of Thumb. We now consider the efficiency of the rule of thumb relative to the optimum configuration of thresholds, for the expected fitness criterion. We demonstrate approximate efficiency, as \( N \to \infty \), for the particular case of \( f \)'s that are step functions, so that

\[
f(y) = \begin{cases} 
\alpha_1 > 0 & \text{if } y \in [0, 1/M) \\
\vdots \\
\alpha_m > 0 & \text{if } y \in [(m-1)/M, m/M) \\
\vdots \\
\alpha_M > 0 & \text{if } y \in [(M-1)/M, 1]
\end{cases}
\]

where \( \sum_{m=1}^{M} \alpha_m = M \).

For each \( N \), there exists a unique positioning of the \( N \) interior thresholds, under the rule of thumb, in the limit as \( G \to \infty \) so that \( \epsilon \to 0 \). Suppose that the expected deficit in \( y \), for the limiting rule of thumb relative to the full information ideal, is given by \( \hat{L}(N) \). (The “full information ideal” entails always choosing the higher outcome.)

**Theorem 3.3.** As \( N \to \infty \), the limiting efficiency of the rule of thumb is characterized by

\[
N^2 \hat{L} \to \sum_m \alpha_m^{2/3}/(1+\beta) (\sum_m \alpha_m^{1/(1+\beta)})^2 /(6M^3).
\]

This expression is uniquely minimized by choice of \( \beta = 1/2 \). Hence the rule of thumb with the best limiting efficiency satisfies \( N^2 \hat{L} \to (\sum \alpha_m^{2/3})^3/(6M^3) \) as \( N \to \infty \).

**Proof.** See the Appendix.

Suppose the expected deficit in \( y \), relative to the full information ideal, for the optimal positioning of thresholds, is given by \( L^*(N) \).

**Theorem 3.4.** The optimal allocation of the thresholds has limiting efficiency characterized by \( N^2 L^* \to (\sum \alpha_m^{2/3})^3/(6M^3) \), as \( N \to \infty \).

**Proof.** See the Appendix.

Hence the rule of thumb, when \( \beta = 1/2 \), has the same limiting efficiency as the optimal allocation of thresholds. That is, the rule of thumb is
approximately optimal for large $N$. Roughly speaking, the efficiencies here can be thought of as Taylor series in powers of $1/N$. The first nonzero term is the term in $1/N^2$, which is the same for $L^*$ and $\hat{L}$. They may then only disagree for terms of higher order.

4. Robustness

The basic results here are Theorems 3.1 and 3.2 which concern limits of invariant distributions as the grid size $\epsilon$ tends to 0. It is important then to show that these results hold approximately for finite time and reasonable positive grid sizes.

We achieve this by simulating the following specific version of the model. Consider the class of cdf’s given by $F(x) = x^\gamma$ with pdf’s $f(x) = \gamma x^{\gamma-1}$, with $\gamma > 0$, for all $x \in [0,1]$. Suppose $\epsilon = 0.0005$. Consider the probability of error case, for example, so that $\beta = 0$, with nine thresholds, so that these thresholds will be optimally positioned at the deciles of the distribution. Take 100,000 periods, where $\gamma = 1$ for the first 20,000 periods and $\gamma = 5$ thereafter, so that probability mass is shifted to the upper end of the interval $[0,1]$. Suppose the thresholds are placed initially at 0.1, 0.2,...,0.9—that is, at the deciles of the distribution for $\gamma = 1$. This is essentially equivalent to supposing that the $\gamma = 1$ regime has been in effect for a long time.

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28 This approximation is additional to those already involved in i) the convergence of the Markov chain to an invariant distribution and ii) taking the limit of the invariant distribution as $\epsilon \to 0$.
29 This is a “rough” argument, only in that the approximation result here remains valid, even if a Taylor series does not exist.
30 The minimized probability of error is easily seen to be $1/2(N + 1)$. The efficiency loss for maximum fitness has a leading term in $1/N^2$ instead because the size of an error is of order $1/N$.
31 Netzer uses the same device of considering a step function $f$. However, he does not consider the adjustment process that is the focus here as the underpinning of utility adaptation. There is, then, no counterpart of the rule of thumb used here. His main result is to show that, in the limit as $M \to \infty$, the density of thresholds is proportional to $f^{2/3}$, in contrast to the limiting density of thresholds in the probability of error case which is simply $f$. This main result of Netzer is an incidental by-product of the present approach. This observation does not, however, help extend our approximation results to a general $f$.
32 Since every detail of the model cannot be taken literally, these details should not be estimated directly, but rather the overall fit of the predictions should be optimized.
The results of simulating this version of the model are presented in Figure 1, confirming the robustness of Theorem 3.1. That is, even with a fixed $\epsilon > 0$, the distribution of thresholds quickly puts most mass near the deciles, as shown by the uniform empirical frequency of outcomes in each interval.

The key results of Theorems 3.3 and 3.4 rely on taking the additional limit as $N \to \infty$, then showing that $\beta = 1/2$ yields a rule of thumb that is approximately optimal. These results are also robust. To show this, we consider the current specific model with $\gamma = 5$, $N = 3$, $\epsilon = 0.0005$ and varying values of $\beta$. We summarize the results in Figure 2, which shows that $\beta = 1/2$ is approximately optimal even for such a small value of $N$.

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\(^{33}\)All the simulations here were done using Excel.

\(^{34}\)It would only be an accident if $\beta = 1/2$ were exactly optimal for $N = 3$ and an arbitrary non-constant pdf.
Figure 2 also demonstrates that, although there is a definite gain from $\beta > 0$, this gain is not overwhelming. The additional complexity cost of rules of thumb with $\beta > 0$ might then outweigh the gain over the rule with $\beta = 0$. This buttresses the case for the rule that minimizes the probability of error and agrees with the efficient coding hypothesis.

5. Immediate Predictions on Economic Behavior

There are straightforward revealed preference implications of the theory that could be tested. One such implication concerns how errors would adapt to a shift in the distribution. That is, if an individual were adapted to high stakes lotteries, there should be a high error rate for gambles involving penny ante amounts, for example. As adaptation to the penny ante regime proceeded, however, the error rate should decline.

For the specific version of the model described in Section 4, but now with $\beta = 1/2$, Figure 3 illustrates that there will be increased error rates in ranges that become less likely, as a result of the upward movement of payoffs. For example, the range of values between 0 and 0.5 that
was originally separated by the first threshold at 0.25 is soon lumped together by the upwards drift of the thresholds. This illustrates how error rates for penny ante decisions will rise if the individual is inured to higher stakes.

The following effective but annoying sales strategy is relevant. When you are buying a car, the salesman suggests that you need various more-or-less-worthless add-ons, undercoating for example, that cost perhaps hundreds of dollars. The salesman is relying on the hundreds of dollars seeming insignificant relative to the thousands that the car costs. The effectiveness of this sales technique is consistent with the shift in utility that larger stakes induce in the present model.\footnote{Khaw, Glimcher, and Louie, (2017), present experimental evidence in favor of such maladaptation. They show that the subjective value of an option in an auction varies inversely with the average value of recently observed items.}

Figure 3 also illustrates how setting $\beta > 0$ increases the density of thresholds in regions where the pdf $f$ is lowest. That is, the first
threshold is set such that the first interval is the largest, but also so that the frequency of outcomes in the first interval is the lowest.

For $\beta = 0$, as in Figure 1, for example, average utility reverts completely to its original level, after a shift in the cdf $F$. This phenomenon, in which adaptation erodes the immediately perceived improvement in well-being arising from improved conditions, is the “hedonic treadmill”. (See Frederick and Lowenstein, 1999, for example.)

For $\beta = 1/2$, however, reversion is generally incomplete or exaggerated. Figure 3 illustrates this, presenting a rolling average of utility, where utility is defined so that average utility for $\gamma = 1$ is normalized to 0.5. This is because expected utility generally depends on the distribution, when $\beta > 0$.

5.1. **Speed Versus Accuracy.** A basic property of the theoretical model is that it generates a trade-off between speed of adjustment and accuracy. This property also has empirical implications.

In the current model, when $\epsilon$ is small, convergence to the invariant distribution is slow, but ultimately precise. This issue could be sharpened by assuming that the underlying cdf, $F$, was subject to occasional change. Suppose, to be more precise, that there is a (finite, say) set of cdfs $\{F_j\}$. With a Poisson arrival rate, the current cdf from this set is switched to a new one, drawn at random from this set. It is intuitively compelling that there should then be an optimal $\epsilon > 0$ and that this should vary with the rate of introduction of novelty, in particular.

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36 Average utility needs to be smoothed to be meaningful. We use a rolling average of the last 1,000 periods.

37 Expected utility also depends on the distribution under the optimal allocation of a finite number of thresholds.

38 Increasing $N$ must also slow convergence, if only because, although there are now more thresholds, in general at most two of these are adjusted in each period.

39 In a similar spirit, the number of thresholds might be allowed to vary with the problem at hand. That is, if a problem has particularly high stakes, $N$ might be allowed to increase, but at a cost.
This tradeoff between speed and accuracy seems bound to be theoretically robust. That is, other models that differ in detail but still capture rapid adaptation seem bound to also produce such a tradeoff.

The model suggests that adaptation should be slow when circumstances change infrequently; but fast when circumstances change frequently. (This would consider the parameter \( \epsilon \) as endogenous, tailored to the circumstances.) This is consistent with adaptation to living in a new locale taking several years; but adaptation to playing a game of penny ante poker being much faster.

Figure 4 illustrates these observations for the specific version of the model in Section 4. It depicts the evolution of the three thresholds over time, now contrasting two different values of the grid size \( \epsilon \); namely 0.002, and 0.000125, top and bottom, respectively. It is evident here that a smaller value of \( \epsilon \) slows down the speed of adjustment but improves the precision of the ultimate allocation of thresholds.
6. Risk-Taking, Prospect Theory and Experiments

A key empirical implication of the model is to generate the S-shaped utility of prospect theory (Kahneman and Tversky 1979). At the same time, and equally important, we show that that experiments involving modest amounts of money may well generate substantial attitudes to risk. Such attitudes cannot be reconciled on the basis of standard expected utility theory with any feasible attitude to risk over more substantial amounts (as shown dramatically by Rabin, 2000).

We need then to extend the interpretation of utility so its expectation represents preferences over gambles. Until now, the cardinal formulation did not matter, since all of the choices considered were essentially deterministic. The interpretation of $J$ and $h$ in terms of “just noticeable differences” in dopamine output is now relevant. Also crucially, Stauffer, Lak, and Schultz (2014), demonstrate empirically that dopamine production is cardinaly related to incremental (“marginal”) von Neumann Morgenstern utility, as derived from behavior. This justifies

\footnote{Netzer, (2009), sketches such an application.}
choosing von Neumann Morgenstern utility to be simply $U_n = n/N$ for $n = 0, \ldots, N$.

Consider now a generalized version of the experiments that Stauffer, Lak and Schultz (2016) use to establish the certainty equivalents of various binary 50-50 gambles. This construction also demonstrates how the approach in the present paper can be extended.

Suppose, in general, that the certain alternative is distributed according to cdf $G$, with pdf $g$, and that each outcome of the 50-50 gamble is distributed according to the cdf, $F$, with pdf $f$. All these outcomes are independent.\(^{41}\) Suppose further that both $F$ and $G$ have full supports $[0, 1]$.

There are, as before, $N$ thresholds $0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq 1$. The individual is confronted with a choice between the binary gamble, with outcomes $y^1$ and $y^2$, say, and a certain reward, $z$, say, but all that is known about the outcomes of the gamble, and the certain reward, is the interval $[x_n, x_{n+1})$ to which each outcome belongs. The individual chooses the gamble or the certain reward according to which of these maximizes expected utility. If these expected utilities are equal, the individual chooses each with probability $1/2$. The thresholds must be chosen subject to the individual’s maximization of expected utility.

The following limiting case demonstrates how the S-shape of utility can readily arise in this setting—

**Lemma 6.1.** Suppose that $F$ is unimodal, with mode $1/2$, and symmetric about $1/2$, so that $F(y) = 1 - F(1 - y)$, for all $y \in [0, 1]$. Suppose $G$ is degenerate, equal to $1/2$ for sure, and that $N$ is even.\(^{42}\) Thresholds that minimize the probability of error, subject to the individual maximizing expected utility, then satisfy $F(x_n) = n/(N + 1)$, for $n = 1, \ldots, N$.

\(^{41}\)The assumptions that the outcomes in the gamble are equally likely and that of independence can be relaxed, at the cost of additional complexity.

\(^{42}\)Choosing $N$ to be even merely ensures that $1/2$ lies in the interior of an interval.
Proof. This is provided in the Appendix.

The utility function is a step function for finite $N$. Since $F$ is unimodal, this step-function utility is S-shaped, roughly speaking, as in prospect theory. This implies risk-preference for $y < 1/2$ and risk-aversion for $y > 1/2$.\(^{43}\) However, in this example, outcomes that are symmetrically located about 1/2 have utilities are also symmetric about 1/2. Hence the concavity or convexity of $U$ is irrelevant, since it is never put to the test.

Suppose, however, that $G$ is non-degenerate, with $g$ being continuous, but $G$ remains close to the degenerate distribution at 1/2. The solution for the optimal thresholds must then be close to the solution found here, and therefore must still exhibit risk-preference and risk-aversion. If $G$ is non-degenerate like this, however, the risk-preference and risk-aversion of $U$ are put to the test.

How can this be, given that risk-aversion or risk-preference is costly when fitness is linear, simply being $y \in [0, 1]$? The placement of the thresholds for finite $N$ is a compromise. On the one hand, accurate assessment of where each outcome has fallen implies there should be a lower density of thresholds where outcomes are less likely. This generates convexity of $U$ at low values of $y$ and eventual concavity at high values of $y$. On the other hand, this has a fitness cost since it induces strictly risk-preferring and strictly risk-averse choices.

That this tension forces risk-preference and risk-aversion to arise is the basis of the explanation provided here for the S-shape of utility, as in prospect theory.

When $N \to \infty$, on the other hand, utility becomes linear on $[0, 1]$. In this case, it is straightforward to eliminate all ambiguity about where outcomes lie, so inappropriate risky choices can be eliminated too. Strict concavity or convexity of utility then directly reflects finite $N$

\(^{43}\)Neglecting the small scale risk-taking arising from the steps themselves.
and a limited ability to make fine distinctions. Lemma 6.2 demonstrates this where \( F \) and \( G \) are arbitrary continuous cdf’s on \([0, 1]\) and thresholds are chosen to maximize expected fitness subject to the individual maximizing expected utility.\(^{44}\)

**Lemma 6.2.** Consider arbitrary reference distributions \( F \) and \( G \) with supports \([0, 1]\). The problem of choosing the thresholds to maximize fitness subject to the individual maximizing expected utility has a solution. In the limit as \( N \to \infty \), utility is linear on \([0, 1]\), with \( U(0) = 0 \) and \( U(1) = 1 \).

**Proof.** See the Appendix. This also formalizes how the thresholds are chosen to maximize expected fitness subject to the individual maximizing expected utility.

To dramatize how the model also generates substantial attitudes to risk over small stakes, consider scaled background cdf’s \( F^k(y) = F(y_0 + k(y - y_0)) \), and \( G^k(y) = G(y_0 + k(y - y_0)) \), for \( k \geq 1 \), where \( y_0 \) is any point in \((0, 1)\). As \( k \) increases, these distributions collapse to a point mass at \( y_0 \). More generally, the shape of the cdf’s \( F^k \) and \( G^k \) retain the shape of \( F \) and \( G \) but over lower stakes.

Lemma 6.3 below shows that the utility also scales so \( U^k(y) = U(y_0 + k(y - y_0)) \), for all \( k \geq 1 \). To consider the implications of this, consider an arbitrary test gamble with cdf \( P \). Preferences are represented by the expected utility \( \int U(y) dP(y) \). Suppose these test gambles also scale with the background cdf’s, so that these test gambles have cdf’s \( P^k(y) = P(y_0 + k(y - y_0)) \). Lemma 6.3 also shows that \( U^k \) ranks the \( P^k \) exactly as \( U \) ranks the \( P \). Altogether, then—

**Lemma 6.3.** Suppose arbitrary reference distributions \( F \) and \( G \) with supports \([0, 1]\) and an arbitrary test distribution \( P \) are subject to scaling as \( F^k \), \( G^k \) and \( P^k \), as described above, for \( k \geq 1 \). Then, utility also scales as \( U^k(y) = U(y_0 + k(y - y_0)) \) and \( \int U^k(y) dP^k(y) = \int U(y) dP(y) \). Hence \( U^k \) ranks the \( P^k \) in exactly the same way that \( U \) ranks the \( P \).

\(^{44}\)Minimizing the probability of error is problematic in that the magnitude of these errors may well be vanishingly small in this model, as \( N \to \infty \).
Proof. See the Appendix.

Whatever the risk attitudes of the individual are under \( U \), this precise relationship is preserved as \( k \) increases. If the first relationship is expressed in dollars, for example, and \( k = 100 \), the second relationship will be identical, now expressed in cents. The model is then perfectly consistent with substantial risk-aversion or preference in experimental situations, as is flatly impossible with the conventional expected utility approach (Rabin, 2000).

Lemma 6.1 already showed how the model can predict the S-shape of von Neumann Morgenstern utility, relative to an endogenous reference point, that is a key characteristic of prospect theory. The S-shape for utility, as in prospect theory, has a solid empirical basis. The strong experimental evidence on this score for monkeys was discussed above and helped motivate the present work (Stauffer, Lak, and Schultz, 2014). Prospect theory also does well in accounting for human behavior, at least in experiments (see Barberis, 2013, for example).

7. FURTHER LITERATURE FROM ECONOMICS

Setting aside the important but tangential issues raised by welfare economics, the hedonic and adaptive qualities of utility raise awkward questions for individual decision-making. Side-stepping these awkward questions was presumably part of the motivation behind the drive in economics to apply Occam’s Razor to trim utility down to its revealed preference nub.

Schkade and Kahneman (1981) address this issue for students at the University of Michigan and at UCLA. Students in the two locations

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45We do not address another salient feature of prospect theory—that there should be a “kink” at the reference point, as is associated with the endowment effect. Neither do we consider that the weights used are nonlinear in the probabilities. Hsu, Krajbich, Zhao, and Camerer, (2009), show that this nonlinearity is reflected in neural response.

46List, (2004), however, gives intriguing evidence that this is more true for naive consumers, with sophisticated consumers being more neoclassical in their behavior.
reported similar degrees of life satisfaction, but Michigan students pro-
jected that UCLA students would be significantly happier. There is a conflict between “decision utility”—which is applied when deciding whether to move from Michigan, and which is based on a substantial increase in life satisfaction in California—and “experienced utility”—which is what is actually ultimately obtained once there. Schkade and Kahneman imply then that “decision utility” is defective.

Robson and Samuelson (2011) revisit these issues. They argue that utility should adapt, so that distinct decision and experienced utilit-
ies are evolutionarily optimal. Individuals cannot maximize expected utility exactly, but make choices that can only come close to maximum expected utility, as in Rayo and Becker (2005). Robson and Samuelson find no sense, however, in which either decision or experienced utility are defective, in contrast to Schkade and Kahneman.

A common feature of all previous models is that the time frame for adaptation is undefined.\(^{47}\) That is, adaptation to the distribution is shown to be optimal, but it is left open how such adaptation occurs. These papers might leave the impression that the utility function is set by evolution, which would be a glacially slow mechanism. It is crucial for most realistic applications that the time frame over which adaptation occurs be short. Even in the case of moving to California, hedonic adaptation would be a matter of a few years at most. Other applications would involve much more rapid adaptation, a matter of days, hours, minutes or less.

This difficulty we resolve here.

8. Conclusions

A key motivation here was to develop a model that is consistent with the burgeoning neuroscience evidence about how decisions are orchestrated in the brain. There is good evidence that economic decisions are made by a neural mechanism with hedonic underpinnings.

\(^{47}\)This observation also applies to Robson (2001), Rayo and Becker (2007), and Netzer (2009).
We present a model where the cardinal levels of hedonic utility shift in response to changing circumstances, as is also consistent with neuroscience. This adaptation acts to reduce the error caused by a limited ability to make fine distinctions, and is evolutionarily optimal.

There is no ultimate conflict with economics, however, since this limited ability is the only reason there are mistakes at all; as this ability improves, behavior converges to that implied by a standard economic model.

These neurological aspects of decision-making are empirical predictions, even if they are predictions of a type that is novel in economics. As neuroscientific evidence accumulates, this increases the demands on a theory—it must be consistent with this neurological evidence, as well as with more traditional evidence on demand behavior, for example.

In addition to the empirical contribution of reconciling the economic and neuroscience views of utility, the model generates predictions concerning observed behavior. The most straightforward of these is that individuals should make more mistakes over small stakes decisions when they are inured to larger stakes (and vice versa). Further, the trade-off between speed and accuracy generates observable consequences.

A key application of the model, however, is to prospect theory. The characteristic S-shape for utility is generated straightforwardly from plausible assumptions. Furthermore, the model provides a resolution of the puzzle that experiments evidence “too much” risk-aversion or risk-preference.

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9. **Appendix—Proofs**

**Proof of Theorem 3.2.**
Suppose that the (finite) Markov chain described by Equation (3.2) is represented by the matrix $A_G$.\footnote{See Mathematical Society of Japan, (1987, 260, p. 963), for example.} This is a $|S_G|$ by $|S_G|$ matrix which is irreducible, so that there exists an integer $P$ such that $(A_G)^P$ has only strictly positive entries.\footnote{Consider any initial configuration, $x^0$, say, and any desired final configuration, $x^T$, say. First move $x^0_t$ to $x^T_t$ by means of outcomes just to the right or left, as required, that do not affect any other thresholds. This might entail $x_1$ crossing the position of other thresholds, but temporarily suspend the usual convention of renumbering the thresholds, if so. This will take at most $G + 1$ periods. Then move $x^0_t$ to $x^T_t$ in an analogous way. And so on. There is a finite time, $(G + 1)N$, such that the probability of all this is positive, given the assumptions in Section 3.2.}

Consider an initial state $x^n_G \in S_G$ where $0 \leq x^n_1 \leq x^n_2 \leq \ldots \leq x^n_N \leq 1$. Consider then the random variable $x^{t+\Delta}_G$ that represents the state of the chain at $t + \Delta$, where $\Delta > 0$ is fixed, for the moment. Suppose there are $R$ iterations of the chain, where $R = \lfloor \Delta/\epsilon \rfloor$. These iterations arise at times $t + r\epsilon$ for $r = 1, \ldots, R$. Suppose the process is constant between iterations, so it is defined for all $t' \in [t, t+\Delta]$.

We consider the limit as $G \to \infty$. Taking this limit implies $\epsilon \to 0$, but also speeds up the process in a compensatory way, in that $R \to \infty$, making the limit non-trivial. This speeding up is only a technical device and has no effect on the invariant distribution, in particular.

We adopt the notational simplification that

$$H(x_n, x_{n+1}) = \xi(x_{n+1} - x_n)\beta(F(x_{n+1}) - F(x_n)), n = 0, \ldots, N.$$ 

Indeed, the key results here only rely on the properties that $H_1(x_n, x_{n+1}) < 0$ and $H_2(x_n, x_{n+1}) > 0$.

We have then that $x^{t+\Delta}_{n,G} = x^n_{n,G} + \sum_{r=1}^R \epsilon_r$ where

$$\epsilon_r = \begin{cases} 
    \epsilon & \text{with probability } H(x^{t+r\epsilon}_{n,G}, x^{t+r\epsilon}_{n+1,G}) \\
    -\epsilon & \text{with probability } H(x^{t+r\epsilon}_{n-1,G}, x^{t+r\epsilon}_{n,G}) \\
    0 & \text{otherwise}
\end{cases} \tag{9.1}$$
It follows that
\[ \frac{x_{n,G}^{t+\Delta} - x_{n,G}^t}{\Delta} = \sum_{r=1}^R \epsilon_r / \epsilon \frac{R(\Delta/(\epsilon R))}{}, \]

where \((\Delta/(\epsilon R)) \to 1\), as \(G \to \infty\).

Since
\[ x_{n,G}^{t+r\epsilon} \in [x_{n,G}^t - \Delta, x_{n,G}^t + \Delta], r = 1, ..., R, \]
it follows that
\[ \Pr\{\epsilon_r / \epsilon = 1\} \in [H(x_{n,G}^t + \Delta, x_{n,G}^{t+1,-\Delta}), H(x_{n,G}^t - \Delta, x_{n,G}^{t+1,\Delta})], r = 1, ..., R \]
and that
\[ \Pr\{\epsilon_r / \epsilon = -1\} \in [H(x_{n,G}^{t-1,-\Delta}, x_{n,G}^{t+1,-\Delta}), H(x_{n,G}^{t-1,\Delta}, x_{n,G}^{t+1,\Delta})], r = 1, ..., R, \]
with probability 1, in the limit as \(G \to \infty\), so that \(\epsilon \to 0\) and \(R \to \infty\).

Hence, if, finally, \(\Delta \to 0\), it follows that
\[ \frac{x_{n,G}^{t+\Delta} - x_{n,G}^t}{\Delta} \to H(x_n^t, x_{n+1}^t) - H(x_{n-1}^t, x_n^t), \]
with probability 1, so that, with probability 1—
\[ dx_n \frac{dt}{dt} = H(x_n, x_{n+1}) - H(x_{n-1}, x_n), n = 1, ..., N. \]

**Lemma 9.1.** There exist unique \(x_n^*, n = 1, ..., N\) such that \(dx_n^* = 0, n = 1, ..., N\).

**Proof.** Choose any \(x_1 > 0\). Then there exist \(x_2 < x_3 < ... < x_{N+1}\) such that \(H(0, x_1) = H(x_1, x_2) = ... = H(x_N, x_{N+1})\). Clearly \(x_n, n = 2, ..., N + 1\) are strictly increasing and continuous in \(x_1\) with \(x_{N+1} \to 0\) if \(x_1 \to 0\) and \(x_{N+1} \to \infty\) as \(x_1 \to \infty\). Hence there exists a unique \(x_1\) such that \(x_{N+1} = 1\). This generates the \(x_n^*, n = 1, ..., N\) as claimed in Theorem 3.2.

**Proposition 9.1.** The differential equation system given by Equation (9.2) is globally asymptotically stable. That is, given any initial \(x(0)\) where \(0 \leq x_1(0) \leq x_2(0) \leq ... \leq x_N(0) \leq 1\), it follows that \(x(t) \to x^*\) as \(t \to \infty\).

---

50This expression is valid even if there are ties so that \(x_n = x_{n+1}\), for example. In this case, \(x_n\) and \(x_{n+1}\) immediately split apart, relying on the convention that \(x_n \leq x_{n+1}\).
**Proof.** The proof proceeds by finding a Lyapunov function. Reversing the usual order of the thresholds, for expositional clarity, the second derivatives are given by

\[
\frac{d^2 x_N}{dt^2} = (H^N_1 - H^{N-1}_2) \frac{dx_N}{dt} - H^{N-1}_1 \frac{dx_{N-1}}{dt}, \ldots,
\]

(9.3) \[
\frac{d^2 x_n}{dt^2} = H^2_n \frac{dx_{n+1}}{dt} + (H^n_1 - H^{n-1}_2) \frac{dx_n}{dt} - H^{n-1}_1 \frac{dx_{n-1}}{dt}, \ldots,
\]

\[
\frac{d^2 x_1}{dt^2} = H^2_1 \frac{dx_2}{dt} + (H^1_1 - H^0_2) \frac{dx_1}{dt},
\]

where \( H^n_1 = H_1(x_n, x_{n+1}) < 0 \) and \( H^n_2 = H_2(x_n, x_{n+1}) > 0 \), for compactness of notation.

Shifting to vector notation and using “dot” notation for derivatives, for further compactness, Equations (9.2) and (9.3) can be written as

\[
\dot{x} = D(x), \text{ and } \ddot{x} = E(x)\dot{x} \text{ respectively,}
\]

where vectors are by default column vectors and “\( T \)” denotes transpose so that, for example, \( x^T = (x_N, x_{N-1}, \ldots, x_1) \).

The vector \( D(x) \) is implied by Equation (9.2); the matrix \( E(x) \) is given as follows—

\[
E(x) = \begin{pmatrix}
H^N_1 - H^{N-1}_2 & -H^{N-1}_1 & 0 & \cdots \\
H^{N-1}_2 & H^{N-2}_1 & -H^{N-2}_1 & \cdots \\
0 & H^{N-2}_2 & H^{N-3}_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Define \( B_n, n = 1, \ldots, N \) as the \( n \)-th principal nested minor of \( E(x) \), where these minors are defined relative to the lower right corner of \( E(x) \).

**Lemma 9.2.** The matrix \( E(x) \) is negative definite because the sign of \( B_n \) is \((-1)^n\) for \( n = 1, 2, \ldots, N \).

---

**Proof.** From the definition of $B_n$, it follows that

$$B_n = (H_1^n - H_2^{n-1})B_{n-1} + H_1^{n-1}H_2^{n-1}B_{n-2},$$

so that, rearranging,

$$B_n - H_1^n B_{n-1} = -H_2^{n-1}(B_{n-1} - H_1^{n-1}B_{n-2}).$$

It also follows that $B_1 = H_1^1 - H_2^0 < 0$ and $B_2 = H_1^2 B_1 = H_2^1 H_2^0 > 0$. Hence the sign of $B_n - H_1^n B_{n-1}$ is $(-1)^n$.

Suppose, as an induction hypothesis, that the sign of $B_{n-1}$ is $(-1)^{n-1}$. Since $B_n = (B_n - H_1^n B_{n-1}) + H_1^n B_{n-1}$, it follows that the sign of $B_n$ is $(-1)^n$, as required to complete the proof of Lemma 9.2.

Global asymptotic stability now follows. Define a Lyapunov function as

$$V(x) = \dot{x}^T \dot{x} = D(x)^T D(x),$$

so that $V(x) \geq 0$ and $V(x) = 0$ iff $x = x^*$. Hence, since $E(x)$ is negative definite,

$$\dot{V} = 2\dot{x}^T \dot{x} = 2\dot{x}^T E(x) \dot{x} \leq 0, \text{ and } \dot{V} = 0 \text{ iff } x = x^*.$$  

It follows that the ordinary differential equation system given by Equation (9.2) is globally asymptotically stable. That is, given any initial $x(0)$ where $0 \leq x_1(0) \leq x_2(0) \ldots \leq x_N(0) \leq 1$, it must be that $x(t) \to x^*$ as $t \to \infty$. This completes the proof of Proposition 9.1.

We can now complete the proof of Theorem 3.2. Suppose that $F_G$ is the cdf representing the unique invariant distribution of the Markov chain with transition matrix $A_G$. Extend $F_G$ to be defined on the entire space $S$. By compactness, it follows that there exists a subsequence of the $F_G$ that converges weakly to a cdf $F$ defined on $S$ (Billingsley, 1968, Chapter 1). That is, $F_G \Rightarrow F$ as $G \to \infty$. We will show that $F$ puts full weight on the singleton $x^*$. Once this is shown, it follows that the full sequence must also converge to this $F$.

Suppose, then by way of contradiction, that $F$ does not put full weight on $x^*$, so that $\int V(\bar{x})dF(\bar{x}) > 0$.  

Reconsider then the construction that led to the differential equation system that approximates the Markov chain, as described from the beginning of this Appendix. Choose any $\bar{x} \in S$, where $\bar{x} \neq x^*$ and $0 \leq \bar{x}_1 < \bar{x}_2 < ... < \bar{x}_N \leq 1$. Now let $\bar{x}_G$ be any of the points in $S_G$ that are closest to $\bar{x}$. Let $x_G^\Delta(\bar{x})$ be the random variable describing the Markov chain at $t = \Delta$ that starts at $\bar{x}_G$ at $t = 0$. Consider now the limit as $G \to \infty$, so that the number of repetitions in the fixed time $\Delta$, given by $R = \lfloor \Delta/\epsilon \rfloor$ also tends to infinity. Suppose $x^\Delta(\bar{x})$ is the solution to Equation (9.2), that is, to $\dot{x} = D(x)$, at $t = \Delta$, given it has initial value $\bar{x}$ at $t = 0$.

Given that $\bar{x} \neq x^*$ and $0 \leq \bar{x}_1 < \bar{x}_2 < ... < \bar{x}_N \leq 1$, it follows that $V(x^\Delta(\bar{x})) < V(\bar{x})$, since we showed that $\dot{V}(x) < 0$ on $[0, \Delta]$. By hypothesis, $\int V(\bar{x})dF(\bar{x}) > 0$. It follows that

$$\int V(x^\Delta(\bar{x}))dF(\bar{x}) < \int V(\bar{x})dF(\bar{x}).$$

That this inequality holds in the limit implies that it must hold for large enough $G$, as follows.

First, the derivation of the approximating system $\dot{x} = D(x)$ implies, in particular, that

$$EV(x_G^\Delta(\bar{x})) \to V(x^\Delta(\bar{x})) \text{ as } G \to \infty.$$  

(9.6)

It now follows that

$$|\int EV(x_G^\Delta(\bar{x}))dF_G(\bar{x}) - \int V(x^\Delta(\bar{x}))dF(\bar{x})| \leq$$

$$|\int EV(x_G^\Delta(\bar{x}))dF_G(\bar{x}) - \int V(x^\Delta(\bar{x}))dF_G(\bar{x})| +$$

$$|\int V(x^\Delta(\bar{x}))dF_G(\bar{x}) - \int V(x^\Delta(\bar{x}))dF(\bar{x})|.$$
$F_G \Rightarrow F$ and the integrand is continuous. Hence

$\int EV(x_G^\Delta(\bar{x}))dF_G \to \int V(x^\Delta(\bar{x}))dF$, as $G \to \infty$.

Secondly, since $F_G \Rightarrow F$, as $G \to \infty$, and $V$ is continuous, it follows that

$\int V(\bar{x})dF_G(\bar{x}) \to \int V(\bar{x})dF(\bar{x})$.

Altogether, then Equations (9.5), (9.7) and (9.8) imply that, whenever $G$ is sufficiently large

$\int EV(x_G^\Delta(\bar{x}))dF_G(\bar{x}) < \int V(\bar{x})dF_G(\bar{x})$,

which is a contradiction, since $F_G$ is the invariant distribution.

To show this explicitly, revert to matrix notation for the finite Markov chain with transition matrix $A_G$. Suppose then that $f_G$ is the column vector describing the associated invariant distribution, so that $f_G^T = f_G^T A_G$. As before, let $R = \lfloor \Delta/\epsilon \rfloor$. We have

$EV(x_G^\Delta(\bar{x})) = \sum_{x \in S_G} e(\bar{x})(A_G)^R(x)V(x),$ 

where $e(\bar{x})$ is the unit vector that assigns 1 to $\bar{x} \in S_G$ and 0 to all other elements of $S_G$. It follows that Equation (9.9) becomes

$\sum_{x \in S_G} f_G^T(A_G)^R(x)V(x) < \sum_{x \in S_G} f_G^T(x)V(x),$ 

which is a contradiction, since $f_G^T(A_G)^R = f_G^T$. This completes the proof of Theorem 3.2.

**Proof of Theorem 3.3.**

**Lemma 9.3.** Consider a uniform distribution with pdf $1/s$ on the interval $[0, s]$. The loss from choosing at random relative to the full information ideal is $s/6$.

The expected payoff from choosing randomly between the two arms is clearly $s/2$. The expected payoff from choosing the higher of the two
arms, on the other hand, as would be the full information ideal, is \(2s/3\).

To see this, suppose

\[
K(y) = \Pr\{\max\{y^1, y^2\} < y\} = \Pr\{y^1 \& y^2 < y\} = (y/s)^2.
\]

Hence \(\int_0^s ydK(y) = 2s/3\). It follows that the expected loss from choosing at random is \(s/6\), proving Lemma 9.3.

Define now the expected fitness loss, relative to the full information ideal, for the step function pdf as in the statement of Theorem 3.4, to be \(\hat{L}(\bar{N})\). It follows that

\[
(9.10) \quad \hat{L} = (1/6) \sum_{m=1}^{M} (n_m - 1)s_m(\alpha_ms_m)^2 + \sum_{m=1}^{M} d_m.
\]

Here, \(n_m\) is the number of thresholds that lie in the subinterval \([(m - 1)/M, m/M]\), which must be evenly spaced apart with a distance \(s_m\) between them, except at the ends of the subinterval. In the intervals that overlap \(m/M\), the expected loss is \(d_m\), say.

This expression for \(\hat{L}(\bar{N})\) holds because the loss between each pair of thresholds in \([(m - 1)/M, m/M]\) is \(s_m/6\), conditional on both outcomes being in that range, there are \(n_m - 1\) such ranges, and the probability of both outcomes lying in each range is \((\alpha_ms_m)^2\).

The limiting equilibrium of the rule of thumb entails

\[
H(x_{n-1}, x_n) = H(x_n, x_{n+1}) = H(x_{n+1}, x_{n+2}).
\]

If \(x_n \leq (m/M) < x_{n+1}\) it follows that \(H(x_{n-1}, x_n) = H(x_{n+1}, x_{n+2})\), so that

\[
\alpha_m(s_m)^{1+\beta} = \alpha_{m+1}(s_{m+1})^{1+\beta},
\]

since \(s_m = x_n - x_{n-1}\) and \(s_{m+1} = x_{n+2} - x_{n+1}\). It follows that there exists \(\lambda\) such that

\[
s_m = \lambda(\alpha_m)^{-1/(1+\beta)}, m = 1, ..., M.
\]

It also follows that

\[
d_m \leq (1/6)(\bar{\alpha})^2(\bar{s})^3, \text{ where } \bar{\alpha} = \max_m \alpha_m \text{ and } \bar{s} = 2 \max_m s_m.
\]
Furthermore,
\[(n_m-1)s_m \leq 1/M\] and, since \((n_m+1)s_m \geq 1/M, (n_m-1)s_m \geq 1/M-2s_m\]
Each value of \(N\) induces a corresponding value of \(\lambda\), further, \(\lambda \to 0\) as \(N \to \infty\).

The foregoing implies that
\[
\hat{L} \geq (1/6) \sum_{m=1}^{M} (1/M-2s_m)(\alpha_m s_m)^2 \text{ and } \hat{L} \leq (1/6) \sum_{m=1}^{M} (1/M)(\alpha_m s_m)^2 + (M/6)s^3 \bar{\alpha}^2.
\]

There exists \(\eta\) such that \(\bar{s} \leq \eta \lambda\). Since it is also true that \(s_m = \lambda (\alpha_m)^{-1/(1+\beta)}, m = 1, \ldots, M\), it follows that
\[
\frac{\hat{L}}{\lambda^2} \to \sum_{m=1}^{M} \alpha_m^{2\beta/(1+\beta)} / (6M) \text{ as } \lambda \to 0.
\]

Furthermore, since \((n_m - 1)s_m \in [1/M - 2s_m, 1/M]\), it follows that
\[
n_m \leq \alpha_m^{1/(1+\beta)}/(M\lambda) + 1 \text{ and } n_m \geq \alpha_m^{1/(1+\beta)}/(M\lambda) - 1.
\]

Since \(\sum_m n_m = N\), it follows that
\[
N\lambda \in \left[\sum_m \alpha_m^{1/(1+\beta)}/M - M\lambda, \sum_m \alpha_m^{1/(1+\beta)}/M + M\lambda\right].
\]
Thus
\[
N\lambda \to \sum_m \alpha_m^{1/(1+\beta)}/M \text{ as } \lambda \to 0.
\]

Hence
\[
N^2 \hat{L} \to \sum_m \alpha_m^{2\beta/(1+\beta)} (\sum_m \alpha_m^{1/(1+\beta)})^2 / (6M^3) \text{ as } N \to \infty.
\]

**Lemma 9.4.** The expression \(\sum_m \alpha_m^{2\beta/(1+\beta)} (\sum_m \alpha_m^{1/(1+\beta)})^2\) is minimized uniquely by choice of \(\beta = 1/2\).

This follows from the Hölder Inequality (Royden, 1988, p. 119) since
\[
\sum_m \alpha_m^{2\beta/(3(1+\beta))} \alpha_m^{2/(3(1+\beta))} = \sum_m \alpha_m^{2/3} \leq \left(\sum_m \alpha_m^{2\beta/(1+\beta)}\right)^{1/3}\left(\sum_m \alpha_m^{1/(1+\beta)}\right)^{2/3}.\]

Furthermore, equality can only hold here if \(\alpha_m^{2\beta/(1+\beta)} = \alpha_m^{1/(1+\beta)}\); that is, only if \(\beta = 1/2\).
It follows that, when $\beta = 1/2$,

$$N^2 \hat{L} \to \left( \sum a_m^{2/3} \right)^3 / (6M^3).$$

This completes the proof of Theorem 3.3

**Proof of Theorem 3.4**

We will now show that the optimal rule has the same leading term as the rule of thumb with $\beta = 1/2$. Suppose that the expected loss from the optimal placement of the thresholds, relative to the full information ideal, is given by $L^*$. An entirely similar argument to that used for $\hat{L}$ shows, upon multiplying by $N^2$, that

$$N^2 L^* \in \left\{ (1/6) \sum_{m=1}^{M} (1/M - 2s_m)(\alpha_m N s_m)^2, (1/6) \sum_{m=1}^{M} (\alpha_m N s_m)^2 + (M N^2/6) \bar{s}^3 \tilde{\alpha}^2 \right\}.$$

Consider the vector $(n_1/N, ..., n_m/N, ..., n_M/M)$. By compactness, there must exist a convergent subsequence such that $n_m/N \to \gamma_m, N \to \infty$. We will characterize the $\gamma_m$ uniquely, so that the entire sequence must then converge to these values as well.

It must be that $s_m \to 0$ for all $m$, as $N \to \infty$, since otherwise it would not be true that $L^* \to 0$, contradicting the optimality of $L^*$. It follows from $(n_m - 1)s_m \leq 1/M$ and $(n_m - 1)s_m \geq 1/M - 2s_m$ that $n_ms_m \to 1/M$.

If $\gamma_m = 0$, it follows from $(n_m/N)(N s_m) \geq 1/M - s_m$ that $N s_m \to \infty$. This implies that $N^2 L^* \to \infty$ which is not optimal. Hence we have $N s_m \to 1/(M \gamma_m)$.

It follows now that $N \bar{s}$ is bounded above and that $\bar{s} \leq 2 \max_m s_m \to 0$, as $G \to \infty$. Hence

$$N^2 L^* \to \left( \sum_m \alpha_m^2 / \gamma_m^2 \right) / (6M^3).$$

The optimal rule must minimize this expression over the choice of the $\gamma_m \geq 0, m = 1, ..., M$ where $\sum_m \gamma_m = 1$. Since this function is convex in the $\gamma_m \geq 0, m = 1, ..., M$ the first-order conditions are necessary and
sufficient for a global minimum. There must then exist a $\lambda$ such that $\alpha_m^2/\gamma_m^3 = \lambda^3$ so $\gamma_m = \alpha_m^{2/3}/\lambda$. It follows that $\lambda = \sum_m \alpha_m^{2/3}$. Hence

$$N^2 L^* \to \left( \sum \alpha_m^{2/3} \right)^3/(6 M^3).$$

This completes the proof of Theorem 3.4.

**Proof of Lemma 6.1** Given the symmetry, there is symmetrical optimal allocation of thresholds. That is, there exists a mirror image to any interval $[x_n, x_{n+1})$, given by $[1 - x_{n+1}, 1 - x_n) = [x_{N-n}, x_{N-n+1})$. Utility is $U(x) = n\delta$ for all $x \in [x_n, x_{n+1})$. Since $N$ is even, $1/2 \in [x_{N/2}, x_{N/2+1})$. Further, if $y^1 \in [x_n, x_{n+1})$ and $y^2 \in [x_{N-n}, x_{N-n+1})$, then the expected utility of the gamble is $(1/2)n\delta + (1/2)(N - n)\delta = (1/2)N\delta$. Hence the individual is indifferent between the certain outcome and any gamble of this form and so is wrong 50% of the time.

In all other cases, where $y^1$ and $y^2$ do not fall into symmetrically located intervals, there is no tie in expected utility for the individual and the option that maximizes expected utility also maximizes expected fitness.

For convenience, let $t_n = F(x_n)$, for $n = 1, ..., N$. It follows that the overall probability of error is

$$P(E) = (1/2)^2(1-t_1)^2+(1/2)^2(1-t_1)^2+...+(1/2)^2(t_{N/2} - t_{(N/2)-1})^2+(1/2)(t_{(N/2)+1} - t_{N/2})^2.$$

Since $t_{(N/2)+1} = 1 - t_{N/2}$ it follows that $t_{(N/2)+1} - t_{N/2} = 1 - 2t_{N/2}$.

The necessary and sufficient conditions for minimizing $P(E)$ over choice of $t_n$ for $n = 1, ..., N/2$ are then

$$t_2 - t_1 = t_1, t_3 - t_2 = t_1, ..., 1/2 - t_{N/2} = t_1/2.$$  

It follows that $t_1 = 1/(N + 1)$ so that $t_n = F(x_n) = n/(N + 1)$ for $n = 1, ..., N$, as asserted.

**Proof of Lemma 6.2**

For any thresholds $0 < x_1 < x_2 < ... < x_N < 1$ the individual maximizing expected utility generates random fitness, $w$, as follows. Whenever

$$z \in [x_m, x_{m+1}), y^1 \in [x_{n_1}, x_{n_1+1}), \text{ and } y^2 \in [x_{n_2}, x_{n_2+1}),$$
then

\[
  w = \begin{cases} 
    z & \text{if } m > (1/2)n_1 + (1/2)n_2 \\
    (1/2)y^1 + (1/2)y^2 & \text{if } m < (1/2)n_1 + (1/2)n_2 \\
    (1/2)z + (1/4)y^1 + (1/4)y^2 & \text{if } m = (1/2)n_1 + (1/2)n_2 
  \end{cases}.
\]

Each allocation of thresholds thus generates expected fitness \( E(w) \). Since the set of thresholds \((x_1, ..., x_N) \in [0, 1]^N\) and \( E(w) \) is continuous in \((x_1, ..., x_N)\), optimal thresholds exist.

Suppose that \( h_N \) is the associated optimal \( h \) function for \( N \). This is also the utility function arising here. The \( h \) functions are essentially cdf’s on \([0, 1]\), and so belong to a compact set. There must then be a subsequence of \( N \) and a \( h^* \) such that \( h_N \Rightarrow h^* \), as \( N \to \infty \) (Billingsley, 1968, Chapter 1). The function \( h^* \), the limiting utility function, is non-decreasing. It is also continuous except for, at most, a countable number of discontinuities.

We show that \( h^*(x) = x \) for all \( x \in [0, 1] \).\(^{52}\) First note the following. Consider thresholds that are equally spaced in \( x \) for all finite \( N \). In the limit as \( N \to \infty \), this yields first-best maximum expected fitness. (That is, the gamble is taken if and only if its expected fitness exceeds that of the certain outcome.) In the limit, the optimal \( h_N \) must also then attain this level.

Suppose now that \( h^* \) jumps up at \( \bar{x} \), for example. Since the left limit and the right limit still exist, with the former strictly less than the latter, there exists a set of positive measure of \((z, y^1, y^2)\) such that the gamble is taken despite having a lower mean fitness than the certain outcome.

Hence \( h^* \) must be continuous. Unless \( h^*(x) = x \) for all \( x \in [0, 1] \), however, it again induces strictly risk-averse or strictly risk-preferring choices with positive probability. This is a contradiction, unless \( h^*(x) = x \) for all \( x \in [0, 1] \).

**Proof of Lemma 6.3**

\(^{52}\)It then follows that \( h_N \Rightarrow h^* \) for the entire sequence of \( h_N \).
Consider any thresholds $0 < x_1 < x_2 < \ldots < x_N < 1$ and associated scaled thresholds $0 < x_{1,k} < x_{2,k} < \ldots < x_{N,k} < 1$, where $x_{n,k} - y_0 = (x_n - y_0)/k$, for $n = 1, \ldots, N$. It follows that each outcome of fitness $w$ under $F$ and $G$ and the $\{x_n\}_{n=1}^N$, defined as in the proof of Lemma 6.2 above, corresponds to an equally likely outcome $w_k = y_0 + \frac{w-y_0}{k}$ under $F^k$ and $G^k$ and the $\{x_{n,k}\}_{n=1}^N$. It follows that

$$E(w_k) = y_0 + \frac{E(w) - y_0}{k},$$

where $E(w)$ is the expected fitness under $F$ and $G$ generated by the $\{x_n\}_{n=1}^N$ and $E(w_k)$ is the expected fitness under $F^k$ and $G^k$ generated by the $\{x_{n,k}\}_{n=1}^N$.

Hence the thresholds $\{x_n^*\}_{n=1}^N$ maximize $E(w)$ if and only if $x_{n,k}^* = y_0 + (x_n^* - y_0)/k$ for $n = 1, \ldots, N$ maximize $E_k(w)$. Thus the utility function derived from $\{x_n^*\}_{n=1}^N$ scales as $U_k(y) = U(y_0 + k(y - y_0))$, where $U$ derives from $\{x_n^*\}_{n=1}^N$.

It is immediate by a change of variable that $\int U_k(y) dP^k(y) = \int U(y) dP(y)$, so that the $U^k$ ranks the test gambles $P^k$ exactly as $U$ ranks the test gambles $P$.

**REFERENCES**


53Finding a suitable rule of thumb to implement this optimum is an open question.


