

Evolution of Time Preference by Natural Selection: Comment

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AN EXTENDED EXAMPLE

This supplement reanalyzes the published example under more general conditions. The basic philosophy adopted here is that individuals may transfer any amount to living relatives and individuals may save any amount they wish. Individuals cannot borrow, however. This is consistent with a stylized version of a primitive agricultural society in which savings can be made by means of reinvesting the harvest, but where borrowing is circumscribed. Although there are then many other transfers and savings that might be made in principle, we show below that all such additional options may optimally be chosen to be zero. Included amongst these additional options are the various “same age transfers.” For the sake of readability, the present treatment is largely self-contained.

0.1 The Example

Individuals are born at age 0 and live to at most to age 3. Let $P_i(c_i)$ denote the survival probability from age i to age $i + 1$, when consumption is c_i , for $i = 0, 1, 2$. In particular, it is assumed that $P_0 \equiv 1$, so that individuals surely survive until the age of one. The functions P_1 and P_2 are assumed to be continuously differentiable, strictly increasing, and strictly concave, and to have infinite derivatives at 0. This last condition favors transfers and savings.

At each age, $i = 1, 2, 3$, each individual is paired with another individual of the same age and the couple has two offspring. The various transfers and savings decisions that must be made are controlled by a single gene, with each possible combined decision implemented by a corresponding variety (or allele) of that gene. Sexual reproduction means that each offspring acquires either the paternal allele or the maternal allele, each with probability $1/2$.

Individuals have zero endowment at ages $i = 0, 1, 3$ but have one unit of endowment at age 2. At age 2, the individual can transfer part of her endowment, in the total amount t_0 , to her two newborn offspring. The transfer from each parent is shared equally between each of the two offspring, since it is not possible for a parent to distinguish between mutant

and non-mutant offspring. The new born offspring will choose to save this transfer, since survival for one period is guaranteed. It is assumed that the technology for intertemporal transformation is linear, with an exogenous interest factor of R . It follows that each of these offspring then receives $Rt_0/2$ at age of one, as the return from a particular parent's transfer, which they are assumed to consume at once.

Each surviving two year old parent also gives up an amount t_1 which is pooled to pay all of the one year old offspring who had one year old parents at birth. These one year old offspring consume the transfer in the current period. It is assumed that one year old parents are offered a deal where a transfer will be made to their current new born in the next period whether or not the parent is alive then. If the parent is alive, then he/she must give up the amount t_1 . This deal lumps together all one year old offspring regardless of the age of their parents. Although this assumption is not particularly realistic, it is made in the interests of simplicity, since it implies that all offspring of one year old parents remain identical. The alternative would be to allow a proliferation of types of offspring of one year old parents, depending on the survival of neither, one, the other or both of their parents. This would introduce complications that are irrelevant to the present purpose.

Finally, we assume that each two year old parent saves an amount s until the next period, when it is transferred to the current new born, who saves it for another period.

The first question is: What values of t_0 , t_1 and s would arise in evolutionary equilibrium? Consider a population with an allele that selects \bar{t}_0 , \bar{t}_1 , and \bar{s} , respectively. A small proportion of the population is then replaced by a *mutant* allele that selects t_0 , t_1 and s . The question becomes: For what values of \bar{t}_0 , \bar{t}_1 , and \bar{s} is it true that, no matter what t_0 , t_1 and s are, the mutant allele grows no faster than the original allele?

0.2 Derivation of Optimal Positive Transfers and Saving

The following survival probabilities obtain. The survival probability to two years of a mutant one year old who had two year old parents is given by $P_1^2 = P_1(R\frac{\bar{t}_0+t_0}{2})$. Since the mutant gene is rare, essentially all mutant individuals with two year old parents have one mutant and one non-mutant parent. Each such couple, furthermore, can be assumed to have one mutant and one non-mutant offspring, under haploid sex. Neither type of parent can tell which is which, however, and the total resources, $\bar{t}_0 + t_0$, are shared equally between the two offspring.

The survival probability of a mutant one year old who had three year old parents is

then $P_1^3 = P_1(RR' \frac{\bar{s}+s}{2})$. The new feature in this case is that R' is the interest rate paid to two year olds who survive to three years old. It is assumed this interest rate is actuarially fair, so that $R'P_2 = R$.

Suppose now that $\mu_i \geq 0$, for $i = 1, 2, 3$, is the proportion of i -year-olds in the mutant gene pool, ignoring newborns, so that $\sum_{i=1}^3 \mu_i = 1$. These proportions need not match those in the general population. Thus $F = [\mu_1 P_1^1 + \mu_2 P_1^2 + \mu_3 P_1^3]$ is the overall probability of survival of a mutant individual from age one to two. Define \bar{F} in strict analogy as the overall probability of survival of individuals from age one to age two in the original population.

The survival probability of a mutant one year who had one year parents is then $P_1^1 = P_1(\bar{F} \frac{\bar{t}_1+t_1}{2})$. The introduction of this factor \bar{F} means that the payments by all surviving two year olds parents balance the receipts by one year olds, as discussed above.

Finally, the survival probability of a mutant two year old to three years old is $P_2 = P_2(1 - t_0 - t_1 - s)$.

If y_t is the number of one year old mutant individuals at date t , it has equation of motion:

$$y_t = y_{t-1} + y_{t-2}F + y_{t-3}P_2F, \quad (1)$$

As before, it is straightforward to show that the mutant population converges to steady state growth with growth factor g satisfying

$$g^3 = g^2 + (g + P_2)F. \quad (2)$$

Since

$$\mu_1 = \frac{1}{g}, \quad \mu_3 = \mu_2 \frac{P_2}{g}, \quad \text{and} \quad \mu_1 + \mu_2 + \mu_3 = 1,$$

it follows that the steady state population proportions are given by

$$\mu_1 = \frac{1}{g}; \quad \mu_2 = \frac{g-1}{g+P_2}, \quad \text{and} \quad \mu_3 = \frac{P_2}{g} \frac{g-1}{g+P_2}. \quad (3)$$

Upon substituting these values for the μ_i 's into (2), we obtain the following equation implicitly determining the growth factor of the mutant population:

$$g^4 = g^3 + (g + P_2)P_1^1 + g(g-1)P_1^2 + (g-1)P_1^3P_2. \quad (4)$$

It is straightforward to show that g as determined by this equation is maximized if and only if the right hand side is maximized over choice of t_0 , t_1 and s for parametric choice of

g. Assuming an interior solution to this maximization problem, the first-order conditions are

$$\begin{aligned}
(P_1^1 + (g-1)P_1^3)P_2' &= (g+P_2)\frac{\bar{F}}{2}P_1^{1'} & (5) \\
&= g(g-1)\frac{R}{2}P_1^{2'} \\
&= (g-1)P_2P_1^{3'}\frac{RR'}{2} = (g-1)P_1^{3'}\frac{R^2}{2}.
\end{aligned}$$

This characterizes the choices of t_0 , t_1 and s that yield the fastest growth rate of the mutant allele, when the mutant alleles are a small fraction of total population of alleles, where most of these choose \bar{t}_0 , \bar{t}_1 and \bar{s} . If $t_0 = \bar{t}_0$, $t_1 = \bar{t}_1$ and $s = \bar{s}$, then the growth rate of the mutant alleles must match that of the original type. Hence, if the maximum of (4) occurs anywhere except at $t_0 = \bar{t}_0$, $t_1 = \bar{t}_1$ and $s = \bar{s}$, the mutants can invade in the sense of initially growing faster than the original population.

0.3 Reproductive Value

The optimum of the previous subsection derives from maximizing the appropriate generalized reproductive value. This value has the form—

$$\bar{F}P_2\left(\frac{v(0_1)}{g} + \frac{P_1^3v(2)}{g^2}\right) + P_1^1v(2) + \frac{\bar{F}P_1^2v(2)}{g}, \quad (6)$$

for appropriate values of two year olds, $v(2)$, and of new borns with one year old parents, $v(0_1)$. These appropriate reproductive values have a concrete interpretation as the relative shares attained in the population, in the limit as the time into the future tends to infinity.

This maximand represents the relevant component of intertemporal preferences for a two year old parent. It has an intuitively appealing interpretation as the “inclusive fitness” of the parent—the expected direct reproductive value of the parent, as influenced by the choice of t_0 , t_1 and s , plus the expected discounted reproductive value of recipients, as also influenced by the choice of t_0 , t_1 and s . These reproductive values generate the utility that underpins the evolutionary equilibrium.

The appropriate perspective for this problem is when the parents are one year old. In particular, one year old parents are offered a deal where a transfer will be made to their current new born in the next period whether or not the parent is alive then. If the parent is alive, then he/she must give up the amount \bar{t}_1 or t_1 . The amount that each one year old offspring will receive from that particular parent is then $\bar{F}\bar{t}_1/2$ or $\bar{F}t_1/2$. The simplifying role of this assumption is discussed above.

The first-order conditions for (6) are

$$P_2' \left(\frac{v(0_1)}{g} + \frac{P_1^3 v(2)}{g^2} \right) = v(2) \frac{1}{2} P_1^{1'} = v(2) \frac{R}{2g} P_1^{2'} = v(2) \frac{R^2}{2g^2} P_1^{3'}.$$

It is desired to show that these conditions are equivalent to (5). The equivalence of the third equality is immediate; that of the second follows from (2). It is sufficient then to show that

$$\frac{v(3)}{v(2)} = \frac{\left(\frac{v(0_1)}{g} + \frac{P_1^3 v(2)}{g^2} \right)}{v(2)} = \frac{(P_1^1 + (g-1)P_1^3)}{g^2(g-1)}.$$

A heuristic approach to finding the correct reproductive values that establish this is as follows. Consider types 0_1 , 0_2 and 0_3 as newborns born to parents of age 1, 2, and 3, respectively. Similarly, types 1_1 , 1_2 and 1_3 are the one year old offspring of parents who were originally of age 1, 2, and 3, respectively. It is only necessary to consider one type, 2, of two year olds and one type, 3, of three year olds. Using $v(\tau)$ to denote the reproductive value of any such type τ , these reproductive values satisfy the following equations—

$$\begin{aligned} v(0_1) &= \frac{v(1_1)}{g}, v(0_2) = \frac{v(1_2)}{g}, v(0_3) = \frac{v(1_3)}{g} \\ v(1_1) &= v(0_1) + \frac{P_1^1 v(2)}{g}, v(1_2) = v(0_1) + \frac{P_1^2 v(2)}{g}, v(1_3) = v(0_1) + \frac{P_1^3 v(2)}{g} \\ v(2) &= v(0_2) + \frac{P_2 v(3)}{g}, v(3) = v(0_3). \end{aligned}$$

It follows that

$$v(3) = \frac{v(1_3)}{g} = \frac{v(0_1)}{g} + \frac{P_1^3 v(2)}{g^2}.$$

Further,

$$v(0_1) = \frac{v(0_1)}{g} + \frac{P_1^1 v(2)}{g^2},$$

so

$$v(0_1) = \frac{P_1^1 v(2)}{g(g-1)}.$$

The desired result that

$$\frac{v(3)}{v(2)} = \frac{(P_1^1 + (g-1)P_1^3)}{g^2(g-1)}$$

then follows.

These reproductive values have a concrete and compelling interpretation as the relative shares attained in the limiting population. More precisely, these values are the right eigenvector of the matrix representing the evolution of the age-structured population,

where the corresponding eigenvalue is the limiting steady state growth factor, g . (The left eigenvector gives the limiting age distribution of the population.) This implies that these reproductive values are indeed the relative shares attained in the population, in the limit as the time into the future tends to infinity.

0.4 Rogers/Fisher Reproductive Value

The notion of reproductive value used by Rogers, due to R.A. Fisher, counts the future expected discounted total fertility of an individual at each age. In the present example, this would mean, using the notation w instead of v :

$$\begin{aligned} w(0_1) &= \frac{w(1_1)}{g} = \frac{1}{g} + \frac{P_1^1}{g^2} + \frac{P_1^1 P_2}{g^3}; & w(0_2) &= \frac{w(1_2)}{g} = \frac{1}{g} + \frac{P_1^2}{g^2} + \frac{P_1^2 P_2}{g^3}; \\ w(0_3) &= \frac{w(1_3)}{g} = \frac{1}{g} + \frac{P_1^3}{g^2} + \frac{P_1^3 P_2}{g^3}; & w(2) &= 1 + \frac{P_2}{g}; & w(3) &= 1. \end{aligned}$$

This implies that, if $R \neq g$

$$\frac{w(3)}{w(2)} = \frac{g}{g + P_2} \neq \frac{v(3)}{v(2)} = \frac{(P_1^1 + (g - 1)P_1^3)}{g^2(g - 1)}$$

since, using (4) and (5),

$$g^3(g - 1) - (g + P_2)(P_1^1 + (g - 1)P_1^3) = (P_1^2 - P_1^3)g(g - 1) \neq 0.$$

Thus the evolutionary equilibrium is not generally the solution to maximizing inclusive fitness derived from these reproductive values—

$$\max_{t_0, t_1, s} \bar{F}P_2 \left(\frac{w(0_1)}{g} + \frac{P_1^3 w(2)}{g^2} \right) + P_1^1 w(2) + \frac{\bar{F}P_1^2 w(2)}{g}.$$

Thus Rogers' formulation of the felicities that support the equilibrium level of savings is incorrect on this account.

0.5 Other Transfers Are Optimally Zero

We now show that the additional transfers and savings that are feasible are optimally chosen to be zero, despite the conditions imposed on the survival functions, given that $R \leq 2^{1/3}g$. All type 1 parents— 1_1 , 1_2 and 1_3 —now have resources available. Consider first the possibility that one year old parents make transfers to their new born offspring.

Consider first the 1_1 case. The new born offspring save all of any transfer and can be assumed to consume it when they are also 1_1 's. The relevant objective to consider the effect of a transfer of $u \geq 0$ to new born offspring is then

$$P_1(\bar{F}\bar{t}_1 - u)v(2) + \frac{P_1(\bar{F}\bar{t}_1 + \frac{Ru}{2})v(2)}{g}.$$

However, given that $R \leq 2g$, it follows that

$$-P_1^{1'}v(2) + \frac{RP_1^{1'}v(2)}{2g} \leq 0,$$

so the optimal $u = 0$. The donor and eventual recipient here are of both of age one, so this is an example of a “same age transfer” as in Rogers. But the conditions that $P_1^{i'}(0) = \infty$, for $i = 1, 2, 3$, do not force the transfer to be positive.

Consider now the 1_2 case, which is another example of a “same age transfer.” Their offspring are still 1_1 's. The relevant criterion is

$$P_1(R\bar{t}_0 - u)v(2) + \frac{P_1(\bar{F}\bar{t}_1 + \frac{Ru}{2})v(2)}{g}.$$

In order that $u = 0$ be optimal, the first-order condition is that

$$-P_1^{2'}v(2) + \frac{RP_1^{1'}v(2)}{2g} \leq 0.$$

It is sufficient for this that $R \leq \sqrt{2}g$, given $P_1^{1'} = \frac{R}{g}P_1^{2'}$.

Consider now transfers by 1_3 's, still another “same age transfer.” The relevant criterion is

$$P_1(RR'\bar{s} - u)v(2) + \frac{P_1(\bar{F}\bar{t}_1 + \frac{Ru}{2})v(2)}{g}.$$

For $u = 0$ to be optimal, it is sufficient that

$$-P_1^{3'}v(2) + \frac{R}{2g}P_1^{1'}v(2) \leq 0.$$

In the light of (5), it is then sufficient that $R \leq 2^{1/3}g$.

Another possibility is that the 1_1 's, 1_2 's or 1_3 's save. If 1_1 's, save the criterion is

$$P_1(\bar{F}\bar{t}_1 - u)v(2) + \frac{P_1(\bar{F}\bar{t}_1)P_2(1 - \bar{t}_0 - \bar{t}_1 - \bar{s} + R'u)v(3)}{g} \leq 0.$$

For $u = 0$ to be optimal, it is enough that

$$-P_1^{1'}v(2) + \frac{R'P_1^1P_2'v(3)}{g} \leq 0.$$

The possibility that a one year old dies before receiving the return on saving must be addressed here. We assume that the interest rate offered to the survivors, R' , is appropriately adjusted on this account, so that $R'P_1^1 = R$. Since, in addition, $P_2'v(3) = v(2)\frac{1}{2}P_1^{1'}$, and $R \leq 2g$, the required inequality follows. Thus, although 1_1 's receive contemporaneous transfers from their parents, these are not enough to induce savings. Such a transfer plus savings would amount to another "same age transfer" as in Rogers. Again, the conditions on the survival functions P_1^i , $i = 1, 2, 3$, do not force such a transfer to be positive.

Consider savings by 1_2 's. All these individuals received a transfer at birth from their two year old parents. If they saved this would complete another "same age transfer." The criterion is

$$P_1(R\bar{t}_0 - u)v(2) + \frac{P_1(R\bar{t}_0)P_2(1 - \bar{t}_0 - \bar{t}_1 - \bar{s} + R''u)v(3)}{g}.$$

For $u = 0$ to be the optimal level here, it is enough that

$$-P_1^{2'}v(2) + \frac{R''P_1^2P_2'v(3)}{g} \leq 0.$$

In this case, the interest rate offered to one year olds who survive to two, R'' , satisfies $R''P_1^2 = R$. Since it is also true that $P_2'v(3) = v(2)\frac{R}{2g}P_1^{2'}$, and $R \leq \sqrt{2}g$, the required inequality follows.

Consider now savings by 1_3 's. These individuals received a transfer from their three parents that was the return on savings from the previous period. If they save, this would complete a still more circuitous "same age transfer." The criterion is now

$$P_1(R'R\bar{s} - u)v(2) + \frac{P_1(R'R\bar{s})P_2(1 - \bar{t}_0 - \bar{t}_1 - \bar{s} + R'''u)v(3)}{g}.$$

For $u = 0$ to be optimal here, it is enough that

$$-P_1^{3'}v(2) + \frac{R'''P_1^3P_2'v(3)}{g} \leq 0.$$

Now the interest rate offered to one year olds who survive to two, R''' , satisfies $R'''P_1^3 = R$. Since it is also true that $v(3)P_2' = v(2)\frac{R^2}{2g^2}P_1^{3'}$, and $R \leq 2^{1/3}g$, the required inequality follows.

The income of age three parents gives them the option of transferring resources to offspring other than the new born. For a transfer to a current one year old, the criterion is

$$\frac{P_1(RR'\bar{s} - Ru/2)v(2)}{g^2} + \frac{P_1(R\bar{t}_0 + u/2)v(2)}{g}.$$

In order that $u = 0$ be optimal it is enough that

$$-\frac{R P_1^{3'} v(2)}{2 g^2} + \frac{P_1^{2'} v(2)}{2g} \leq 0.$$

However, (5) implies this expression is zero. In this case, there are two equivalent ways to transfer resources to current one year olds—giving them a transfer the period before and saving from the period before to make the transfer now.

Consider finally a transfer by an age three parent to a current two year old offspring. Considered from the point of view of the two year old parent in the period before, this constitutes a “same age transfer.” The criterion is

$$\frac{P_1(RR'\bar{s} - Ru/2)v(2)}{g^2} + \frac{P_2(1 - \bar{t}_0 - \bar{t}_1 - \bar{s} + u/2)v(3)}{g}.$$

In order that $u = 0$ be optimal it is enough that

$$-\frac{R P_1^{3'} v(2)}{2 g^2} + \frac{P_2^{2'} v(3)}{2g} \leq 0.$$

This condition is implied by (5) and $R \leq 2g$.

The above discussion covers all potential savings and direct transfers to children. There are still other transfers that are possible. For example, a two year parent could transfer resources to the the new born offspring of his/her one year old child. That is, a two year old parent could transfer resources to a grandchild. However, the optimal level of such a transfer is zero. Indeed, the optimal transfer to such a grandchild is zero *because* it is optimal for his/her one year old parent to make a zero transfer. That is, the MRS of the grandparent between resources for the child and resources for the grandchild is the same as the child’s MRS. Further, although a three year old parent could also transfer income to grandchildren or even to great grandchildren, this is not optimal, for similar reasons.

The intuitive reason that interior solutions may not exist for the same age transfers here is that the evolutionary criterion considers the net income position of all individuals of a given age. Such transfers appear both as a debit, and as a credit, the latter multiplied by the appropriate interest and growth factor, in the single argument of the same survival function.

Note that, although the growth rate, g , is endogenous, it is independent of R . For example, here it must be that $g \geq \underline{g}$ for some $\underline{g} > 1$, since all offspring survive to be one year old and have one offspring at that age, and have a positive probability of having further offspring. If the exogenous parameter $R \leq 2^{1/3}\underline{g}$, then $R \leq 2^{1/3}g$.

The discussion so far was based on a linear intertemporal technology. But even if this technology were strictly convex, so that corner solutions for total savings were indeed ruled out, it is clearly possible that the endogenous interest rate would be determined by strictly positive savings elsewhere, would satisfy $R \leq 2^{1/3}g$, and so would still choke off all “same age transfers.” There is then no robust prediction here concerning the real rate of interest, contrary to Rogers claim.