Appendix B for "The Evolution of Strategic Sophistication" (Intended for Online Publication)

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Appendix B: Proof of Theorem 2

This appendix contains the proof of Theorem 2 which is the main result for the general version of the model presented in Section II of the paper. The numbering here (of equations, result, and definitions, etc.) follows the numbering in the published paper. Assumptions 6-10, which are stated in the paper, are assumed implicitly throughout.

A single subscript will be used to denote the number of accumulated iterations of the game (as in the proof of Theorem 1 in Appendix A). For example, instead of writing $H_{n,t}$ for the period n, iteration t history, H_s will be used, where s is the total number of iterations of the game along the history. The notation $s(n)$ denotes $\sum_{m<n} \kappa(m) + 1$. Notice, in particular, that for each period $n = 1, 2, \ldots,$ iteration $s = s(n)$ corresponds to the arrival of the n-th novel outcome.

B1. ToPs Learn Preferences Whenever $\alpha > 1$.

A key step in the proof of Theorem 2 is to show that the *ToP*s learn their opponents' preferences completely in the limit whenever $\alpha > 1$. (Recall that α determines the number of iterations within in each period as in Assumption 8.) This sets the stage for the ultimate dominance of the *SR-ToP* strategy. It implies, in particular, that the *SR-ToP*s will eventually make the subgame perfect choice in every game. Although in general this choice is sub-optimal initially, it is the appropriate choice in the long run (this will be shown in a later section).

We begin by establishing what it means for a player with *ToP* to learn an opponent's preferences. With this in mind consider the following.

DEFINITION 9: *$\mathcal{Q}_n^i(\mathbf{z}, \mathbf{z}')$ is the set of i role subgames available in period n that satisfy the following. The subgame $\mathbf{q}$ is in $\mathcal{Q}_n^i(\mathbf{z}, \mathbf{z}')$ if and only if for two actions, say a, $a' \in A$, $\mathbf{z}$ is the unique subgame perfect equilibrium outcome of the continuation game following i's choice of a, $\mathbf{z}'$ is the unique equilibrium outcome of the subgame following i's choice of a', and one of the actions a, a' is strictly dominant for role i.*

Say that H_s *reveals* role I's preference over the (unordered) pair of outcomes $(\mathbf{z}, \mathbf{z}')$ if a subgame $\mathbf{q} \in \mathcal{Q}_n^I(\mathbf{z}, \mathbf{z}')$ was reached along H_s.[35] Proceeding inductively,

[35] Assumption 9, requiring that every player make the dominant choice when one is available, ensures that it is at least possible for preferences to be revealed by history in this manner. That is, if the Is reach $\mathbf{q} \in \mathcal{Q}_n^I(\mathbf{z}, \mathbf{z}')$, and they prefer $\mathbf{z}$ to $\mathbf{z}'$, then all the Is at $\mathbf{q}$ will make the choice yielding $\mathbf{z}$. Any *ToP* observing this will "know" that the Is prefer $\mathbf{z}$ to $\mathbf{z}'$.

for $i < I$ say that H_s *reveals* role i's preference over $(\mathbf{z}, \mathbf{z}')$ if some $\mathbf{q} \in \mathcal{Q}_n^i(\mathbf{z}, \mathbf{z}')$ was reached along H_s, after the pairwise preferences of roles $i+1, \ldots, I$ in $\mathbf{q}$ had been revealed.[36] Notice that, by definition, $\mathcal{Q}_n^i(\mathbf{z}, \mathbf{z}')$ is empty whenever role i is indifferent between $\mathbf{z}$, and $\mathbf{z}'$. If $\mathbf{z} \neq \mathbf{z}'$ such indifference arises within the available set of outcomes with probability zero. In the short run, however, there can be ties, as there is some positive probability that an outcome will be repeated within a particular stage game. We assume, in order to simplify the notation required in the proof, and as is natural, moreover, that every H_s reveals automatically role i's indifference over $(\mathbf{z}, \mathbf{z})$, for every outcome $\mathbf{z}$ available during iteration s, for every role $i = 1, \ldots, I$.

In order to keep an account of how much information about preferences is conveyed by the history we define—

DEFINITION 10: *For each iteration $s = 1, 2, \ldots,$ the random variable K_s^i is number of outcome pairs for which H_s reveals role i's binary preference. For each $s = s(n), \ldots, s(n+1)-1$, define $L_s^i = K_s^i / |\mathcal{Z}_n|^2$—this is the fraction of i pairwise preferences revealed along H_s.*[37]

DEFINITION 11: *For each role i, let $I_s^i \in \{0, 1\}$ be such that $I_s^i = 1$ if and only if each i subgame at iteration s is in $\mathcal{Q}_n^i(\mathbf{z}, \mathbf{z}')$, for some pair of outcomes $\mathbf{z}$, and $\mathbf{z}'$ over which i's preference is not revealed by H_s. For each $i < I$, let $J_s^i \in \{0, 1\}$ be such that $J_s^i = 1$ if and only if the game drawn at iteration s is such that all the pairwise choices of roles $j = i + i, \ldots, I$ are revealed along H_s. For the case $i = I$ (the last player role), let $J_s^i = 1$ for all $s = 1, 2, \ldots$.*

Notice that whenever $I_s^i \cdot J_s^i = 1$, role i reveals at least one binary preference that had not already been revealed by history. Hence, $K_{s+1}^i - K_s^i \geq I_s^i \cdot J_s^i$. Since $A \cdot B \geq A + B - 1$ for any binary variables $A, B \in \{0, 1\}$, it follows that

$$E(K_{s+1}^i \mid H_s) - K_s^i \geq E(I_s^i + J_s^i - 1 \mid H_s). \tag{B1}$$

Notice that J_s^i converges in probability to one whenever L_s^j converges in probability to one for each j after role i. (By definition, J_s^i is identically equal to one for $i = I$.) Equation (B1) therefore implies that if L_s^j converges in probability to one, for every role j after i, then the probability of revealing something new about i's preferences is small only if $E(I_s^i \mid H_s)$ is small. The following result is simply that in the limit $E(I_s^i \mid H_s)$ is small only if L_s^i is close to one. All of this together implies that whenever L_s^j converges to one for each j after i, in the long run, the probability of revealing something new about i's preferences is small only if L_s^i is already close to one (i.e., if much is already known about i's preferences).

[36] Here again, Assumption 9 ensures role i reveals preference when reaching the subgame in question, since all the is there will choose into the subgame that delivers $\mathbf{z}$ in the subgame perfect equilibrium of the continuation game.

[37] That is, for each $n = 1, 2, \ldots,$ and $s = s(n), \ldots, s(n+1) - 1$, $K_s^i = |\mathcal{Z}_n| + 2 \cdot \tilde{K}_s^i$, where $\tilde{K}_s^i$, is the number of pairs of outcomes $(\mathbf{z}, \mathbf{z}')$, $\mathbf{z} \neq \mathbf{z}'$, such that H_s reveals i's preference over $(\mathbf{z}, \mathbf{z}')$.

LEMMA 7: *Consider role i. For each $\eta > 0$ there is almost surely a $\varepsilon > 0$ and an integer M such that for all iterations, $s \geq r \geq M$, if $E(I_s^i \mid H_r) < \varepsilon$, then $E(L_s^i \mid H_r) > 1 - \eta$.*

PROOF:

For each iteration $s = 1, 2, \ldots$, let $n(s)$ denote the period prevailing during the iteration. Let $\mathcal{U}_s$ denote the outcome pairs $(\mathbf{z}, \mathbf{z}') \in \mathcal{Z}_{n(s)} \times \mathcal{Z}_{n(s)}$ such that i's preference over $(\mathbf{z}, \mathbf{z}')$ has not been revealed by history. Notice, in particular, that $L_s^i = 1 - |\mathcal{U}_s|/|\mathcal{Z}_{n(s)}|^2$ for each $s = 1, 2, \ldots$. Notice that $I_s^i = 1$ whenever every i subgame is in $\bigcup_{(\mathbf{z},\mathbf{z}')\in\mathcal{U}_s} \mathcal{Q}_{n(s)}^i(\mathbf{z}, \mathbf{z}')$ (see Definition 9 where $\mathcal{Q}_{n(s)}^i(\mathbf{z}, \mathbf{z}')$ is defined). The set of i subgames available during period s is $|\mathcal{Z}_{n(s)}|^{T^i}$, where T^i is the number of end nodes in these subgames. There are A^{i-1} role i subgames, and thus it follows that

$$\text{(B2)} \qquad E(I_s^i \mid H_s) = P\left\{I_s^i = 1 \mid H_s\right\} \geq \left[\sum_{(\mathbf{z},\mathbf{z}')\in\mathcal{U}_s} \frac{|\mathcal{Q}_{n(s)}^i(\mathbf{z}, \mathbf{z}')|}{|\mathcal{Z}_{n(s)}|^{T^i}}\right]^{A^{i-1}}.$$

For each $\xi > 0$, and $s = 1, 2, \ldots$, write

$$\mathcal{S}_s^i(\xi) = \left\{(\mathbf{z}, \mathbf{z}') \in \mathcal{Z}_{n(s)} \times \mathcal{Z}_{n(s)} \,:\, \frac{|\mathcal{Q}_{n(s)}^i(\mathbf{z}, \mathbf{z}')|}{|\mathcal{Z}_{n(s)}|^{T^i-2}} < \xi\right\}.$$

Using this in equation (B2) gives, for all $\xi > 0$,

$$\begin{aligned}
E(I_s^i \mid H_s) &\geq \left(\sum_{(\mathbf{z},\mathbf{z}')\in\mathcal{U}_s\setminus\mathcal{S}_s^i(\xi)} \frac{|\mathcal{Q}_{n(s)}^i(\mathbf{z}, \mathbf{z}')|}{|\mathcal{Z}_{n(s)}|^{T^i}}\right)^{A^{i-1}} \\
&\geq \left(\xi \cdot \frac{|\mathcal{U}_s \setminus \mathcal{S}_s^i(\xi)|}{|\mathcal{Z}_{n(s)}|^2}\right)^{A^{i-1}} \\
&\geq \left[\xi \cdot \max\left\{0\,,\ \frac{|\mathcal{U}_s|}{|\mathcal{Z}_{n(s)}|^2} - \frac{|\mathcal{S}_s^i(\xi)|}{|\mathcal{Z}_{n(s)}|^2}\right\}\right]^{A^{i-1}} \\
&= \left[\xi \cdot \max\left\{0\,,\ 1 - L_s^i - \frac{|\mathcal{S}_s^i(\xi)|}{|\mathcal{Z}_{n(s)}|^2}\right\}\right]^{A^{i-1}}.
\end{aligned}$$

After some algebra we see that

$$\text{(B3)} \qquad L_s^i \geq 1 - \frac{\left[E(I_s^i \mid H_s)\right]^{\frac{1}{A^{i-1}}}}{\xi} - \frac{|\mathcal{S}_s^i(\xi)|}{|\mathcal{Z}_{n(s)}|^2}.$$

Take the expectation with respect to H_r, for $r \leq s$, to obtain (after applying

Jensen's inequality to the resulting term, $E(E(I_s^i \mid H_s)^{\frac{1}{A^{i-1}}} \mid H_r))$

$$\text{(B4)} \qquad E(L_s^i \mid H_r) \geq 1 - \frac{\left[E(I_s^i \mid H_r)\right]^{\frac{1}{A^{i-1}}}}{\xi} - E(|\mathcal{S}_s^i(\xi)|/|\mathcal{Z}_{n(s)}|^2 \mid H_r).$$

The Glivenko-Cantelli Lemma (see Lemma 1 in the paper) implies that for each $\xi > 0$, $|\mathcal{S}_s^i(\xi)|/|\mathcal{Z}_n|^2$ almost surely converges to some $s(\xi)$. The regularity of the distribution over introduced outcomes (Assumption 7) implies that $s(\xi)$ tends to zero as ξ tends to zero. In the light of equation (B4) this completes the proof of Lemma 7.

The goal of the remainder of this section is to prove the following.

PROPOSITION 1: *If $\alpha > 1$, then L_s^i converges in probability to one for each $i = 1, \ldots, I$.*

The first step toward proving Proposition 1 is establishing—

PROPOSITION 2: *Suppose L_s^I converges in probability to some random variable, L^I. If $\alpha > 1$, then $L^I = 1$ a.e. Consider a player role $i < I$. Suppose L_s^j converges in probability to one for each $j = i+1, \ldots, I$, and that L_s^i converges in probability to some random variable, L^i. If $\alpha > 1$, then $L^i = 1$ a.e.*

PROOF:

Fix a player role $i \leq I$. Sum equation (B1) over $s = 1, \ldots, \tau - 1$, and take the unconditional expectation of the result to obtain

$$\text{(B5)} \qquad \sum_{s=1}^{\tau-1} E\left(E(K_{s+1}^i \mid H_s) - K_s^i\right) \geq \sum_{s=1}^{\tau-1} E\left(I_s^i + J_s^i - 1\right).$$

The expression on the left hand side here is $E(K_\tau^i - K_1^i)$, and since $E(K_1^i) = N$ (the number of outcomes initially), this is just $E(K_\tau^i) - N$.

Let $n(s)$ again denote the period prevailing during iteration s. Since $L_\tau^i = K_\tau^i/|\mathcal{Z}_{n(\tau)}|^2$, equation (B5) gives

$$\text{(B6)} \qquad E(L_\tau^i) - N/|\mathcal{Z}_{n(\tau)}|^2 \geq \frac{\tau - 1}{|\mathcal{Z}_{n(\tau)}|^2} \cdot \left[\frac{1}{\tau - 1}\sum_{s=1}^{\tau-1} E\left(I_s^i + J_s^i - 1\right)\right].$$

Now suppose $\alpha > 1$, and consider (B6) as τ tends to infinity. The first thing to notice is that the $(\tau - 1)/|\mathcal{Z}_{n(\tau)}|^2$ term in the expression diverges to infinity. To see this observe the following. The iteration corresponding to the arrival of the n-th novel outcome, $s(n) = \sum_{m=1}^{n-1} \kappa(m) + 1$, is non-decreasing in n, and has order of $n^{1+\alpha}$. Since each iteration $\tau = 1, 2, \ldots,$ satisfies $s(n(\tau)) \leq \tau \leq s(n(\tau) + 1)$, it follows that $n(\tau)$ has order of $\tau^{\frac{1}{1+\alpha}}$, and hence that $|\mathcal{Z}_{n(\tau)}|^2 = (N + n(\tau))^2$ has order of $\tau^{\frac{2}{1+\alpha}}$. Clearly if $\alpha > 1$, then τ grows at a faster rate than $|\mathcal{Z}_{n(\tau)}|^2$.

Next, notice that the quantity on the right hand side of (B6) must surely be bounded above by one, uniformly in τ (since surely $L_s^i \leq 1$). The limit inferior of the bracketed term in the expression must then be bounded above by zero, since otherwise the quantity on the right hand side would diverge to infinity.

If $i = I$, then $J_s^i = 1$, surely (by definition). If $i < I$, then the hypothesis that L_s^j converges to one in probability for each $j = i+1, \ldots, I$, implies that J_s^i converges in probability to one. In any case, we see now that $\liminf\{\sum_{s=0}^{\tau-1} E\left(I_s^i\right)/(\tau-1)\} = 0$, and therefore that $\liminf E(I_s^i) = 0$, since surely $I_s^i \geq 0$. Applying the result of Lemma 7 gives $\limsup E(L_s^i) = 1$. But if L_s^i converges in probability to L^i (as hypothesized), then $E(L_s^i) \longrightarrow E(L^i)$, and thus $E(L^i) = 1$, and therefore L^i must equal one a.e. This completes the proof of Proposition 2.

In the light of Proposition 2 what is needed now in order to prove Proposition 1 is to establish that the L_s^i processes converge when $\alpha > 1$. We proceed by proving that when $\alpha > 1$ the L_s^i sequences belong to a class of generalized sub-martingales. In particular, we rely on the following definition and result.

DEFINITION 12: *The $\{H_s\}$ adapted process $\{X_s\}$ is a weak sub-martingale in the limit (W-submil) if $P\{\inf_{s\geq r} E(X_s \mid H_r) - X_r \geq -\eta\}$ converges to one as r tends to infinity, for each $\eta > 0$. If X_s is a W-submil, with $|X_s|$ bounded almost surely, then there is a random variable X such that X_s converges in probability to X.*[38]

We will also need the following related technical result, the proof of which is deferred to Section B.B3.

LEMMA 8: *Suppose the $\{H_s\}$ adapted sequence $\{Y_s\}$, with $Y_s \in [0,1]$ a.e., is a W-submil converging in probability to one. Consider another $\{H_s\}$ adapted sequence $\{X_s\}$ with $X_s \in [0,1]$ a.e. The process X_s is a W-submil if either of the following are true.*

(1) For each $\eta > 0$ there is almost surely a $\varepsilon > 0$, and an integer M, such that for every $s \geq M$, if $Y_s > 1-\varepsilon$, then $X_s > 1-\eta$.

(2) For each $\eta > 0$ there is almost surely a $\varepsilon > 0$, and an integer M, such that for every $s > r \geq M$: If $E(X_{s+1} - X_s \mid H_r) < 0$ and $E(Y_s \mid H_r) > 1-\varepsilon$, then $E(X_{s+1} \mid H_r) > 1-\eta$.

The first enumerated condition in the statement of Lemma 8 is that in the limit X_s is close to one whenever Y_s is close to one. The second condition is that in the limit if Y_s is close to one, then X_s drifts downward only if it is also close to one.

The proof of Proposition 1 follows by induction using Proposition 2 in conjunction with the next result (the induction argument is presented formally in Section B.B2 below).

[38] A W-submil is a stronger version of a type of process called a “game that becomes fairer with time” (GFT) (in particular, every W-submil is a GFT). The claimed convergence of W-submils follows from Theorem VIII.1.22 in Egghe (1984) which establishes the convergence in probability of GFTs.

PROPOSITION 3: *If $\alpha > 1$, then L_s^I converges in probability to some random variable, L^I. For each $i < I$, if $\alpha > 1$, and L_s^j converges in probability to one for each $j = i+1, \ldots, I$, then L_s^i converges in probability to some random variable, L^i.*

PROOF:

The proof is broken up into three parts. In order to ease on notation we fix throughout the proof a role $i \leq I$, and suppress the i superscript.

Part 1. *Suppose the subsequence $\{L_{s(n)}\}$ is a W-submil, then so is the overall sequence $\{L_s\}$. Moreover, if $L_{s(n)}$ converges in probability to L, say, then so does L_s.*

Proof of Part 1. Recall that L_s is surely non-decreasing in between arrival dates. Notice, moreover, that the amount by which L_s decreases at arrivals of novelty tends surely to zero. That is, $L_{s(n)} = [K_{s(n)-1} + \Delta_{s(n)-1}]/|\mathcal{Z}_n|^2$, where $\Delta_{s(n)-1}$ denotes the number of i pairwise choices revealed at iteration $s(n)-1$ (just before the arrival of the n-novel outcome), and thus $L_{s(n)} = L_{s(n)-1} \cdot |\mathcal{Z}_{n-1}|^2/|\mathcal{Z}_n|^2 + \Delta_{s(n)-1}/|\mathcal{Z}_n|^2$. The $\Delta_{s(n)-1}/|\mathcal{Z}_n|^2$ term here tends surely to zero, and $|\mathcal{Z}_{n-1}|^2/|\mathcal{Z}_n|^2$ tends surely to one. Thus, surely $\lim\{L_{s(n)-1} - L_{s(n)}\} = 0$. Since L_s is nondecreasing in $s = s(n), \ldots, s(n+1)-1$, for each $n = 1, 2, \ldots$, and since the drops in L_s at arrivals of novelty tend to zero, it follows naturally that if $\{L_{s(n)}\}$ is a W-submil, then the overall sequence $\{L_s\}$ must also be a W-submil. If both $L_{s(n)}$, and L_s are W-submils, then they converge in probability to some limits. They must, however, possess the same limit since $\{L_{s(n)}\}$ is a subsequence of $\{L_s\}$.

Part 2. *Suppose $i < I$, and that for each role j after i, L_s^j is W-submil converging in probability to one. Then $\{J_s^i\}$ is a W-submil and converges in probability to one.*

Proof of Part 2. Fix $i < I$. In order to prove the result we proceed to show that almost surely in the limit $E(J_s^i \,|\, H_s)$ is close to one whenever L_s^j is close to one for each $j = i+1, \ldots, I$ (the result then follows from Lemma 8-(1)). We use $n(s)$ to denote the period corresponding to iteration s. For each role j after i, let $c_s^j \leq |\mathcal{Z}_{n(s)}|$ be the size of the largest subset of $\mathcal{Z}_{n(s)}$ such that, for every pair of outcomes in this subset, j's pairwise preference is revealed along H_s. The probability of the iteration s game being such that all of j's pairwise choices in the game have been revealed is bounded below by $(c_s^j/|\mathcal{Z}_{n(s)}|)^T$, where T is the number of terminal nodes in the fixed game tree. Recall that J_s^i is equal to one if history has revealed all the $\mathbf{q}_s$ pairwise choices for all the roles after i (Definition 11). It follows that

(B7) $$E(J_s^i \,|\, H_s) \geq 1 + \sum_{j=i+1}^{I} \left[\left(\frac{c_s^j}{|\mathcal{Z}_{n(s)}|} \right)^T - 1 \right].$$

To see that (B7) is true notice that $J^i_s \geq 1 + (B^{i+1}_s - 1) + \cdots + (B^I_s - 1)$, for $B^j_s \in \{0,1\}$ such that $B^j_s = 1$ if and only if all of role j's pairwise choices in $\mathbf{q}_s$ have been revealed, then observe that $E(B^j_s \mid H_s) \geq (c^j_s/|\mathcal{Z}_{n(s)}|)^T$, for each j. We proceed next to obtain a lower bound on $c^j_s/|\mathcal{Z}_{n(s)}|$ in terms of L^j_s.

Consider role j after i, and an iteration s. Let $C \subseteq \mathcal{Z}_{n(s)}$ be such that $\mathbf{z}$ is in C if and only if for every $\mathbf{z}' \in \mathcal{Z}_{n(s)}$ role j's preferences over $(\mathbf{z}, \mathbf{z}')$ are revealed by H_s. Clearly $c^j_s \geq \max\{|C|, 1\}$ (the lower bound of one follows because every H_s reveals preference on $(\mathbf{z}, \mathbf{z})$ for every outcome, $\mathbf{z}$, available at s). On the other hand, if C is non-empty, then $|C|$ is the largest integer such that

$$|\mathcal{Z}_{n(s)}| - 1 + |\mathcal{Z}_{n(s)}| - 2 + \cdots + |\mathcal{Z}_{n(s)}| - |C| \leq \frac{K^j_s - |\mathcal{Z}_{n(s)}|}{2}.$$

The term on the right hand side is the number of outcomes pairs $(\mathbf{z}, \mathbf{z}')$, $\mathbf{z} \neq \mathbf{z}'$ with j preference on $(\mathbf{z}, \mathbf{z}')$ revealed by H_s. After some algebra, and after dividing through by $|\mathcal{Z}_{n(s)}|^2$, we obtain

$$\text{(B8)} \qquad \frac{|C|}{|\mathcal{Z}_{n(s)}|}\left(2 - \frac{|C|}{|\mathcal{Z}_{n(s)}|}\right) \leq L^j_s - \frac{|\mathcal{Z}_{n(s)}| - |C|}{|\mathcal{Z}_{n(s)}|^2}.$$

The term $\frac{|\mathcal{Z}_{n(s)}| - |C|}{|\mathcal{Z}_{n(s)}|^2}$ surely converges to zero, and thus we see that surely in the limit $|C|/|\mathcal{Z}_{n(s)}|$ is close to one whenever L^j_s is close to one. Since $c^j_s \geq |C|$ it also follows that in the limit $c^j_s/|\mathcal{Z}_{n(s)}|$ is close to one whenever L^j_s is close to one. Bringing this together with (B7) we see that in the limit $E(J^i_s \mid H_s)$ is close to one if L^j_s is close to one, for each $j = i+1, \ldots, I$. Applying Lemma 8-(1) gives that J^i_s is a W-submil whenever L^j_s is a W-submil converging in probability to one, for each role j following i. Obviously, J^i_s converges in probability to one, under these conditions.

Part 3. *Suppose $\alpha > 1$. If $i = I$, then $\{L^I_{s(n)}\}$ is a W-submil. If $i < I$, and L^j_s is a W-submil converging in probability to one, for each role j after i, then $L^i_{s(n)}$ is a W-submil.*

Proof of Part 3. Sum equation (B1) over $s = s(n-1), \ldots, s(n) - 1$ to obtain

$$\text{(B9)} \qquad \begin{aligned} E(K_{s(n)} \mid H_{s(n-1)}) - K_{s(n-1)} &= \sum_{s=s(n-1)}^{s(n)-1} E(E(K_{s+1} \mid H_s) - K_s \mid H_{s(n-1)}) \\ &\geq \sum_{s=s(n-1)}^{s(n)-1} E(I^i_s + J^i_s - 1 \mid H_{s(n-1)}). \end{aligned}$$

Notice that for every period, $n = 1, 2, \ldots$, $E(I^i_s \mid H_s) = P\left\{I^i_s = 1 \mid H_s\right\}$ is non-increasing in $s = s(n-1), \ldots, s(n)-1$. The reason is that the set of games is fixed in between arrival dates, and thus as more is learned about i preferences between arrivals it becomes less likely that $I_s^i = 1$. (see Definition 11). Notice, moreover, that $E(J_s^i \mid H_s) = P\left\{J_s^i = 1 \mid H_s\right\}$ is non-decreasing in $s = s(n-1), \ldots, s(n)-1$, for every period $n = 1, 2, \ldots$. This is because the set of outcomes is fixed throughout these iterations, and hence $J_s^i = 1$ is more likely as more is learned about role $j = i+1, \ldots, I$ preferences. Using these facts in equation (B9) gives (recall that there are $k(n-1)$ iterations between the arrival of the $n-1$-th and the n-th new outcomes)

$$\text{(B10)} \qquad \begin{aligned} &E(K_{s(n)} \mid H_{s(n-1)}) - K_{s(n-1)} \\ &\geq \kappa(n-1) \cdot E(I^i_{s(n)-1} + J^i_{s(n-1)} - 1 \mid H_{s(n-1)}). \end{aligned}$$

Since $L_s^i = K_s^i / |\mathcal{Z}_{n-1}|^2$, for each iteration $s = s(n-1), \ldots, s(n)-1$, it also follows that for every $n > m$,

$$\begin{aligned} &E(L_{s(n)} - L_{s(n-1)} \mid H_{s(m)}) < 0 \quad \Longrightarrow \\ &E(K_{s(n)} - K_{s(n-1)} \mid H_{s(m)}) < |\mathcal{Z}_n|^2 - |\mathcal{Z}_{n-1}|^2. \end{aligned}$$

Using this in (B10) gives, for all $n > m$,

$$\text{(B11)} \qquad \begin{aligned} &E(L_{s(n)} - L_{s(n-1)} \mid H_{s(m)}) < 0 \quad \Longrightarrow \\ &E(I^i_{s(n)-1} \mid H_{s(m)}) \;<\; E(1 - J^i_{s(n-1)} \mid H_{s(m)}) \;+\; \frac{|\mathcal{Z}_n|^2 - |\mathcal{Z}_{n-1}|^2}{\kappa(n-1)}. \end{aligned}$$

Now fix $\eta > 0$. Notice that if $\alpha > 1$, then $(|\mathcal{Z}_n|^2 - |\mathcal{Z}_{n-1}|^2)/\kappa(n-1)$ surely converges to zero. With this, and with Lemma 7 in mind, let M and $\varepsilon > 0$ be such that if $m \geq M$, then the following hold: 1) $(|\mathcal{Z}_m|^2 - |\mathcal{Z}_{m-1}|^2)/\kappa(m-1) < \varepsilon/2$, 2) for all $n > m$, if $E(I^i_{s(n)-1} \mid H_{s(m)}) < \varepsilon$, then $E(L_{s(n)-1} \mid H_{s(m)}) > 1 - \eta/2$, and 3) $|\mathcal{Z}_{m-1}|^2 / |\mathcal{Z}_m|^2 > 1 - \eta/2$. Given equation (B11), requirements 1) and 2) together imply that for all $n > m \geq M$, if $E(L_{s(n)} - L_{s(n-1)} \mid H_{s(m)}) < 0$ and $E(1 - J^i_{s(n-1)} \mid H_{s(m)}) < \eta/2$, then $E(I^i_{s(n)-1} \mid H_{s(m)}) < \varepsilon$, and therefore $E(L_{s(n)-1} \mid H_{s(m)}) > 1 - \eta/2$. Notice that $L_{s(n)} \geq L_{s(n)-1} \cdot |\mathcal{Z}_{n-1}|^2 / |\mathcal{Z}_n|^2$ and thus requirement 3) implies $L_{s(n)} \geq L_{s(n)-1} - \eta/2$. We therefore have that for each $n > m \geq M$—

$$\text{(B12)} \qquad \begin{aligned} &\text{If } E(L_{s(n)} - L_{s(n-1)} \mid H_{s(m)}) < 0 \ \text{ and } E(J^i_{s(n-1)} \mid H_{s(m)}) > 1 - \varepsilon/2, \\ &\text{then } E(L_{s(n)} \mid H_{s(m)}) > 1 - \eta. \end{aligned}$$

For $i = I$, the last player role, J_s^i is equal to one for all s, and therefore 8-(2) implies immediately that $\{L^I_{s(n)}\}$ is a W-submil. For $i < I$, suppose L_s^j is a W-submil converging in probability to one for each $j = i+1, \ldots, I$. Part 2 above gives that J_s^i is a W-submil converging in probability to one. Recall that in (B12) η is any positive number. It thus follows from Lemma 8-(2) again that $\{L_{s(n)}\}$ is a W-submil. In view of Part 1 above this completes the proof of Proposition 3 since W-submils converge in probability (Definition 12).

B2. Proof of Proposition 1

Proceed by induction using Proposition 2 and Proposition 3. Suppose $\alpha > 1$. Proposition 3 gives that L_s^I converges in probability to some random variable. Proposition 2 implies this limit is equal to one a.e. Next consider $i < I$. If L_s^j converges in probability to one, for each role j after i, Proposition 3 implies that L_s^i converges in probability to some random variable L^i, say. Proposition 2 then gives that $L^i = 1$ a.e.

B3. Proof of Lemma 8

Let Y_s be a W-submil converging in probability to one. Notice first that, for all $\varepsilon > 0$,

$$\lim_{r\to\infty} P\left\{\inf_{s\geq r} E(Y_s \mid H_r) > 1-\varepsilon\right\} = 1. \tag{B13}$$

Now consider the first implication of Lemma 8: If for every $\eta > 0$ there is almost surely a $\varepsilon > 0$ and an integer M such that $Y_s > 1-\varepsilon \Longrightarrow X_s > 1-\eta$, for all $s \geq M$, then X_s is a W-submil. Fix an $\eta > 0$, and then choose $\varepsilon \leq \eta/2$, and M so that almost surely for all $s \geq M$, $Y_s > 1-\varepsilon$ implies $X_s > 1-\eta/2$. Next observe that $Y_s \leq 1$ (almost surely, by hypothesis), and thus for each $\varepsilon > 0$,

$$\begin{aligned} E(Y_s \mid H_r) &\leq P\{Y_s > 1-\varepsilon \mid H_r\} + (1-\varepsilon)\cdot P\{Y_s \leq 1-\varepsilon \mid H_r\} \\ &= 1-\varepsilon\cdot P\{Y_s \leq 1-\varepsilon \mid H_r\}. \end{aligned}$$

It thus follows that $E(Y_s \mid H_r) > 1-\varepsilon^2$ implies $P\{Y_s \leq 1-\varepsilon \mid H_r\} \leq \varepsilon$. But, by choice of M, we have that for each $s \geq r \geq M$, $P\{Y_s \leq 1-\varepsilon \mid H_r\} \leq \varepsilon$ implies $P\{X_s \leq 1-\eta/2 \mid H_r\} \leq \varepsilon$, which in turn implies, since $X_s \geq 0$ a.e. (recall here that $\varepsilon \leq \eta/2$ by choice),

$$E(X_s \mid H_r) > (1-\eta/2)\cdot(1-\varepsilon) > 1-\eta/2-\varepsilon \geq 1-\eta. \tag{B14}$$

Since $X_s \leq 1$ by hypothesis it follows that $E(Y_s \mid H_r) > 1-\varepsilon^2$ implies $E(X_s \mid H_r) - X_r > -\eta$, for all $s \geq r \geq M$. In the light of equation (B13) this proves that X_s is a W-submil.

Now consider the second implication of Lemma 8: If Y_s is a W-submil that converges in probability to one, then X_s is a W-submil if for each $\eta > 0$ there is almost surely a $\varepsilon > 0$, and an integer M, such that for every $s > r \geq M$: $E(X_{s+1} - X_s \mid H_r) < 0$, and $E(Y_s \mid H_r) > 1-\varepsilon$ implies $E(X_{s+1} \mid H_r) > 1-\eta$. Fix η, and let ε be as hypothesized, given the choice of η. Suppose $\inf_{s\geq r} E(Y_s \mid H_r) > 1-\varepsilon$. If, for some $s \geq r$, $E(X_s \mid H_r) \leq 1-\eta$, then it must be that $E(X_{s-1} \mid H_r) \leq E(X_s \mid H_r)$, and thus that $E(X_{s-1} \mid H_r) \leq 1-\eta$. Proceeding recursively, we see that for all $s \geq r$, $E(X_s \mid H_r) \leq 1-\eta \Longrightarrow E(X_s \mid H_r) - X_r \geq 0$, whenever $\inf_{s\geq r} E(Y_s \mid H_r) > 1-\varepsilon$. If instead we have that $E(X_s \mid H_r) > 1-\eta$, then $X_s \leq 1$ (as hypothesized) implies $E(X_s \mid H_r) - X_r > -\eta$. We have thus shown that if $\inf_{s \geq r} E(Y_s \mid H_r) > 1 - \varepsilon$, then $\inf_{s \geq r} E(X_s \mid H_r) - X_r > -\eta$. This proves that X_s is a W-submil given the limit in equation (B13).

B4. The SR-ToP Strategy Dominates if $\alpha \in (1, A^2 - 1)$.

In the light of Proposition 1 we see that the *SR-ToP* makes the sequentially rational choice essentially always in the limit. Since role I makes the sequentially rational choice always (Assumption 9), the main result, Theorem 2, is established by induction given the following result.

PROPOSITION 4: *Consider a role $i < I$. Let γ_s denote the fraction of iteration s matchings where the players in roles $i+1$ through I use one-shot strategies that correspond to a subgame perfect equilibrium profile for those roles. If $\alpha \in (1, A^2 - 1)$, and γ_s converges in probability to one, then the proportion of role i players that use the SR-ToP strategy tends to one in probability.*

The remainder of this appendix is devoted to proving Proposition 4. Fix, for all that follows, a player role $i < I$. We suppress the i superscript throughout in order to ease the notation. Consider first a definition and an auxiliary result.

DEFINITION 13: *For each strategy r of role i that is not the SR-ToP strategy let $A_s(r, \varepsilon) \in \{0, 1\}$ equal one if and only if the iteration s average payoff to the SR-ToP strategy exceeds that of r by at least $\varepsilon > 0$, and no other strategy obtains a higher average payoff than the SR-ToP strategy.*

LEMMA 9: *Let γ_s denote the fraction of iteration s matchings where the players in roles $i+1$ through I use one-shot strategies that correspond to a subgame perfect equilibrium profile for those roles. If $\alpha \in (1, A^2 - 1)$, and γ_s converges in probability to one, then for each sufficiently small $\varepsilon > 0$ there is a $\mu > 0$ such that $P\{E(A_s(r, \varepsilon) \mid H_s) < \mu\}$ converges to zero, for every strategy r that is an alternative to the SR-ToP strategy in role i.*

PROOF:

Let $\bar{z}_s^*$ denote the iteration s expected payoff to a role i player given that he chooses a one-shot strategy that is part of the subgame perfect equilibrium of the game, while all the remaining players also choose as in an equilibrium.[39] If r is an alternative to the *SR-ToP* strategy of role i, let $\bar{z}_s^*(r)$ denote the expected payoff obtained by r given that all the remaining players play as in a subgame perfect equilibrium of the game. With probability tending to one the game has a unique subgame perfect equilibrium, and thus Proposition 1 gives (since $\alpha > 1$, by hypothesis) that the *SR-ToP* in role i plays this equilibrium with probability

[39]These are expected payoffs, in particular, for any $i > 1$ because of the randomness in the choices made by the roles before i. Notice, moreover, that in general the game will not have a unique equilibrium (although games with multiple equilibria will crop up with vanishing probability). In such cases, for concreteness, we let $\bar{z}_s^*$ be the worst payoff to i given that each role $i, \ldots, I$ uses a component of some equilibrium strategy profile.

tending to one. If γ_s converges in probability to one, then the average payoff to the *SR-ToP* in role i must converge in probability to $\bar{z}_s^*$. It also follows that the payoff to the alternative strategy r must converge in probability to $\bar{z}_s^*(r)$. In order to prove Lemma 9 it therefore suffices to show that for each sufficiently small $\varepsilon > 0$ there is a $\mu > 0$ such that $P\{P\{\bar{z}_s^* - \bar{z}_s^*(r) > \varepsilon \,|\, H_s\} < \mu\}$ converges to zero for every alternative, r, to the *SR-ToP* strategy in role i. With this in mind consider the following.

For each alternative r to the *SR-ToP* strategy of role i let $\mathcal{Q}(r,\varepsilon) \subseteq \mathcal{Q}$ be the set of games that satisfy the following conditions: 1) the absolute payoff difference between any of i's payoffs is at least ε, and 2) if r is a *ToP* strategy, the choice of r differs from the subgame perfect choice in every i subgame, given that the choices of the remaining roles are revealed by the history; if r is a naive strategy, the initial reaction of the naive strategy in each subgame differs from the subgame perfect choice. Now, if r is a *ToP* alternative to the *SR-ToP* strategy, then

$$P\{\bar{z}_s^* - \bar{z}_s^*(r) > \varepsilon \,|\, H_s\} \geq P\{\mathbf{q}_s \in \mathcal{Q}(r,\varepsilon) \,|\, H_s\}. \tag{B15}$$

If r is a naive strategy, then

$$P\{\bar{z}_s^* - \bar{z}_s^*(r) > \varepsilon \,|\, H_s\} \geq P\{\mathbf{q}_s \in \mathcal{Q}(r,\varepsilon), N_s = 1 \,|\, H_s\}, \tag{B16}$$

where $N_s \in \{0,1\}$ is equal to one if and only if none of the i subgames of $\mathbf{q}_s$ has been seen along H_s.

The probability of seeing a familiar subgame tends surely to zero whenever $\alpha < A^2 - 1$. To see this notice first that the number of i subgames available at iteration s is $|\mathcal{Z}_{n(s)}|^{T^i}$, where $n(s)$ is the period corresponding to s, and T^i is the number of end nodes in an i subgame. At most A^{i-1} role i subgames are seen at any given iteration, and therefore the fraction of familiar i subgames at s is at most $A^{i-1} \cdot s/|\mathcal{Z}_{n(s)}|^{T^i}$. As argued in Appendix A of the paper, $|\mathcal{Z}_{n(s)}|$ has order of $s^{\frac{1}{1+\alpha}}$, and therefore $s/|\mathcal{Z}_{n(s)}|^{T^i}$ converges surely to zero whenever $T^i/(1+\alpha) > 1$, i.e, when $T^i - 1 > \alpha$ (note that $T^i \geq A^2$ for all $i < I$). We have thus argued that $P\{\mathbf{q}_s \in \mathcal{Q}(r,\varepsilon), N_s = 1 \,|\, H_s\}$ converges almost surely to $P\{\mathbf{q}_s \in \mathcal{Q}(r,\varepsilon) \,|\, H_s\}$ for every naive strategy r.

Next, notice that the Glivenko-Cantelli Lemma (see Lemma 1) implies that $P\{\mathbf{q}_s \in \mathcal{Q}(r,\varepsilon) \,|\, H_s\}$ converges almost surely to the measure of $\mathcal{Q}(r,\varepsilon)$ under G.[40] In view of (B15) and (B16) what is needed to complete the proof is therefore to show that for each sufficiently small $\varepsilon > 0$ the measure of $\mathcal{Q}(r,\varepsilon)$ under G is positive for each alternative r to the *SR-ToP* strategy. Consider the following.

The set of games with absolute payoff differences exceeding ε, for any role, has positive measure for each small enough but strictly positive ε. When r is a *ToP* alternative to the *SR-ToP* it thus follows, by assumption (cf. Definition 6), that

[40] Recall that G is the distribution of games implied by F, where F is the cdf according to which new outcomes are introduced in every period (F's properties are as in Assumption 7).

for each sufficiently small $\varepsilon > 0$, $\mathcal{Q}(r, \varepsilon)$ has positive measure under G.[41] For the case in which r is a naive strategy, that $\mathcal{Q}(r, \varepsilon)$ has positive measure for small $\varepsilon > 0$ follows from the proof of Lemma 5 in Appendix A. This proves Lemma 9.

We use the following elementary result concerning sequences in the proof of Proposition 4.

LEMMA 10: *Let x_s, $s = 1, 2, \ldots$, be a sequence taking values in $[0, 1]$. Suppose $\liminf\{x_{s+1} - x_s\} \geq 0$. Suppose further that for each $\xi > 0$, there is an integer M such that if $s \geq M$, then $x_{s+1} - x_s < 0$ only if $x_s > 1 - \xi$. Then x_s converges to some limit $\bar{x} \in [0, 1]$.*

PROOF:

Fix $\xi > 0$. Consider the subset of indices, $N \subseteq \{1, 2, \ldots,\}$, such that $s \in N$ if and only if $x_{s+1} - x_s < 0$. If N is empty or finite, then obviously x_s possesses a limit. Therefore, suppose N contains infinitely many terms. Let s_r, $r = 1, 2, \ldots$, denote these terms, enumerated so that s_r increases in r. The hypotheses of Lemma 10 imply the subsequence $\{x_{s_r}\}$ converges to one. By assumption, $\liminf\{x_{s+1} - x_s\} \geq 0$, and therefore the subsequence $\{x_{s_r+1}\}$ also converges to one. This implies the entire sequence $\{x_s\}$ converges to one, since x_s is non-decreasing in $s = s_r + 1, \ldots, s_{r+1}$, for every $r = 1, 2, \ldots$.

B5. Proof of Proposition 4

Let f_s denote the fraction of *SR-ToP*s in role i at iteration s. We will first show that $E(f_s)$ satisfies the hypotheses of Lemma 10, and thus converges to some limit. With this in mind fix ξ (to play the role of the ξ from Lemma 10). Given this choice of ξ, choose $\eta > 0$, and $\varepsilon > 0$ so that the following hold. 1) If $P\{f_s(r) > \eta\} < \eta$ for each alternative r to the *SR-ToP* strategy, then $E(f_s) > 1 - \xi$. 2) There is a $\mu > 0$ such that $P\{E(A_s(r, \varepsilon) \mid H_s) < \mu\}$ converges to zero, for each i role alternative, r, to the *SR-ToP* (cf. Lemma 9). Fix this μ for the remainder of this proof. Then, given the choice of η and ε, let $\Delta > 0$ be such that for each iteration s the following holds. If at iteration s no strategy obtains a higher average payoff for role i than the *SR-ToP*, and moreover the average payoff to the *SR-ToP* exceeds that of an alternative strategy r by ε, for which $f_s(r) > \eta$, then $f_{s+1} - f_s > \Delta$. (That this can be done is assured by Assumption 10).

Notice that if $A_s(r, \varepsilon) = 1$ (Definition 13), and $f_s(r) > \eta$ for some alternative r to the *SR-ToP*, then $f_{s+1} - f_s > \Delta$. Let $C_s \in \{0, 1\}$ equal one if and only some alternative to the *SR-ToP* obtains a higher payoff than the *SR-ToP* at iteration s. Since $f_{s+1} - f_s \geq -1$, surely, we have, for every alternative r to the *SR-ToP* strategy,

[41] Specifically the assumption is that the *ToP* alternative for role i makes the non-equilibrium choice in at least one i subgame defined by i's ordinal preferences, and those of the remaining players.

(B17)

$$
\begin{aligned}
E(f_{s+1} - f_s \mid H_s) &\geq \Delta \cdot E(A_s(r,\varepsilon) \mid H_s) \cdot I\{f_s(r) > \eta\} - E(C_s \mid H_s) \\
&= \Delta \cdot \mu \cdot I\{f_s(r) > \eta\} \\
&\quad - E(C_s \mid H_s) - \Delta \cdot (\mu - E(A_s(r,\varepsilon) \mid H_s)) \cdot I\{f_s(r) > \eta\},
\end{aligned}
$$

where $I\{\}$ denotes the indicator function. With (B17) in mind write

$$Y_s = E(C_s \mid H_s) + \Delta \cdot \max_r \{(\mu - E(A_s(r,\varepsilon) \mid H_s)) \cdot I\{f_s(r) > \eta\}\}.$$

Taking the expectation in (B17) gives that for every alternative r to the *SR-ToP* in role i,

$$\text{(B18)} \qquad E(f_{s+1}) - E(f_s) \geq \Delta \cdot \mu \cdot P\{f_s(r) > \eta\} - E(Y_s).$$

A rearrangement of equation (B18) gives,

$$\text{(B19)} \qquad E(f_{s+1}) - E(f_s) \ < 0 \ \Longrightarrow P\{f_s(r) > \eta\} \ < \ \frac{E(Y_s)}{\Delta \cdot \mu}, \ \text{ for all } r > 1.$$

Now, suppose $\alpha \in (1, A^2 - 1)$. Proposition 1 implies that L_s^i converges in probability to one for each $i = 1, \ldots, I$. By hypothesis, the fraction of *SR-ToP*s in role $j = i+1, \ldots, I-1$ converges in probability to one. It thus follows that C_s converges in probability to zero. Our choice of ε then gives that $\liminf E(Y_s) \leq 0$ (recall that $P\{E(A_s(r,\varepsilon) \mid H_s) < \mu\} \to 0$). Then, with (B19) in mind, choose M so that for all $s \geq M$, $E(Y_s)/(\Delta \cdot \mu) < \eta$, for each $r > 1$. Recall that our choice of η implies that if $P\{f_s(r) > \eta\} < \eta$ for each alternative r to the *SR-ToP* strategy, then $E(f_s) > 1 - \xi$. From (B19) it then follows that for every $s \geq M$, if $E(f_{s+1}) - E(f_s) < 0$, then $E(f_s) > 1 - \xi$. Moreover, given that $\liminf E(Y_s) \leq 0$, equation (B18) implies that $\liminf\{E(f_{s+1}) - E(f_s)\} \geq 0$. We see now that $E(f_s)$ satisfies the hypotheses of Lemma 10, and thus converges to some limit. Taking the limit as s tends to infinity in (B18) gives, since the convergence of $E(Y_s)$ implies $\lim\{E(f_{s+1}) - E(f_s)\} = 0$, while $\liminf E(Y_s) \leq 0$, that $\lim P\{f_s(r) > \eta\} = 0$, for each $r > 1$. Since $\eta > 0$ is arbitrarily it follows that $f_s(r)$ converges in probability to zero for each alternative r to the *SR-ToP*, and thus that f_s converges in probability to one. This completes the proof of Proposition 4, and thus completes the proof of Theorem 2.

References

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