SUPPLEMENTARY ONLINE MATERIAL FOR ROBSON AND KAPLAN

NOTES ON THE ANALYSIS

It follows readily that any solution to the problem described in Section III must also solve the following problem

(5) $$\max_{K \geq K_0, w(.) \geq 0} \int_0^{\infty} e^{-(\mu+r)t}\left(F - \beta K w\right)dt - \alpha(K - K_0) = \gamma \text{ subject to } \frac{dQ}{dt} = w - \rho Q.$$

Simplification is possible since the optimal fertility profile is indeterminate in this example, subject only to satisfying the Euler-Lotka equation, (4).

Given that $Q_0 = c/2d$ and that the cost of quality maintenance is linear in w, the optimal trajectory in (5) for w involves allowing quality to run down until it reaches its optimal steady state level, and then maintaining it at that level. That is, for some $Q^* \geq 0$, $w = 0$ until $t = t^*$ but $w = \rho Q^*$ thereafter, where $Q_0 e^{-\rho t^*} = Q^*$. Let $Q(t)$ be the overall time path of quality this implies. Thus, the problem (5) reduces to

$$\max_{K^*, Q^*} V(K^*, Q^*) = \gamma \quad \text{where}$$

$$V(K^*, Q^*) = \int_0^{\infty} F(K^*, Q(t))e^{-(\mu+r)t}dt - \frac{\beta\rho K^* Q^*}{\mu + r}\left(\frac{Q^*}{Q_0}\right)^{\frac{\mu+r}{\rho}} - \alpha(K^* - K_0)$$

Dropping the asterisks on K and Q, and since the upper bound is never binding, the first-order Kuhn-Tucker condition for choice of *K* for this problem is then

$V_K \leq 0$ and $V_K(K-K_0)=0$ so $a-2bK-\left(\frac{Q}{Q_0}\right)^{\frac{\mu+r}{\rho}}\beta\rho Q-\alpha(\mu+r)\leq 0$ and

$$\left(a-2bK-\left(\frac{Q}{Q_0}\right)^{\frac{\mu+r}{\rho}}\beta\rho Q-\alpha(\mu+r)\right)(K-K_0)=0$$

The first-order condition for choice of Q is similarly

$V_Q \leq 0$ and $V_Q Q=0$ so that $c-2dQ-(\mu+r+\rho)\beta K\leq 0$ and

$(c-2dQ-(\mu+r+\rho)\beta K)Q=0$.

There cannot be an interior solution for Q if $\beta > \frac{c}{(\mu+r+\rho)K_0}$; rather $Q=0$ in this case. There must, on the other hand, be an interior solution for K as long as α and K_0 are small enough to satisfy $a>\alpha(\mu+r)+2bK_0$.

Finally, for completeness, it is shown that any solution of (5) is also a solution to the problem posed in Section III. Note first that

$$\frac{d}{dr}\left(\max_{K\geq K_0, w(.)\geq 0}\left[\int_0^\infty e^{-(\mu+r)t}(F-\beta Kw)dt-\alpha(K-K_0)\right]\right)=-\int_0^\infty te^{-(\mu+r)t}(F-\beta Kw)dt$$

Since, furthermore

$$\frac{d}{dt}\left(t\int_t^\infty e^{-(\mu+r)\tau}(F-\beta Kw)d\tau\right)=\int_t^\infty e^{-(\mu+r)\tau}(F-\beta Kw)d\tau-te^{-(\mu+r)t}(F-\beta Kw)$$

then

$$\int_0^\infty te^{-(\mu+r)t}(F-\beta Kw)dt=\int_0^\infty dt\int_t^\infty e^{-(\mu+r)\tau}(F-\beta Kw)d\tau>0$$

because choosing $w=0$ ensures that $\int_t^\infty e^{-(\mu+r)\tau}Fd\tau>0$. Hence

(6) $$\frac{d}{dr}\left(\max_{K \geq K_0, w(.) \geq 0}\left[\int_0^{\infty} e^{-(\mu+r)t}\left(F - \beta K w\right)dt - \alpha(K - K_0)\right]\right) < 0.$$

Consider now any solution of the problem in (5). Since increasing *r* must be infeasible given (6), the desired result follows.