Online Appendix

Status, Intertemporal Choice and Risk-Taking

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This Online Appendix contains the proofs of all the lemmas that do not appear in the printed Appendix to the paper. It also contains the statements of all these lemmas and details of the intervening arguments in the proofs of the propositions. These arguments appeared in the printed Appendix. It also contains a proof of existence for a more general model that implies Proposition 3(ii). This Online Appendix can be used without reference to the printed Appendix.

We maintain Assumptions 1–3 until further notice. We begin with a central lemma.

Lemma 1. *At any date with equilibrium distribution of consumption F, $\mu(c) \equiv u(c, \bar{F}(c))$ must be concave.*

Proof. This property is an essentially static result that holds within each period independently of the intertemporal links. We first claim that $\mu(c) \equiv u(c, \bar{F}(c))$ must be continuous at all $c > 0$. If not, F has a mass point at $c > 0$. Now choose $\epsilon \in (0, 1)$ smaller than c such that $(1 - \epsilon)u(c, F(c)) > u(c, \bar{F}(c))$ (by (2), this is always possible). Consider a fair consumption bet which pays $c + \epsilon^2$ with probability $1 - \epsilon$ and $c - \epsilon(1 - \epsilon)$ with probability ϵ. The expected payoff from this bet is then given by

$$(1 - \epsilon)\mu(c + \epsilon^2) + \epsilon\mu\left(c - \epsilon[1 - \epsilon]\right) > (1 - \epsilon)u(c, F(c)) > \mu(c). \tag{A.1}$$

There must be a positive measure of individuals who take gambles that have a positive probability of generating c. All such individuals would be strictly better off replacing the realization c with this gamble, so such an atom cannot arise in equilibrium.

Now we show that μ must be concave. Suppose not; then by the already-established continuity of μ for all positive consumption, there exist c, c_1, c_2, with $0 < c_1 < c < c_2$,[1] and $\pi(c) \in (0, 1)$ such that

$$c = \pi(c)c_1 + (1 - \pi(c))c_2 \text{ and } \pi(c)u(c_1, \bar{F}(c_1)) + (1 - \pi(c))u(c_2, \bar{F}(c_2)) > u(c, \bar{F}(c)) \tag{A.2}$$

Indeed, it is not hard to see that it must be possible to choose such c_1 and c_2 such that (A.2) holds *for all* $c \in (c_1, c_2)$.[2]

Now, there must be such a positive measure of individuals who have a positive probability of consuming in the interval (c_1, c_2). For, if not, μ would be concave on (c_1, c_2), contradicting (A.2). Consider any such individual; she takes a (possibly degenerate) lottery F'. Suppose

[1]Because $\mu(0) = u(0, \bar{F}(0)) \leq u(0, F(0))$, the presumed lack of concavity of μ must allow us to choose c_1 and c_2 to be strictly positive.

[2]That is, consider the chord from $(c_1, \mu(c_1))$ to $(c_2, \mu(c_2))$. There must then exist $c_1' < c_2'$ such that $\mu(c)$ intersects the chord at c_1' and at c_2' but is strictly beneath the chord for all $c \in (c_1', c_2')$. Redefining c_1' and c_2' as c_1 and c_2 then yields (A.2). Note further that $\pi(c)$ is clearly a continuous function of c, so the integral in (A.3) exists.

that F' is modified by the addition of the simple lottery $(c_1, c_2; \pi(c), 1-\pi(c))$, conditional on any $c \in (c_1, c_2)$. The increment in expected utility from this change is then

$$\text{(A.3)} \qquad \int_{c_1}^{c_2} [\pi(c)u(c_1, \bar{F}(c_1)) + (1-\pi(c))u(c_2, \bar{F}(c_2)) - u(c, \bar{F}(c))]dF(c) > 0$$

which shows a positive measure of individuals strictly prefer to deviate from their equilibrium strategies, and thereby creates a contradiction. □

Proof of Proposition 1. First suppose that H has finitely many mass points. For any "initial point" a such that $H(a) < 1$, and for any "terminal point" $d > a$, let $[aHd]$ be the affine segment that connects $u(a, H(a))$ to $u(d, H(d))$. Associated with $[aHd]$ is a positive slope α, given by

$$\alpha \equiv \frac{u(d, H(d)) - u(a, H(a))}{d-a},$$

Say that $[aHd]$ is *allowable* if $\alpha \geq u_c(a, H(a))$.

Lemma **2.** *If $[aHd]$ is allowable, then the following distribution function F is well-defined and strictly increasing: $F(c) = H(c)$ for all $c \notin (a, d)$, and*

$$\text{(A.4)} \qquad u(c, F(c)) = u(a, H(a)) + \alpha(c-a)$$

for all $c \in (a, d)$.

Proof. It is obvious that the number $F(c)$ is well-defined in $[0,1]$ for every $c \in (a, d)$, because $H(a) \leq F(c) \leq H(d)$. So we only need to show that F is strictly increasing. Because $[aHd]$ is allowable and u is strictly concave in c, F is strictly increasing at a. Consider any interval $[a, b]$ on which F is strictly increasing. Then for all $x < b$, $\alpha \geq u_c(x, F(x))$. Passing to the limit and using the continuity of u_c in (c, s), we see that $\alpha \geq u_c(b, F(b))$, so again by strict concavity of u in c, F must continue to strictly increase just to the right of b. This proves that F is strictly increasing *everywhere* on $[a, d]$. □

For allowable $[aHd]$ with associated distribution function F as described in Lemma 2, define

$$I_{[aHd]}(x) \equiv \int_a^x [F(z) - H(z)]dz$$

for $x \geq a$. Say that the allowable segment $[aHd]$ is *feasible* if

$$\text{(A.5)} \qquad I_{[aHd]}(x) \geq 0$$

for all $x \in [a, d)$, with equality holding at $x = d$:

$$\text{(A.6)} \qquad I_{[aHd]}(d) = 0$$

Because H has finitely many jumps and is flat otherwise, and because u is concave in c, it is easy to see that from any a, there are at best finitely many feasible segments (there may not be any). Construct a function $d(a)$ in the following way. If, from a, there is no feasible segment with $d > a$, set $d(a) = a$. Otherwise, set $d(a)$ to be the largest value of d among all d's that attain the highest value of α.

Lemma **3.** *Let $[aHd]$ and $[aHd']$ be two feasible segments. If $\alpha' > \alpha$, then $d' > d$.*

Proof. Let F and F' be the distributions associated with $[aHd]$ and $[aHd']$ respectively. By (A.6),

$$\int_a^{d'} [F'(x) - H(x)]dx = 0,$$

It follows that

$$\int_a^{d'} [F(x) - H(x)]dx < 0.$$

Because (A.5) holds for every $b \in (a, d]$, we must conclude that $d' > d$. □

LEMMA **4.** *For every a with $H(a) < 1$, $\alpha(a)$ and $d(a)$ are well-defined.*

Proof. Given that $u_c(c, s) \to 0$ as $c \to \infty$ for every s, it is easy to see that if a feasible solution exists at a, then the supremum over all feasible slopes from a — call it α^* — is finite.

If α^* is itself feasible, we are done. Otherwise there is a sequence $\{d^n\}$ of feasible solutions with associated distributions F^n, and slopes α^n *strictly* increasing to α^*. By Lemma 3, d^n must strictly increase as well, to some finite limit d^*.[3]

We claim that $F(d^n)$ must converge to $F(d^*)$. The only way in which this may not happen is if d^n converges up to some point of discontinuity of F. But it is easy to see that in this case, α^n has to *decline* in n, a contradiction.

Therefore $[aHd^*]$ is a segment. It has slope α^*. It is easy to see that allowability, (A.5) and (A.6) hold for this segment, so it is feasible. □

LEMMA **5.** *Suppose that $a^n \downarrow a$ with $d(a^n) > a^n$ for all n. Then $d(a) > a$.*[4]

Proof. Because $u(c, H(c))$ is strictly concave as long as H is unchanging in c, every feasible segment $[aHd]$ must possess a jump point of H in the *interior* of $[a, d]$. Let c^n be the first jump point within $[a^n, d^n]$. Because there are finitely many jumps and $a^n \geq a^{n+1}$, we can set $c^n = c$ for all n large enough. It follows that $d(a) \geq c > a$. □

LEMMA **6.** *Suppose that $[aHd]$ with slope α is allowable, but* (A.5) *fails at $x = d$. Then the maximum slope $\alpha(a)$ from a strictly exceeds α.*

Proof. Because $[aHd]$ (with associated F) is allowable, $F(x) > H(x)$ just to the right of a, so that (A.5) must hold over this range. Let d_1 be the *largest* value in $[a, d]$ such that (A.5) holds for all $x \in [a, d_1]$. Clearly, $d_1 < d$.

We claim that $H(d_1) > F(d_1)$. Certainly, $H(d_1) \geq F(d_1)$ (if the opposite inequality held, it would contradict the definition of d_1). If equality holds, then because $H(x)$ is flat just to the right of d_1 and F strictly increasing (Lemma 2), $F(x)$ must strictly exceed $H(x)$ for x in an interval just to the right of d_1, once again contradicting the definition of d_1.

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[3]The sequence d^n is bounded because the slopes α^n are bounded away from 0, and [U1] holds.

[4]This assertion is false for arbitrary sequences a^n; consider a distribution H with a unique mass point at a. It is clear that $d(a) = 0$, while $d(a') > 0$ for all $a' < a$.

For any segment $[aHb]$ with corresponding distribution F_b, define

$$I(b) \equiv I_{[aHb]}(b) = \int_a^b [F_b(z) - H(z)]dz.$$

Consider this function on $[d_1, d]$. Because $H(d_1) > F(d_1)$, we have $I(d_1) > 0$, while of course $I(d) < 0$. Moreover, it is easy to verify that $I(b)$ is right-continuous, and jumps upward at every discontinuity point. Therefore there exists a *smallest value* of $b \in (d_1, d)$ for which $I(b) = 0$; call it d^*. Consider the segment $[aHd^*]$ with distribution F^* and slope α^*.

Because $d^* > d_1$, it follows from the definition of d_1 that

$$I_{[aHd]}(d^*) < 0,$$

while at the same time

$$I_{[aHd^*]}(d^*) = 0.$$

It follows from these two expressions that $\alpha^* > \alpha$.

To complete the proof, we show that $[aHd^*]$ is feasible. It is certainly allowable because $\alpha^* > \alpha$. Next, $I(d^*) = 0$, so (A.6) holds. It remains to verify (A.5). Given $\alpha^* > \alpha$, (A.5) certainly holds for all $b \in [a, d_1]$, so in the rest of the proof we focus on $b \in (d_1, d^*)$. Suppose on the contrary that for some such b,

$$I_{[aHd]}(b) = \int_a^b [F^*(x) - H(x)]dx < 0.$$

There are three possibilities.

(a) $F^*(b) > H(b)$. In this case, there exists $c \in [d_1, b)$ such that $F^*(c) = H(c)$, and $F^*(x) > H(x)$ for all $x \in (c, b]$.[5] It is obvious that $F^*(x)$ coincides (on $[a, c]$) with the distribution associated with the segment on $[aHc]$, and we must conclude that $I(c) < 0$.[6]

(b) $F^*(b) < H(b)$. In this case, there exists $c \in (b, d^*]$ such that $F^*(c) = H(c)$, and $F^*(x) < H(x)$ for all $x \in [b, c)$.[7] Again, it is obvious that $F^*(x)$ — suitably extended on $[b, c]$ — is the distribution associated with the segment $[aHc]$, so that $I(c) < 0$.[8]

(c) $F^*(b) = H(b)$. Set $c = b$; then $F^*(x)$ is the distribution associated with the segment $[aHc]$, and $I(c) < 0$.

In all three cases, then, there exists $c < d^*$ with $I(c) < 0$. But now — noting that $[aHc]$ is allowable — we can apply the same argument as above for $I(b)$ on $[d_1, c]$ to conclude that there exists $\hat{d} \in [d_1, c)$ with $I(\hat{d}) = 0$. But $\hat{d} < d^*$, which contradicts our choice of d^* as the *smallest* value of $b \in [d', d]$ with $I(b) = 0$.

Therefore we must conclude that (A.5) holds for all $b \in (d_1, d^*)$, which completes the proof. □

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[5]This follows from the observations that (A.5) holds for F^* at d_1 and fails at b, and the right-continuity of H.

[6]$\int_a^b [F^*(x) - H(x)]dx < 0$, while $F^*(x) > H(x)$ for all $x \in (c, b]$.

[7]Note that (A.6) holds for F^* at d_1, so the inequality $F^*(x) < H(x)$ must be reversed somewhere in the interval $[b, d^*)$.

[8]After all, $I(d^*) = \int_a^{d^*} [F^*(x) - H(x)]dx = 0$, while $F^*(x) > H(x)$ for all $x \in (c, d^*]$. Therefore $0 = \int_a^{d^*} [F^*(x) - H(x)]dx > \int_a^c [F^*(x) - H(x)]dx = I(c)$.

Now construct a utility function μ^* on consumption alone. In the sequel this will be the unique reduced-form utility [RFU] satisfying [R1]–[R3] for the distribution H.

The construction is always in one of two phases: "on the curve" or "off the curve", referring informally to whether we are "currently" following the original function $u(x, H(x))$ or are changing it in some way. Start at $a = 0$, and follow the original function $u(a, H(a))$ as long as $d(a) = a$ (stay "on the curve"); at the first point at which $d(a) > a$ — and Lemma 5 guarantees that if any $d(a) > a$ exists, there is a *first* such a — move along the line segment $[aHd(a)]$ (go "off the curve"). Repeat the same process once back again "on the curve" at $d(a)$.[9] The reduced-form function — call it μ^* — will be made up of affine segments in the regions in which $d(a) > a$, and when $d(a) = a$, of stretches that locally coincide with $u(c, H(c))$. It is easy to see that there are at most finitely many affine segments involved in the construction of μ^*.[10]

When H has finite support, our construction generates a reduced-form utility for H:

Lemma 7. *μ^*, as given by the construction, satisfies* [R1]–[R3].

Proof. It is very easy to see that μ^* satisfies [R1] and [R3]. The heart of the argument is the verification of [R2]. Begin with concavity. We proceed in steps.

Step 1. *No segment is followed by a stretch of $u(x, H(x))$ with higher slope.* Consider a terminal point d of a segment that is used in the construction of μ^*. It must hit $u(c, H(c))$ "from below"; i.e., $\alpha \geq u_c(d, H(d))$. Because $H(x)$ is locally constant at the value $H(d)$ to the right of d, the step follows.

Step 2. *No stretch of $u(x, H(x))$ is followed by a segment with higher slope, or a discontinuity.* Suppose this is false, so that a stretch of $u(x, H(x))$ ending at a^*, with $H(x)$ constant at its step-function value — call it h — is followed by an allowable segment $[a^*Hd^*]$ with slope α strictly higher than $u_c(a^*, h)$. This includes a possible discontinuity at a^*, which may be viewed for the sake of the argument that follows as a segment with infinite slope, in which case $d^* = a^*$.

Because $\alpha > u_c(a^*, h)$, it is easy to see that for $d > d^*$ but close to it, $[a^*Hd]$ is allowable as well, with associated slope $\alpha' > u_c(a^*, h)$, but (A.5) must fail at d.[11] Because $\alpha' > u_c(a^*, h)$, we can use continuity to assert the existence of $a < a^*$ such that the segment $[aHd]$ is allowable, with (A.5) again failing at d. By Lemma 6, a feasible segment $[aHd']$ exists from a. This contradicts the construction of μ^*, which demands that a feasible segment be followed whenever available, at any point such as a where the construction is "on the line".

Step 3. *No adjacent segments $[aHd]$ and $[a'Hd']$ (with $a' = d$) in the construction of μ^*, with $\alpha < \alpha'$.* For suppose $\alpha < \alpha'$. Extend $[aHd]$ with same slope to form segment $[aHd'']$, where d'' is

[9] It could be that $d(d(a)) > d(a)$ so that we immediately leave the curve again at $d(a)$.

[10] Indeed, the number of affine segments cannot exceed the number of atoms in H.

[11] This follows from three observations: (i) $I_{[a^*Hd^*]}(d^*) = 0$ (this includes the case of a discontinuity at a^*), (ii) $u(c, H(c))$ is strictly concave just to the right of d^*, and (iii) by feasibility, the segment $[a^*Hd^*]$ hits $u(c, H(c))$ "from below" at d^*.

the first number greater than d' with $u(a, H(a)) + \alpha(d'' - a) = u(x, H(x))$.[12] The conditions of Lemma 6 are satisfied for $[aHd'']$. It follows that $\alpha(a) > \alpha$, a contradiction.

It is immediate that these three steps complete the proof of concavity. □

Now we prove that a larger class of consumption budget distributions all admit reduced-form utilities satisfying [R1]–[R3]. We begin by proving the uniqueness of such functions.

Lemma 8. *For every distribution of consumption budgets H, there is at most one RFU.*

Proof. Suppose on the contrary that μ and μ' are two distinct RFUs for H, both satisfying the needed properties. Let a^* be the infimum value of c such that $\mu^*(c) \neq \mu(c)$. Without loss of generality, $\mu(a^*) \geq \mu'(a^*)$ and $\mu(c) > \mu'(c)$ for all $c \in (a^*, b)$, for some $b > a$.[13] Let F and F' be the associated distribution of consumption realizations. Then $F(c) = F'(c)$ for all $c < a^*$, $F(a^*) \geq F'(a^*)$, and $F(c) > F'(c)$ for all $c \in (a^*, b)$, so that

$$\int_0^c F(x)dx > \int_0^c F'(x)dx \geq \int_0^c H(x)dx.$$

for all $c \in (a^*, b)$, where the second inequality follows from second-order stochastic dominance (Condition R1 applied to F'). Using [R3] applied to F, we must conclude that μ is affine over (a^*, b). Let $[a^*, d]$ be the maximal interval over which μ is affine. Then

$$\int_0^d \Big[F(t) - H(t)\Big]dt = 0.$$

At the same time, because μ is linear and μ' is concave on $(a^*, d]$ (Condition R2), it must be that $\mu(c) > \mu'(c)$ for all $c \in (a^*, d]$, so that $F(c) \geq F'(c)$ for all $c \in [0, d]$ with strict inequality on $(a^*, d]$. It follows that

$$\int_0^d \Big[F'(t) - H(t)\Big]dt < 0,$$

which contradicts the fact that F' must satisfy [R1]. □

To complete the proof of the first part of the Proposition, we use an extension argument.

Consider the collection $\mathcal{H}$ of all cdfs H on $[0, M]$, where $M < \infty$. We seek the existence of a mapping ϕ that assigns to each $H \in \mathcal{H}$ its unique RFU μ. Let $\mathcal{H}^{\text{fin}}$ be the subspace of $\mathcal{H}$ containing all H with finite support. Then $\mathcal{H}^{\text{fin}}$ is dense in $\mathcal{H}$ in the weak topology. Lemma 7 tells us that the mapping ϕ is already well-defined on $\mathcal{H}^{\text{fin}}$. The required extension is provided below.

Lemma 9. *Let G^n converge weakly to G, and (a^n, b^n) to (a, b). Then*

$$\int_{a^n}^{b^n} G^n(x)dx \to \int_a^b G(x)dx \text{ as } n \to \infty.$$

[12] Such a d'' must exist, because $u(a, H(a)) + \alpha(d' - a) < u(d', H(d'))$ (use $\alpha < \alpha'$), while $u_c(c, 1) \to 0$ as $c \to \infty$, by [U1].

[13] That such an interval must exist follows from the concavity of both μ and μ'.

Proof. Note that

$$\int_{a^n}^{b^n} G^n(x)dx = \int_a^b G^n(x)dx + \int_{a^n}^{a} G^n(x)dx + \int_b^{b^n} G(x)dx.$$

Weak convergence and monotonicity of F ensures that $G^n(x)$ converges to $G(x)$ a.e. on $[a, b]$, so that by dominated convergence,

$$\int_a^b G^n(x)dx \to \int_a^b G(x)dx,$$

while at the same time,

$$\int_{a^n}^{a} G^n(x)dx \to 0 \text{ and } \int_{b^n}^{b} G^n(x)dx \to 0$$

as $n \to \infty$. □

Lemma **10.** *Consider any sequence $H^n \in \mathcal{H}$ converging weakly to $H \in \mathcal{H}$, and suppose that there exist associated RFUs μ^n, along with distributions of realized consumption F^n. If F^n converges weakly to F, then μ given by $\mu(c) \equiv u(c, F(c))$ for all c is the RFU for H.*

Proof. We verify that μ satisfies [R1]–[R3], given H. First we show that F (second-order stochastically) dominates H. Because F^n dominates H^n, it follows that

$$\int_0^c [F^n(x) - H^n(x)]dx \geq 0$$

for every c. It is immediate (e.g., an immediate corollary of Lemma 9) that

$$\int_0^c [F(x) - H(x)]dx \geq 0$$

for every c, so that [R1] is verified.

Next, we claim that μ is concave. For if not, there exist $c_1, c_2 \in (0, \infty)$ and $\theta \in (0, 1)$ such that

$$\mu(\bar{c})(\bar{c}, F(\bar{c})) < \theta u(c_1, F(c_1)) + (1 - \theta)u(c_2, F(c_2)),$$

where $\bar{c} \equiv \theta c_1 + (1 - \theta)c_2$. Now, there exist sequences $\bar{c}^m \downarrow \bar{c}$, $c_1^m \downarrow c_1$, $c_2^m \downarrow c_2$, and $\theta^m \to \theta$, where $\bar{c}^m = \theta^m c_1^m + (1 - \theta^m)c_2^m$, and $\bar{c}^m$, c_1^m, and c_2^m are all points of continuity of F. Because F is right continuous, it must be that

$$F(\bar{c}^m) \to F(\bar{c}), F(c_1^m) \to F(c_1) \text{ and } F(c_2^m) \to F(c_2) \text{ as } m \to \infty.$$

It follows that there exists m such that

$$u(\bar{c}^m, F(\bar{c}^m)) < \theta^m u(c_1^m, F(c_1^m)) + (1 - \theta^m)u(c_2^m, F(c_2^m)). \tag{A.7}$$

Moreover, because F^n converges weakly to F,

$$F^n(\bar{c}^m) \to F(\bar{c}^m), F^n(c_1^m) \to F(c_1^m), \text{ and } F^n(c_2^m) \to F(c_2^m), \text{ as } n \to \infty.$$

Using this information in (A.7), we see that there exists n such that

$$u(\bar{c}^m, F^n(\bar{c}^m)) < \theta^m u(c_1^m, F^n(c_1^m)) + (1 - \theta^m)u(c_2^m, F^n(c_2^m)),$$

but this contradicts the concavity of μ^n, and so establishes [R2].

Finally, to establish [R3], we must show that if

$$\int_0^c F(x)dx > \int_0^c H(x)dx \text{ for all } c \in (\underline{c}, \bar{c}), \tag{A.8}$$

then $\mu(c) = u(c, F(c))$ must be affine on $(\underline{c}, \bar{c})$. To this end, we first claim that if (A.8) holds, then for all $\epsilon > 0$, there exists N such that $n > N$ implies

$$\int_0^c F^n(x)dx > \int_0^c H^n(x)dx \text{ for all } c \in [\underline{c} + \epsilon, \bar{c} - \epsilon]. \tag{A.9}$$

Suppose the claim is false; then there is $\epsilon > 0$ and a sequence $c^n \in [\underline{c} + \epsilon, \bar{c} - \epsilon]$ such that

$$\int_0^{c^n} H^n(x)dx = \int_0^{c^n} F^n(x)dx.$$

Pick a subsequence such that $c^n \to c \in [\underline{c}+\epsilon, \bar{c}-\epsilon]$ as $n \to \infty$. By Lemma 9, we must conclude that

$$\int_0^c H(x)dx = \int_0^c F(x)dx,$$

which contradicts (A.8), and so verifies (A.9). Because $\mu^n(c) = u(c, F^n(c))$ satisfies [R3], it follows that there exist $a^n > 0$ and $\alpha^n > 0$ such that

$$\mu^n(c) = a^n + \alpha^n(c - \underline{c})$$

for all $c \in [\underline{c} + \epsilon, \bar{c} - \epsilon]$. Because μ satisfies [R2], it is concave and therefore continuous on $[\underline{c} + \epsilon, \bar{c} - \epsilon]$ and consequently, so is F. Therefore $u(c, F^n(c)) \to u(c, F(c))$ as $n \to \infty$, and so there must exist $a > 0$ and $\alpha > 0$ such that $a^n \to a$, $\alpha^n \to \alpha$ and

$$u(c, F(c)) = a + \alpha(c - \underline{c})$$

for all $c \in [\underline{c}+\epsilon, \bar{c}-\epsilon]$. Since $\epsilon > 0$ is arbitrary and $u(., F(.))$ is continuous, this must then hold for all $c \in [\underline{c}, \bar{c}]$, and the proof is complete. □

Lemma **11.** *Every sequence in $\phi(\mathcal{H}^{\text{fin}})$, the space of all RFUs for distributions in $\mathcal{H}^{\text{fin}}$, admits a weakly convergent subsequence.*

Proof. Pick any distribution $H \in \mathcal{H}^{\text{fin}}$, and let μ be the RFU associated with it, with distribution function F. If the supremum of the support of F exceeds M, then it is easy to see, using [R3], that μ must be affine beyond M. Let a be the starting point of the corresponding segment $[aHd]$. Because $[aHd]$ must be allowable, the slope of the segment can be no smaller than $u_c(a, H(a))$, and therefore no smaller than the minimum h of all these partial derivatives. It follows that the affine segment must end before M', where M' is the intersection of the affine ray with slope h emanating from $u(M, 0)$ with the function $u(c, 1)$. Notice that $F(M') = 1$, and that M' is independent of the particular distribution $H \in \mathcal{H}^{\text{fin}}$. □

Now proceed as follows. Pick any budget distribution $H \in \mathcal{H}$. We know that there is a sequence $H^n \in \mathcal{H}^{\text{fin}}$ that converges weakly to H. Each H^n has its (unique) RFU μ^n, with associated distribution of realized consumptions F^n. By Lemma 11, the sequence $\{F^n\}$ admits a convergent subsequence that weakly converges to some distribution F. By Lemma 10, this is an RFU for H. By Lemma 8, it is the only one, so the proof of Proposition 1(i) is complete.

To prove Proposition 1(ii), let $\{H_t, F_t\}$ be a equilibrium sequence of consumption budgets and realizations. We observe that $H_t(0) = 0$ for all $t \geq 0$. This follows easily from Assumption 3 combined with the unbounded steepness of utility $u(c, \bar{F}_t(c))$ at every date (recall that $u_c(c,s) \to \infty$ as $c \to 0$ for any s); initial wealth positive implies optimal consumption is positive at all dates.[14]

It follows that $F_t(0) = 0$ for all t. For if $F_t(0) > 0$, there must be a positive measure of individuals who take gambles that have a positive probability of generating 0. All of them have strictly positive budgets, so each such person would be better off by replacing her gamble by one that avoids 0, which yields a status payoff $\bar{F}_t(0)$ that is discontinuously lower than $F_t(0)$. This contradicts that fact that we have an equilibrium to begin with.

It is now easy to prove [R1]–[R3], noting that [R2] is a direct corollary of Lemma 1. □

Proof of Corollary 1. Simply verify that the conditions in the statement of the corollary correspond to [R1]–[R3] when the consumption budget is degenerate, and then apply Proposition 1(i) for the case of a degenerate distribution H.[15]

To establish the very last assertion in the corollary, suppose that b is increased. Then, by (4) in the main paper, the new distribution function F must have a higher mean. It is easy to conclude that a and d must both increase. By the concavity of $u_c(.,0)$, $\alpha = u_c(a, 0)$ cannot increase. □

Proof of Proposition 2. Let F^* be any steady state distribution of consumption. Then we know that the RFU $\mu^*(c) = u(c, F^*(c))$ is concave. By Assumption 2 and the fact that $\mu^*(c) \geq u(c, 0)$ for all c, μ^* has unbounded steepness at 0.

Consider the following problem:

$$\max \sum_{t=0}^{\infty} \delta^t \mu^*(b_t(i))$$

subject to

$$w_t(i) = b_t(i) + k_t(i)$$

for all t, and

$$w_{t+1}(i) = f(k_t(i))$$

for all t, with $w_0(i)$ given. Because μ^* is concave and f is strictly concave, this problem has a unique optimal investment strategy associated with it, assigning an investment k and consumption budget b for every starting wealth w.

One can check (see, e.g., Mitra and Ray (1984)) that for each individual, k_t must converge to a steady state. Because μ^* has unbounded steepness at 0, it is easy to see that if initial wealth is positive, this steady state value must equal k^*, defined by $\delta f'(k^*) = 1$. Finally, F^* must be the distribution associated with the degenerate consumption budget $b^* = f(k^*) - k^*$.

[14]We record this observation formally in Lemma 13 below.

[15]For part (iii) in particular, use the fact that $u_c(c,s) \to \infty$ as $c \to 0$ to argue that $a > 0$, and the concavity of the RFU to argue that $\alpha = u_c(a, 0)$.

That verifies that if there is any steady state with positive wealth for all individuals, it must be the one described in the Proposition.

We need to complete the formalities of showing that this outcome is indeed a steady state. All we need to do is exhibit an optimal consumption policy. If the consumption budget b at any date equals $b^* \equiv f(k^*) - k^*$, take a fair bet with cdf F^*, consuming the proceeds entirely.

We already know that the investment policy is optimal. So is the consumption policy, because utilities are linear in realized consumption over the support of F^*. □

Proof of Proposition 3. In what follows, we assume throughout that Assumptions 1 and 2 hold, and that the initial wealth distribution has bounded support with infimum wealth positive (as assumed in the statement of the proposition).

Part (i): Convergence.

We first review the main argument to follow. The first step is Lemma 12, which is based on the Mitra-Zilcha turnpike theorem (see Mitra and Zilcha (1981) and the version we use, which is Mitra (2009)). It states that in any equilibrium, the paths followed by all agents converge *to one another*. Lemmas 13 and 14 ensure that convergence occurs to some common sequence which has a strictly positive limit point (over time). The second step is Lemma 16, which states that when all stocks cluster sufficiently close to this common limit point, a bout of endogenous risk-taking must force all consumption budgets to lie in the same affine segment of the "reduced-form" utility function μ at that date. Lemma 17 states that all individual capital stocks must fully coincide thereafter. The remainder of the proof shows that this common path must, in turn, converge over time to k^*, with consumption distributions converging to F^*, the unique cdf associated (as in Corollary 1) with $d^* = f(k^*) - k^*$.

Lemma **12.** *In any equilibrium,* $\sup_{i,j} |k_t(i) - k_t(j)| \to 0$ *and* $\sup_{i,j} |b_t(i) - b_t(j)| \to 0$ *as* $t \to \infty$*.*

Proof. Pick numbers $\underline{w} > 0$ and $\bar{w} > 0$ such that $\underline{w} \leq w \leq \bar{w}$ for every initial wealth w. Denote by $\{\underline{k}_t\}$ the optimal path followed by a hypothetical individual with initial wealth $\underline{w}$, and by $\{\bar{k}_t\}$ the optimal path followed by a hypothetical individual with initial wealth $\bar{w}$. By a non-crossing argument (see Mitra (2009), Proposition 1),

$$\underline{k}_t \leq k_i(t) \leq \bar{k}_t \tag{A.10}$$

for all individuals i and dates t. Moreover, Lemma 1 together with our assumptions on f guarantee that all[16] the assumptions of Theorem 2 in Mitra (2009) are satisfied, so that

$$\bar{k}_t - \underline{k}_t \to 0 \text{ as } t \to \infty. \tag{A.11}$$

Combining (A.10) and (A.11), we must conclude that for any two optimal paths for individuals i and j from the given set of initial wealths,

$$\sup_{i,j} |k_t(i) - k_t(j)| \to 0 \text{ as } t \to \infty. \tag{A.12}$$

which proves the first part of the assertion. Using the fact that for every i, $b_t(i) = f(k_t(i)) - k_{t+1}(i)$, the second part follows directly from (A.12). □

[16] Mitra assumes that $f(0) = 0$. It can be checked that this assumption is not needed for any of the propositions in Mitra (2009) that we invoke.

Lemma **13.** *In any equilibrium, for any i with initial wealth strictly positive, $b_t(i) > 0$ for every t and* $\limsup_t b_t(i) > 0$.

Proof. Because $u_c(c, s) \to \infty$ as $c \to 0$ (for any s), the same is true of the steepness of the RFU $\mu_t(c)$ as $c \to 0$. Moreover, μ_t is concave. The observation that $b_t(i) > 0$ for all t (when initial wealth is positive) now follows from a standard argument.

To prove the second part of the lemma, suppose, on the contrary, that $b_t(i) \to 0$ as $t \to \infty$. Then it must be that $w_t(i) \to 0$ as $t \to \infty$.[17] It follows from Assumption 1 that there is T such that $\delta f'(k_t(i)) \geq \rho > 1$ for all $t \geq T$. For each such t, the Euler equation assures us that

$$\beta_t(b_t(i)) = \delta f'(k_t(i))\beta_{t+1}(b_{t+1}(i)) \geq \rho\beta_{t+1}(b_{t+1}(i)),$$

where β_t and β_{t+1} are appropriately chosen supports to the functions μ_t and μ_{t+1} respectively. Because $\rho > 1$, it follows that the right-hand derivatives of μ_t evaluated at $b_t(i)$ — $\mu_t^+(b_t(i))$ — are bounded in t (in fact, these derivatives converge to 0). On the other hand,

$$\mu_t^+(b_t(i)) \geq u_b\left(b_t(i), F_t(b_t(i))\right),$$

but the latter term goes to infinity as $t \to \infty$, because $b_t(i) \to 0$. This contradiction establishes the lemma. □

Lemma **14.** *There exists $\sigma > 0$ so that for every $\epsilon > 0$, there is a date T*

$$b_T(i) \in [\sigma - \epsilon, \sigma + \epsilon] \tag{A.13}$$

for all i.

Proof. Let $\sigma \equiv \limsup_t b_t$, where b_t is the average of all $b_t(i)$'s. Then $\sigma > 0$, by Lemma 13. By Lemma 12, $b_t(i)$ must converge — uniformly in i, and along a common subsequence independent of i — to σ for all i. But this means that for every $\epsilon > 0$, there exists a date T such that $b_T(i) \in [\sigma - \epsilon, \sigma + \epsilon]$ for all i. □

Lemma **15.** *For any $\sigma > 0$, there exists $\psi > 0$ such that for all $\epsilon < \sigma/2$,*

$$F_t(\sigma + \epsilon) - F_t(\sigma - \epsilon) \leq \psi\epsilon \tag{A.14}$$

independently of t.

Proof. Denote by m the smallest value of $u_s(c, s)$ for $c \in [\sigma/2, 3\sigma/2]$ and $s \in [0, 1]$. By Assumption 2, $m > 0$. Let $\psi \equiv 4u(\sigma/2, 1)/\sigma m$. Then by the concavity of μ_t, for all $\epsilon < \sigma/2$,

$$\begin{aligned}
\psi\epsilon m \geq 4\mu_t(\sigma/2)/\sigma \geq 2\epsilon\mu_t^+(\sigma/2) \geq 2\epsilon\mu_t^+(\sigma - \epsilon) &\geq \mu_t(\sigma + \epsilon) - \mu_t(\sigma - \epsilon) \\
&= u(\sigma + \epsilon, F_t(\sigma + \epsilon)) - u(\sigma - \epsilon, F_t(\sigma - \epsilon)) \\
&\geq u(\sigma - \epsilon, F_t(\sigma + \epsilon)) - u(\sigma - \epsilon, F_t(\sigma - \epsilon)) \\
&\geq m[F_t(\sigma + \epsilon) - F_t(\sigma - \epsilon)],
\end{aligned}$$

which is what we needed to prove. □

We now combine Lemmas 14 and 15 to prove

[17] For if $\limsup w_t(i) > 0$ while $b_t(i) \to 0$, we contradict the assumption that u is strictly increasing in c.

Lemma **16.** *There exists a date T such that for every i, $b_T(i)$ belongs to the interior of the same affine segment of μ_T; in particular, $\mu_T'(b_T(i))$ is a constant independent of i.*

Proof. Fix σ as given in Lemma 14, and then ψ from Lemma 15. Pick $\epsilon' \in (0, \sigma/2)$ so that $\psi\epsilon' < 1/2$. By Lemma 15,

$$F_t(\sigma - \epsilon') + [1 - F_t(\sigma + \epsilon')] > 1/2. \tag{A.15}$$

Choose $\epsilon < \epsilon'/7$. By Lemma 14, there is a date T so that (A.13) is satisfied. We claim that at that date, some individual must take a fair bet F with $\inf F < \sigma - \epsilon \leq \sigma + \epsilon < \sup F$.

Suppose the claim is false. Observe from (A.15) that either $F_t(\sigma-\epsilon')$ or $1-F_t(\sigma+\epsilon')$ exceeds $1/4$. Suppose the former. Then there is *some* individual with consumption budget in $[\sigma - \epsilon, \sigma + \epsilon]$ who accepts a bet F with $F(\sigma - \epsilon') > 1/4$, so that $\inf F < \sigma - \epsilon$. The expected utility of this individual is

$$\begin{aligned} \int \mu_T(c) dF(c) &\leq (1/4)\mu_T(\sigma - \epsilon') + (3/4)\mu_T(\sigma + \epsilon) \\ &\leq \mu_T\left(\sigma + \frac{3\epsilon - \epsilon'}{4}\right) \\ &< \mu_T(\sigma - \epsilon) \end{aligned} \tag{A.16}$$

where the first inequality follows from $\inf F < \sigma - \epsilon$ and our supposition that the claim is false, so that $\sigma + \epsilon \geq \sup F$, the second inequality follows from the concavity of μ_T, and the last inequality from our choice of ϵ. But (A.16) contradicts the supposed willingness of the individual to accept such a bet.

Now suppose, on the other hand, we have $1 - F_t(\sigma + \epsilon') > 1/4$. Then there is some individual with consumption budget b in $[\sigma - \epsilon, \sigma + \epsilon]$ who accepts a fair bet F with $1 - F(\sigma + \epsilon') > 1/4$, so that $\sigma + \epsilon < \sup F$. If the claim is false, then $\inf F \geq \sigma - \epsilon$ and

$$\int c dF(c) \geq (1/4)(\sigma + \epsilon') + (3/4)(\sigma - \epsilon) > \sigma + \epsilon \geq b,$$

which contradicts the fairness of the bet.

This proves the claim: at date T, some individual must take a fair bet F with $\inf F < \sigma - \epsilon \leq \sigma + \epsilon < \sup F$. Because μ_T satisfies [R3], it must be affine on $[\inf F, \sup F]$. □

Lemma **17.** *For every date $t \geq T + 1$, where T is given by Lemma 16, the wealths, investments and consumption budgets of all agents must fully coincide.*

Proof. By Lemma 16, we see that $\mu_T'(b_T(i)) = \alpha_T$ for all i, where $\alpha_T > 0$ is independent of i. Let $V_{T+1}(w)$ be the value function at date $T + 1$; it is concave because μ_t is concave for all t. Therefore $V_{T+1}(f(k))$ is *strictly* concave.

Note that $b_T(i) > 0$ for all i by Lemma 13. So using the Bellman equation between dates T and $T + 1$, and writing the first order condition, we see that

$$\alpha_T \geq \delta\beta_i(k_T(i)), \text{ with equality if } k_T(i) = 0$$

for every agent i, where $\beta_i(k)$ denotes some supporting hyperplane to $V_{T+1} \circ f$ at k. It follows that the wealths of all agents fully coincide at date $T+1$. It is easy to see that optimal programs are unique starting from any initial wealth at any date,[18] so all wealths, investments, and consumption budgets must fully coincide from date $T+1$ onward. □

In what follows, we consider only dates $t > T$. By Lemma 17, the equilibrium program has common values at all dates thereafter: (w_t, b_t), where all these values are strictly positive. By Proposition 1 and Corollary 1, the distribution F_t is also fully pinned down at all these dates. Denote by α_t the corresponding slopes of the affine segments of μ_t, given by (5); these too are all strictly positive.

Lemma 18. *Suppose that for some $t \geq T+1$, we have $k_t \leq k_{t+1}$ and $\alpha_t \leq \alpha_{t+1}$. Then $k_s \leq k_{s+1}$ for all $s \geq t$.*

Proof. Suppose that for some t, we have $k_t \leq k_{t+1}$ and $\alpha_t \leq \alpha_{t+1}$. If $k_{t+1} = 0$, then so is k_t, and then $b_{t+1} = f(0) \geq b_{t+2}$, so that by then very last part of Corollary 1, $\alpha_{t+1} \leq \alpha_{t+2}$. Otherwise, $k_{t+1} > 0$, and using the Euler equations for utility maximization (with appropriate inequality at date t, and with equality at date $t+1$) and combining them with the concavity of f,

$$\frac{\alpha_t}{\alpha_{t+1}} \geq \delta f'(k_t) \geq \delta f'(k_{t+1}) = \frac{\alpha_{t+1}}{\alpha_{t+2}},$$

which permits us to conclude that $\alpha_{t+1} \leq \alpha_{t+2}$ once more. Using again the very last part of Corollary 1, we must also conclude that $b_{t+1} \geq b_{t+2}$. Therefore

$$k_{t+1} = f(k_t) - b_{t+1} \leq f(k_{t+1}) - b_{t+2} = k_{t+2}.$$

We have therefore shown unambiguously that $\alpha_{t+1} \leq \alpha_{t+2}$ and $k_{t+1} \leq k_{t+2}$. We can continue the recursive argument indefinitely to show that $k_s \leq k_{s+1}$ for all $s \geq t$. □

Lemma 19. *The common sequence of investments $\{k_t\}$, defined for $t \geq T+1$, must converge to k^*, which solves $\delta f'(k^*) = 1$.*

Proof. First we establish convergence. If the sequence $\{k_t\}$ is either eventually nondecreasing or eventually nonincreasing, it must converge. Otherwise, there is some date $t \geq T+1$ with $k_t \geq k_{t+1} < k_{t+2}$. Then

$$b_{t+1} = f(k_t) - k_{t+1} > f(k_{t+1}) - k_{t+2} = b_{t+2},$$

which implies (by Corollary 1) that $\alpha_{t+1} \leq \alpha_{t+2}$. But now all the conditions of Lemma 18 are satisfied, so that in this case $\{k_t\}$ must eventually be nondecreasing. By Assumption 1, k_t is bounded and so must converge. It follows that b_t and therefore α_t also converge. Passing to the limit using the Euler equations, and noting from Assumption 1 that $\delta f'(0) > 1$, we must conclude that $\lim k_t = k^*$. □

The proof of Proposition 3(i) then proceeds as follows. Lemma 16 assures us that there exists a date T at which consumption budgets $b_T(i)$ belong to the same affine segment of μ_T for every i. Lemma 17 states that for every date $t \geq T+1$, the wealths, investments and consumption budgets of all agents must fully coincide. Lemma 19 states that the common sequence of investments $\{k_t\}$, defined for $t \geq T+1$, must converge to k^*, which solves $\delta f'(k^*) = 1$.

[18] The optimization problem facing each individual is strictly concave.

At the same time, Corollary 1 asserts that for all $t \geq T+1$, the equilibrium distribution of consumptions must be the unique cdf associated with the common consumption budget b_t, where "association" is defined (and uniqueness established) in Proposition 1. Therefore the sequence of consumption distributions must converge to the unique cdf associated with $b^* = f(k^*) - k^*$. This is the unique steady state of Proposition 2, so the proof is complete. □

Part (ii): Existence will follow as a corollary of Proposition 6 below. □

We turn now to consideration of the special model of Section 4. For this purpose, we specialize to the case of $u(s)$ and adopt Assumptions 4–6 instead of Assumptions 1–3. It is not hard to see that Lemma 1 remains true in this new setting.

Proof of Proposition 4. Part (i). Suppose that all individuals in a set of unit measure use the policy function (6). Let $\mathbf{G} = \{G_t\}$ be the resulting sequence of wealth distributions. It is obvious that for every date t and for every w in the support of G_t,

$$G_{t+1}\left(f(\delta w)\right) = G_t(w),$$

so that for every w in the support of G_{t+1},

$$G_{t+1}(w) = G_t\left(f^{-1}(w)/\delta\right). \tag{A.17}$$

Lemma **20.** *$u(G_t(w))$ is concave for all dates t on the support of G_t.*

Proof. Because f is increasing and convex, and $f(0) = 0$, $f^{-1}(w)$ is increasing and concave in w. Using (A.17), we therefore see that $u(G_{t+1}(w)) = u(G_t(f^{-1}(w)))$ is concave provided that $u \circ G_t$ is concave. Now proceed recursively from date 0, using Assumption 6. □

Fix a date t. Suppose that a particular individual employs the policy (6) for all dates $s \geq t+1$, and that every other individual employs the policy (6) at all dates. Define $V_{t+1}(w')$ to be the discounted value to our individual under these conditions, starting from wealth w' and date $t+1$. Then status at every $s \geq t+1$ is simply

$$\bar{F}_s(c_s) = G_{t+1}(w'),$$

so that

$$V_{t+1}(w') = (1-\delta)^{-1} u\left(G_{t+1}(w')\right). \tag{A.18}$$

Now suppose that at date t, our individual has starting wealth w, does not randomize, and chooses $k \in [0, w]$. Then her lifetime payoff at that date is given by

$$\begin{aligned} u\left(\bar{F}_t(w-k)\right) + \delta V_{t+1}\left(f(k)\right) &= u\left(\bar{F}_t(w-k)\right) + \delta(1-\delta)^{-1} u\left(G_{t+1}\left(f(k)\right)\right) \\ &= u\left(G_t\left([w-k]/(1-\delta)\right)\right) + \delta(1-\delta)^{-1} u\left(G_{t+1}\left(f(k)\right)\right) \\ &= u\left(G_t\left([w-k]/(1-\delta)\right)\right) + \delta(1-\delta)^{-1} u\left(G_t\left(k/\delta\right)\right), \end{aligned}$$

where the first equality uses (A.18), the second uses the fact that $\bar{F}_t(c) = G_t\left(c/(1-\delta)\right)$ for every $c \geq 0$, and the last uses (A.17).

By Lemma 20, this expression is concave in both w and k so no randomization is necessary. Moreover, given the concavity of $u(G_t(w))$ and the assumption that u is C^1, G_t must have left-hand and right-hand derivatives everywhere ($G_t^-(w)$ and $G_t^+(w)$ respectively), with

$$G_t^-(w) \geq G_t^+(w) \tag{A.19}$$

for all w. So a solution to the first-order condition

$$\begin{aligned} & -u'(r_t)G_t^+\left([w-k]/(1-\delta)\right)(1-\delta)^{-1} + \delta(1-\delta)^{-1}u'(r_{t+1})G_t^-\left(k/\delta\right)\delta^{-1} \geq 0 \\ \geq\ & -u'(r_t)G_t^-\left([w-k]/(1-\delta)\right)(1-\delta)^{-1} + \delta(1-\delta)^{-1}u'(r_{t+1})G_t^+\left(k/\delta\right)\delta^{-1} \end{aligned} \tag{A.20}$$

(where r_s is the resulting status in date s, for $s = t, t+1$) is an optimum. Using (A.19), we see that $k = \delta w$ is indeed a solution to (A.20), so that by the one-shot deviation principle and the fact that t and w are arbitrary, (6) is an equilibrium policy.

Part (ii).[19] Notice that each individual is atomless and therefore has the same intertemporal utility criterion as any other. Because the equilibrium is regular, we see that at any date, the solution to the optimization problem is unique except at countably many wealth levels. But it is easy to see that such a solution cannot admit more than one differentiable selection. Therefore all individuals must use the same savings policy, which we denote by $\{c_t\}$. Given this environment, let $V_t(w)$ be the (lifetime) value to a person with wealth w at date t. By using exactly the same steps as in Part (i), we see that for every w, $c = c_t(w)$ must maximize

$$u\left(G_t\left(c_t^{-1}(c)\right)\right) + \delta(1-\delta)^{-1}u\left(G_t\left(s_t^{-1}(w-c)\right)\right), \tag{A.21}$$

where $s_t(w) \equiv w - c_t(w)$ is also strictly increasing and differentiable, by regularity.

Notice that under our assumptions, $\bar{F}_t(c) = F_t(c)$ for all c. Therefore by Lemma 1, $u(F_t(c))$ is concave in c, and so F_t is differentiable almost everywhere. Consequently, because

$$G_t(w)) = F_t(c_t(w))$$

and c_t is differentiable and strictly increasing, G_t is also differentiable at a.e. w. Using the fact that optimal c and $w - c$ are both strictly increasing in w, we may therefore differentiate the expression (A.21) with respect to c at almost every w, set the resulting expression equal to zero (it is the first-order condition) and cancel common terms all evaluated at the same rank or same wealth to obtain

$$\frac{1}{c_t'(w)} = \frac{\delta}{1-\delta}\frac{1}{s_t'(w)} = \frac{\delta}{1-\delta}\frac{1}{1-c_t'(w)},$$

or $c_t'(w) = (1-\delta)$ for every t and for a.e. w. This completes the proof of the proposition. □

Proof of Proposition 5. We now revert to Assumptions 1–3 instead of Assumptions 4–6. Consider the steady state F^*; we may equivalently express it as a mapping from realized status $s \in [0,1]$ to realized consumption $c^*(s)$ at status s, given by $c^*(s) = (F^*)^{-1}(s)$. If the outcome is Pareto-efficient, that mapping must maximize the integral

$$\int u(c(s), s)ds$$

over all continuous and increasing functions c on $[0,1]$ with $\int c(s)ds = b$. But it is easy to see that a necessary condition for such maximization is that $u_c(c^*(s), s)$ is constant as s varies

[19] We are indebted to a referee for suggesting this line of proof, which is simpler than the one we had.

over $[0,1]$, or equivalently, that

$$u_c(c, F^*(c)) = \lambda \text{ for some } \lambda > 0, \tag{A.22}$$

for all $c \in [a, d]$. Now, recall from (5) that

$$u(c, F^*(c)) = u(a, 0) + \alpha[c - a]$$

for all $c \in [a, d]$. Because $\alpha = u_c(c, 0)$, it follows that $F^{*\prime}(a) = 0$. Consequently,

$$\frac{du_c(c, F^*(c))}{dc}|_{c=a} = u_{cc}(a, F^*(a)) + u_{cs}(a, F^*(a))F^{*\prime}(a) = u_{cc}(a, F^*(a)) < 0,$$

which contradicts (A.22). □

Existence in a More General Model. The analysis that follows establishes existence for a more general model than the one in the paper, in that the production function can have convex segments and there are possibly stochastic shocks to production. It is worth noting that we establish the existence of an equilibrium in Markovian policies, where each individual's current action is independent of her own past actions and of past distributions.

[U.1] The one-period utility function satisfies Assumption 2 in the paper, and is normalized so that $u(c, s) \geq 0$ for all c and s.

Turning to production, we suppose that bequests k_t produce fresh wealth according to a stochastic production function

$$w_{t+1} = f(k_t, \theta_{t+1}),$$

where θ_t is a sequence of random shocks, iid across time and across individuals,[20] with compact support; say $[0,1]$. As per the discussion in the paper, we assume without loss of generality that all random shocks are uninsurable. We maintain the following assumption on the technology:

[F.1] f is continuous. It is continuously differentiable and strictly increasing in k, with $f'(k, \theta) > 0$ for every k and θ, though the derivative is always finite for $k > 0$. f is nondecreasing in θ.

We also place (possibly time-varying) upper bounds on the size of any investment gamble, and therefore on wealth and consumption budgets:

[B.1] There is a sequence $\{M_t\}$ with $M_t < \infty$ for every t and $M_{t+1} \geq f(M_t, 1)$, such that no wealth or investment can exceed M_t at date t.

Notice that if no investment gamble is considered, as in the central model of the paper with strictly concave f, then [B.1] is automatically satisfied.

A Remark on Risk-Taking. Each individual has access to any fair gamble over her wealth, and she can divide the realizations of that gamble over consumption and investment. A special case is as follows: an agent divides realized output into a consumption budget b and an investment budget k, which are expended on *independent* consumption and investment

[20]We will follow imprecise convention and sidestep the issues of independence across a continuum population.

gambles. This "independence property" can be imposed without any essential loss of generality. It is true that the most general gamble is one taken on starting wealth, with realizations divided *ex post* into consumption and investment. (This is tantamount to a gamble with *correlated* realizations across consumption and investment.) However, a moment's reflection indicates there is nothing extra to be gained from this. Say V is the value function defined on investment at some date; then an individual divides wealth w into consumption c and investment x to maximize

$$\mu(c) + V(x),$$

subject to $w = c + x$, where μ is some reduced-form utility defined on consumption alone. Suppose that w is subjected to randomization with the proceeds divided for every realization into c and x. This joint gamble induces marginals on c and x, say with means b and k, where $b + k$ must sum to w. Because utility is additive over time, we can replace the joint gamble by two *independent* gambles on c and x (with the same marginals), and expected utility will be unaffected thereby.

In what follows, we look for (Markovian) equilibria that have this *independence property*: at every date wealth is allocated to consumption and investment budgets, and randomization of either budget (if any) occurs independently. In the paper, all equilibria have the independence property.

RFU revisited. In any equilibrium with the independence property, the "indirect utility function"

$$\mu_t(c) = u(c, F_t(c)),$$

where F_t is the distribution of consumptions at date t, must be a RFU given the distribution of consumption budgets at that date. The following lemma highlights a basic property of RFUs:

Lemma **21.** *For every $M > 0$, there exists $M' < \infty$ such that if the distribution of consumption budgets has support contained in $[0, M]$, the distribution of realized consumptions under the RFU has support contained in $[0, M']$.*

Proof. The same argument as in the proof of Lemma 11 tells us that the current assertion is true if the distribution of consumption budgets has finite support. Using Lemma 10, we can extend the assertion to arbitrary distributions of consumption budgets on $[0, M]$. □

With [B.1] in mind, we can conclude from Lemma 21 that no consumption will exceed M'_t at any date. Our last assumption imposes an "insignificant future" condition using these bounds:

[B.2] For $\{M_t\}$ given by [B.1], and the corresponding sequence $\{M'_t\}$ given by Lemma 21,

$$U_t \equiv \sum_{s=t}^{\infty} \delta^s u(M'_s, 1) \text{ is finite, and converges to 0 as } t \to \infty. \tag{A.23}$$

When arbitrary randomization in investments is permitted, it is possible (in principle) to make unboundedly large investments with very low probabilities; the conditions [F.1] and [B.2] place a bound on the support of such randomizations.

We make two remarks on [B.2]. First, it is automatically satisfied for the central model of the paper, as all wealth, and therefore consumption, is *uniformly* bounded over time. (Simply take M_t to be the maximum of K and largest initial wealth.) Second, in many cases one can make the "more primitive" assumption

$$\text{(A.24)} \qquad \sum_{s=t}^{\infty} \delta^s u(M_s, 1) \text{ is finite, and converges to 0 as } t \to \infty,$$

and deduce (A.23) from it. As an example, take $u(c, s) = h(s)c^\alpha$, where $h(s)$ is a strictly positive, increasing, continuous function, and $\alpha \in (0, 1)$. Then it is easy to see that $u(M', 1) \leq Ku(M, 1)$ for some K independent of M, so that if (A.24) is satisfied, so must (A.23) be.

We now state and prove

Proposition **6.** *There exists an equilibrium.*

Proof. It will be useful to break up the study of equilibria (satisfying the independence property) into two parts. In the first part, we presume that a sequence $\boldsymbol{\mu} = \{\mu_t\}$ of concave utility functions on consumption *alone* is already given. In the sequel these will be RFUs. We study joint distributions over investments and consumption *budgets* that form a best-response to $\boldsymbol{\mu}$. We call these $\boldsymbol{\mu}$*-optimal sequences*.

A $\boldsymbol{\mu}$-optimal sequence will generate a sequence of consumption budget distributions $\mathbf{H}$. Later — this is the second part — we complete the description by requiring that μ_t be an RFU for each H_t.

We need to consider a space of functions large enough to serve as potential RFUs. Noting that no consumption budget at date t can exceed M_t, define M'_t as the corresponding bound for realized consumptions, using Lemma 21. Let $\mathcal{U}_t$ be the space of all increasing, concave continuous utility functions such that for every $\mu \in \mathcal{U}_t$,

$$\mu(c) \in [u(c, 0), u(c, 1)]$$

for all $c \in [0, M'_t]$, and $\mu(c) = u(c, 1)$ for $c > M'_t$. Then $\mathcal{U}$, the infinite product of $\mathcal{U}_t$, contains all *sequences* $\boldsymbol{\mu} = \{\mu_t\}$ of continuous and concave one-period utilities, with μ_t drawn from $\mathcal{U}_t$ at every t.

Fix $\boldsymbol{\mu} \in \mathcal{U}$. For $t \geq 0$, define a sequence of value functions $\{W_t\}$, defined on wealth in $[0, M_t]$:

$$W_t(w) \equiv \max_{\{b,\zeta\}} \{\mu_t(b) + \delta \mathbb{E}_\zeta \mathbb{E}_\theta W_{t+1}(f(x, \theta))\},$$

where ζ is a cdf (generating investments x) with mean $k = w - b$, and constrained (by [F2]) to have support within $[0, M_t]$.

Standard contraction mapping arguments guarantee the existence of a unique sequence of such value functions, each continuous.[21] It is obvious that when all randomizations of investments on $[0, M_t]$ are permitted, and μ_t is concave (by assumption), W_t must be concave

[21] For each t, consider the complete space $\mathcal{W}_t$ of all continuous functions on $[0, M_t]$, each bounded by U_t (where U_t is given in (A.23)), and equipped with the sup-norm metric. Now the product space equipped with a suitable metric is complete (because each $\mathcal{W}_t$ is complete). Carry out the contraction argument on the product space to yield the desired result.

for every t. Maximizing the values for every date and initial wealth generates an optimal policy correspondence, one for each date t — call it Γ_t — which collects, for every initial wealth $w \in [0, M_t]$, all the optimal (b, ζ) choices that maximize $W_t(w)$.[22]

This correspondence allows us to define $\boldsymbol{\mu}$-*optimal sequences* $\mathbf{Z} = \{Z_t\}$ of joint distributions of wealth, consumption budgets[23] and capital investments over all dates, given the initial distribution G and the utility sequence $\boldsymbol{\mu}$. Define any such sequence recursively as follows. Let G_t be the distribution of wealth at date t, with support within $[0, M_t]$. Pick any function γ_w that is (a) measurable in w, and (b) for G_t-a.e.w, assigns a probability measure on the set $\Gamma_t(w)$. Such functions, interpretable as regular conditional probabilities (see, e.g., Ash (1972)), exist.[24] Now define the cdf Z_t by

$$Z_t(w, b, x) = \int_0^w \int_{(\tilde{b}, \tilde{\zeta})} \mathbf{1}_{[0,b]}(\tilde{b})\tilde{\zeta}(x) d\gamma_{\tilde{w}}(\tilde{b}, \tilde{\zeta}) dG_t(\tilde{w})$$

where $\mathbf{1}_{[0,b]}$ is the indicator function for the set $[0, b]$.

Uniquely associated with this construction is a consumption distribution H_t, given by

$$H_t(b) = Z_t(\infty, b, \infty)$$

for each b, and an investment distribution X_t, given by

$$X_t(x) = Z_t(\infty, \infty, x).$$

for each x. All these distributions have support within $[0, M_t]$, by our restriction. Complete the recursive step by defining next period's distribution of wealth:

$$G_{t+1}(w) = \int \text{Prob}_\theta[f(x, \theta) \le w] dX_t(x), \tag{A.25}$$

for all w. By our restriction, G_{t+1} will have support contained within $[0, M_{t+1}]$, and the recursion can continue. Begin the iteration from the initial distribution $G_0 = G$.

The assignment of *distributions* of the form γ_w over optimal consumption budgets and investment gambles, given wealth w, is an implicit use of Lyapunov's theorem, which allows us to use the continuum of agents to generate all possible convexifications over choices.

Because our primary interest is in the distribution of consumption, we shall refer to a sequence $\mathbf{H} = \{H_t\}$ of consumption distributions as $\boldsymbol{\mu}$-*optimal* if there is a $\boldsymbol{\mu}$-optimal sequence $\mathbf{Z}$ of joint wealth-consumption-investment distributions, with consumption marginals given by $\{H_t\}$. Denote by $\mathcal{H}(\boldsymbol{\mu})$ the collection of all $\boldsymbol{\mu}$-optimal sequences of consumption distributions.

We now note a useful restriction on the space of possible consumption budget distributions in equilibrium.

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[22] This dynamic programming formulation is the key to establishing that the equilibrium is Markov perfect. Note that the μ_t will also be constructed to depend only on the current distribution of consumption budgets.

[23] We deliberately omit the distribution of consumption *realizations*, which will be embedded in the RFUs $\{\mu_t\}$.

[24] Standard arguments tell us that Γ_t is nonempty-valued and uhc, where we use weak convergence on probability measures. The same properties are inherited by the set Γ_t^*, which assigns to each initial wealth $w \in [0, M_t]$, all the *probability measures* over the set $\Gamma_t(w)$. Therefore a measurable selection γ_w exists from Γ^*.

Lemma **22.** *There exists a strictly positive sequence $\{\beta_t\}$ such that for any $\boldsymbol{\mu}$-optimal sequence $\{H_t\}$ of consumption budget distributions, $b_t \geq \beta_t$ for any consumption budget in the support of H_t.*

Proof. We begin by showing that there exist functions $e(w)$ and $E(w)$ with $0 < e(w) \leq E(w) < w$ such that for any $\boldsymbol{\mu}$-optimal sequence $\mathbf{Z}$, an optimal consumption budget choice at any date t must be sandwiched between these two numbers; i.e., for every t,

$$e(w) \leq b \leq E(w) \text{ for any } (b, \zeta) \in \Gamma_t(w). \tag{A.26}$$

For any $w > 0$, $W_t(w) \leq \sum_{s=t}^{\infty} \delta^t u(M_s, 1) \leq U_t < \infty$, where U_t is given by (A.23). Because W_t is concave, it admits a left derivative W_t^-, and it must be that

$$W_t^-(w) \leq U_t/w. \tag{A.27}$$

At the same time, and again because W_t is concave, it admits a right derivative W_t^+, and

$$W_t^+(w) \geq \min_s u_c(b, s) \geq \min_s u_c(w, s) \tag{A.28}$$

where b is any optimal choice at w.

Pick any $w > 0$, and date t, and consider the problem of choosing b to maximize

$$\mu_t(b) + \delta \mathbb{E}_\theta W_{t+1}(f(k, \theta)).$$

Pick $e(w) \in (0, w)$ such that

$$\min_s u_c(b, s) \geq \delta \frac{U_t}{f(w-b, 0)} \max_\theta f'(w-b, \theta)$$

for all $b \in (0, e(w))$. By Assumptions 2 and [F.1], such an $e(w) > 0$ must exist. Now we claim that any optimal b must be at least as large as $e(w)$. For if not, a tiny increase of Δ in b will create a first-order gain of at least

$$\min_s u_c(b, s)\Delta > \min_s u_c(e(w), s)\Delta$$

while the corresponding drop (by Δ) in k will cause a future first-order value loss of at most

$$\delta \mathbb{E}_\theta W_{t+1}^-(f(k, \theta)) f'(k, \theta)\Delta \leq \delta \frac{U}{f(w - e(w), 0)} \max_\theta f'(w - e, \theta)\Delta,$$

using (A.27), thus contradicting maximization.

For the second bound, pick $E(w) \in (0, w)$ such that

$$\max_s u_c(b, s) \leq \delta[\min_s u_c(f(w-b, 1), s)][\min_\theta f'(w-b, \theta)].$$

Assumptions 2 and [F.2] guarantee that such a number must exist. Now we claim that any optimal b cannot exceed $E(w)$. For if it did, a tiny decrease of Δ in b will create a first-order loss of at most

$$\max_s u_c(b, s)\Delta < \max_s u_c(E(w), s)\Delta$$

while the corresponding increase (by Δ) in k will cause a future first-order value gain of at least

$$\delta \mathbb{E}_\theta W_{t+1}^+(f(k, \theta)) f'(k, \theta)\Delta > \delta[\min_s u_c(f(w - E(w), 1), s)][\min_\theta f'(w - E(w), \theta)]\Delta,$$

using (A.28), once again a contradiction. This establishes (A.26).

Now observe that at any date s, if $w_s > 0$, then so are $w_s - E(w_s)$ and $f(w_s - E(w_s), 0)$, the latter by [F1], so that

$$w_{s+1} = f(k_s, \theta) \geq f(w_s - E(w_s), 0) > 0.$$

Moreover, $\inf w_0 = \underline{w} > 0$. It follows that there exists $\underline{w}_t$ such that $w_t \geq \underline{w}_t$ in any $\boldsymbol{\mu}$-optimal **Z** independently of $\boldsymbol{\mu}$. Define $\beta_t \equiv e(\underline{w}_t) > 0$ to complete the proof. □

This lemma tells us that we can focus only those on those budget distributions with infimum consumption budget at least as great as the β_t identified in Lemma 22. Define, then, $\mathcal{H}_t$ to be the space of consumption budget distributions with support contained in $[\beta_t, M_t]$.

The next observation places a corresponding restriction on RFUs of $H \in \mathcal{H}_t$:

Lemma **23.** *There exists $\eta_t > 0$ such that for any distribution $H \in \mathcal{H}_t$, the associated reduced form μ has $\mu(b) = u(b, 0)$ for all $b \in [0, \eta_t]$.*

Proof. Suppose not; then there is a sequence of distributions $H^n \in \mathcal{H}_t$, a corresponding sequence of reduced-form utilities μ^n, and a sequence $\epsilon^n \to 0$ such that $\mu^n(\epsilon^n) \neq u(\epsilon^n, 0)$ for all n. That is, the distribution of realizations F^n satisfies $F^n(\epsilon^n) \neq H^n(\epsilon^n)$ for all n. It follows from [R3] that $u(c, F^n(c))$ must be locally affine around ϵ^n. That is, $u(c, F^n(c))$ must coincide with a segment of the form $[a^n H^n d^n]$, with $\epsilon^n \in (a^n, d^n)$, for all $c \in [a^n, d^n]$.

Certainly, $a^n \leq \epsilon^n < \beta_t/2$ for all large n, where β_t is given by Lemma 22. However, the termination d^n must be at least as large as β_t.[25]

It follows that

$$\begin{aligned} \mu^n(\beta_t) &\geq u(a^n, H^n(a^n)) + u_c(a^n, H^n(a^n))(\beta_t - a^n) \\ &\geq u_c(a^n, H^n(a^n))\beta_t/2 \end{aligned}$$

for all n large enough. Because $u_c(a^n, H^n(a^n)) \to \infty$ as $n \to \infty$, it follows that $\mu^n(\beta_t) \to \infty$ as $n \to \infty$, which contradicts the fact that $\mu^n(\beta_t) \leq u(\beta_t, 1)$ for all n. □

In view of Lemma 23, let $\mathcal{U}_t^*$ be the space of all functions μ in $\mathcal{U}_t$ such that $\mu(b) = u(b, 0)$ for all $b \in [0, \eta_t]$. Give $\mathcal{U}_t^*$ the topology of uniform convergence. Let $\mathcal{U}^*$ be the infinite product of all the $\mathcal{U}_t^*$'s with the product topology. Let $\mathcal{H}$ be the infinite product of all the H_t's with the product topology. Let Φ be the map that takes a sequence of distributions $\mathbf{H} = \{H_t\}$ in $\mathcal{H}$ and assigns the sequence $\boldsymbol{\mu} = \{\mu_t\}$, where for each t, μ_t is the unique RFU associated with H_t.

Lemma **24.** *Φ maps $\mathcal{H}$ into $\mathcal{U}^*$, and it is continuous.*

Proof. First, observe that for each t, H_t has support contained in $[0, M_t]$. By Lemma 21, the RFU for H_t must have support $[0, M_t']$. Therefore Φ maps $\mathcal{H}$ into $\mathcal{U}$. That Φ additionally maps into $\mathcal{U}^*$ follows from Lemma 23. Lemma 10 tells us that if $H^n \Rightarrow H$ in $\mathcal{H}_t$, and if the sequence of distributions of realized consumptions $\{F^n\}$ associated with the sequence of RFUs $\{\mu^n\}$ weakly converge to F, then μ defined by $\mu(c, F(c))$ must be the RFU for H.

[25] There are *no* consumption budgets below β_t. For F^n to serve as a fair randomization, it must pay out more than β_t for some realizations, and moreover we know that $u(c, F^n(c))$ must be affine over the entire range of the randomization.

However, by Proposition 1, H has just one RFU and the sequence $\{F^n\}$ always has a weakly convergent subsequence (Lemma 11). So the *entire* sequence $\{F^n\}$ must indeed weakly converge.

Now, μ is concave and so it is continuous for $c > 0$, but given Lemma 23 we must conclude that it is continuous everywhere. Therefore F is continuous as well, so F^n must actually converge pointwise, which means that μ^n also converges pointwise to μ. But now it is easy to see that *uniform* convergence is obtained free of charge, because the family $\mathcal{U}_t^*$ is equicontinuous, by the concavity of each of its elements and Lemma 23. Finally, the continuity of Φ in the appropriate product topology follows immediately. □

For each $\boldsymbol{\mu} \in \mathcal{U}^*$, let $\mathcal{H}(\boldsymbol{\mu}) \subseteq \mathcal{H}$ be the collection of all $\boldsymbol{\mu}$-optimal sequences of consumption distributions.

Lemma **25.** *The mapping* $\boldsymbol{\mu} \mapsto \mathcal{H}(\boldsymbol{\mu})$ *on* $\mathcal{U}^*$ *is nonempty-valued, convex-valued, and upperhemicontinuous.*

Proof. The mapping is nonempty-valued because $\Gamma_t(w)$ is nonempty for each t and w. To prove convex-valuedness, pick $\mathbf{H}$ and $\mathbf{H'}$ both in $\mathcal{H}(\boldsymbol{\mu})$. We must show that for any $\lambda \in [0, 1]$, the sequence of consumption distributions $\{H_t^\lambda\}$ given by

$$H_t^\lambda(x) = \lambda H_t(x) + (1 - \lambda)H_t'(x)$$

for every x and t, also belongs to $\mathcal{H}(\boldsymbol{\mu})$.

Let $\{Z_t\}$ and $\{Z_t'\}$ be the $\boldsymbol{\mu}$-optimal sequences of joint wealth-consumption-investment distributions that correspond to $\mathbf{H}$ and $\mathbf{H'}$, with associated sequences of wealth distributions $\{G_t\}$ and $\{G_t'\}$, and investment distributions $\{X_t\}$ and $\{X_t'\}$. Denote with superscript λ the relevant convex combinations.

It is trivial to see that for every t, H_t^λ is the consumption marginal of Z_t^λ, G_t^λ is the wealth marginal of Z_t^λ, and X_t^λ is the investment marginal of Z_t^λ. To complete the claim of convex-valuedness, it remains to show that $\{Z_t^\lambda\}$ is $\boldsymbol{\mu}$-optimal. For every t and for Z_t-a.e. w, the conditional distribution $Z_t(w, \cdot)$ assigns some measure γ on $\Gamma_t(w)$ (and the same is true — using a measure γ' — of Z_t'), but then the convex combination $Z_t^\lambda(w, \cdot)$ clearly does the same (simply use the measure $\lambda\gamma + (1 - \lambda)\gamma'$), for Z_t^λ-a.e. w. Finally, $G_0^\lambda = \lambda G_0 + (1 - \lambda)G_0' = G$, while for all t and w,

$$\begin{aligned} G_{t+1}^\lambda(w) &= \lambda G_{t+1}(w) + (1 - \lambda)G_{t+1}'(w) \\ &= \lambda \int \text{Prob}_\theta[f(x, \theta) \le w] dX_t(x) + (1 - \lambda) \int \text{Prob}_\theta[f(x, \theta) \le w] dX_t'(x) \\ &= \int \text{Prob}_\theta[f(x, \theta) \le w] dX_t^\lambda(x), \end{aligned}$$

so that (A.25) holds. This establishes convex-valuedness.

Finally, we show that $\mathcal{H}(\boldsymbol{\mu})$ is uhc. To this end, let $\boldsymbol{\mu}^n$ be a sequence of utility streams that converges to some limit $\boldsymbol{\mu}$ in the product topology on $\mathcal{U}^*$.[26] Let $\mathbf{H}^n \in \mathcal{H}(\boldsymbol{\mu}^n)$ be a

[26] It is enough to employ the sequential definition of uhc, as the underlying space is metrizable.

corresponding collection of $\boldsymbol{\mu}^n$-optimal consumption distribution sequences. We need to show that every limit point of $\mathbf{H}^n$ is contained in $\mathcal{H}(\boldsymbol{\mu})$.

Recall that there exists a collection of $\boldsymbol{\mu}^n$-optimal joint distribution sequences $\{Z_t^n\}$ such that H_t^n is the consumption marginal of Z_t^n for every n and t. Because the support of Z_t^n is compact for every t (it is contained in $[0, M_t]^3$), we can use a diagonal argument to extract a subsequence (of n) such that

$$Z_t^n \to Z_t$$

weakly for every t, where the sequence of consumption marginals attached to $\{Z_t\}$ is the limit point of interest. It only remains to prove that at every date t and at each (i.e., Z_t-a.e.) w, the conditional distribution of Z_t on (b, x) can be generated by a probability measure γ on $\Gamma_t(w)$, the set of $\boldsymbol{\mu}$-optimal choices at the node (t, w).

Certainly, for each n, this is true of $\mathbf{Z}^n$: at each date t and at each (i.e., Z_t^n-a.e.) w, the conditional distribution of Z_t^n on (b, x) is generated by some γ^n on $\Gamma_t^n(w)$, the set of $\boldsymbol{\mu}^n$-optimal choices at the node (t, w). The γ^n's have uniformly compact support, so we can extract a convergent subsequence that converges weakly to some γ. We will be done, therefore, if we can show that the set $\Gamma_t^n(w)$ has all its limit points contained in $\Gamma_t(w)$ as $n \to \infty$.

Specifically, for every sequence (b^n, ζ^n) in $\Gamma_t^n(w)$ with $(b^n, \zeta^n) \to (b, \zeta)$ as $n \to \infty$, we need to show that $(b, \zeta) \in \Gamma_t(w)$.

By uniform convergence of the μ_s^n to μ_s for every date s, there exists a collection $\{\epsilon_s^n\}$ such that $\epsilon_s^n \to 0$ as $n \to \infty$ for every s, and

$$|\mu_s(b^n) - \mu_s^n(b^n)| \le \epsilon_s^n \le u(M_s', 1) \tag{A.29}$$

over *any* sequence b^n in $[0, M_s]$. (The last inequality follows from the fact that $u \ge 0$ by normalization, and the definition of the space $\boldsymbol{\mathcal{U}}_s$.)

It follows that for each date t and $w \in [0, M_t]$, if $(b^*, \zeta^*) \in \Gamma_t(w)$,

$$\begin{aligned} W_t(w) - W_t^n(w) &\le \mu_t(b^*) - \mu_t^n(b^*) + \delta \mathbb{E}_{\zeta^*} \mathbb{E}_\theta [W_{t+1}(f(x, \theta)) - W_{t+1}^n(f(x, \theta))] \\ &\le \epsilon_t^n + \delta \|W_{t+1} - W_{t+1}^n\|, \end{aligned}$$

while by the same token, recalling that $(b^n, \zeta^n) \in \Gamma_t^n(w)$,

$$\begin{aligned} W_t^n(w) - W_t(w) &\le \mu_t^n(b^n) - \mu_t(b^n) + \delta \mathbb{E}_{\zeta^n} \mathbb{E}_\theta [W_{t+1}^n(f(w - x, \theta)) - W_{t+1}(f(w - x, \theta))] \\ &\le \epsilon_t^n + \delta \|W_{t+1} - W_{t+1}^n\|, \end{aligned}$$

where both inequalities employ (A.29). Combining the two, we must conclude that

$$\|W_t - W_t^n\| \le \epsilon_t^n + \delta \|W_{t+1} - W_{t+1}^n\|.$$

Using this inequality repeatedly, we see that for all positive integers T,

$$\|W_t - W_t^n\| \le \sum_{s=0}^{T} \delta^s \epsilon_{t+s}^n + \delta^{T+1} \|W_{t+T+1} - W_{t+T+1}^n\|.$$

By virtue of (A.23), it must be the case that $\delta^{T+1}\|W_{t+T+1} - W^n_{t+T+1}\| \leq U_{t+T+1}/\delta^t \to 0$ as $T \to \infty$. Therefore, fixing n and passing to the limit as $T \to \infty$, we must conclude that

$$\|W_t - W^n_t\| \leq \sum_{s=0}^{\infty} \delta^s \epsilon^n_{t+s},$$

The right hand side of this inequality is a well-defined, finite sum, using the last inequality in (A.29), and using again the restriction (A.23). Moreover, that sum converges to 0 as $n \to \infty$. We have therefore shown that for each t,

$$\|W_t - W^n_t\| \to 0 \text{ as } n \to \infty. \tag{A.30}$$

To complete the proof that $(b, \zeta) \in \Gamma_t(w)$, suppose on the contrary that there exists a feasible choice (b', ζ') such that

$$\mu_t(b') + \delta \mathbb{E}_{\zeta'} \mathbb{E}_\theta W_{t+1}(f(x, \theta)) > \mu_t(b) + \delta \mathbb{E}_\zeta \mathbb{E}_\theta W_{t+1}(f(x, \theta)).$$

Then using the continuity of μ_t and W_{t+1}, and the fact that $(b^n, \zeta^n) \to (b, \zeta)$ as $n \to \infty$, there exists $\epsilon > 0$ and integer N such that for all $n \geq N$,

$$\mu_t(b') + \delta \mathbb{E}_{\zeta'} \mathbb{E}_\theta W_{t+1}(f(x, \theta)) > \mu_t(b^n) + \delta \mathbb{E}_{\zeta^n} \mathbb{E}_\theta W_{t+1}(f(x, \theta)) + \epsilon. \tag{A.31}$$

But

$$\mu_t(b') + \delta \mathbb{E}_{\zeta'} \mathbb{E}_\theta W_{t+1}(f(x, \theta)) \leq \mu^n_t(b') + \delta \mathbb{E}_{\zeta'} \mathbb{E}_\theta W^n_{t+1}(f(x, \theta)) + \|\mu_t - \mu^n_t\| + \delta\|W_{t+1} - W^n_{t+1}\|,$$

while

$$\mu_t(b^n) + \delta \mathbb{E}_{\zeta^n} \mathbb{E}_\theta W_{t+1}(f(x, \theta)) \geq \mu^n_t(b^n) + \delta \mathbb{E}_{\zeta^n} \mathbb{E}_\theta W^n_{t+1}(f(x, \theta)) - \|\mu_t - \mu^n_t\| - \delta\|W_{t+1} - W^n_{t+1}\|.$$

Combining these two inequalities with (A.31), we must conclude that for all $n \geq N$ and large enough so that —using (A.30) — $2\|\mu_t - \mu^n_t\| + 2\delta\|W_t - W^n_t\| < \epsilon$,

$$\begin{aligned} \mu^n_t(b') + \delta \mathbb{E}_{\zeta'} \mathbb{E}_\theta W^n_{t+1}(f(x, \theta)) &> W^n_t(w) + \epsilon - 2\|\mu_t - \mu^n_t\| - 2\delta\|W_{t+1} - W^n_{t+1}\| \\ &> W^n_t(w), \end{aligned}$$

which contradicts the Bellman equation for W^n_t. □

Define a correspondence Ψ from $\mathcal{H}$ to itself by

$$\mathbf{H} \underset{\Phi}{\longmapsto} \boldsymbol{\mu} \longmapsto \mathcal{H}(\boldsymbol{\mu}) \subseteq \mathcal{H}.$$

Let $\mathcal{H}^*_t$ be the collection of *all* probability distributions on $[0, M_t]$, with the weak topology, and let $\mathcal{H}^*$ be its infinite product. It is easy to see that $\mathcal{H}$ is a compact, convex subset of this locally convex space $\mathcal{H}^*$. This observation, along with Lemmas 24 and 25, guarantees that all the assumptions of the Fan-Glicksberg-Kakutani fixed point theorem for locally convex spaces is satisfied. Therefore a fixed point $\mathbf{H}$ exists. This, together with $\boldsymbol{\mu} = \Phi(\mathbf{H})$, constitutes an equilibrium. □