A Brief History of Mathematics

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What is mathematics?

What do mathematicians do?

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What is mathematics?

What do mathematicians do?

http://www.sfu.ca/~rpyke/presentations.html

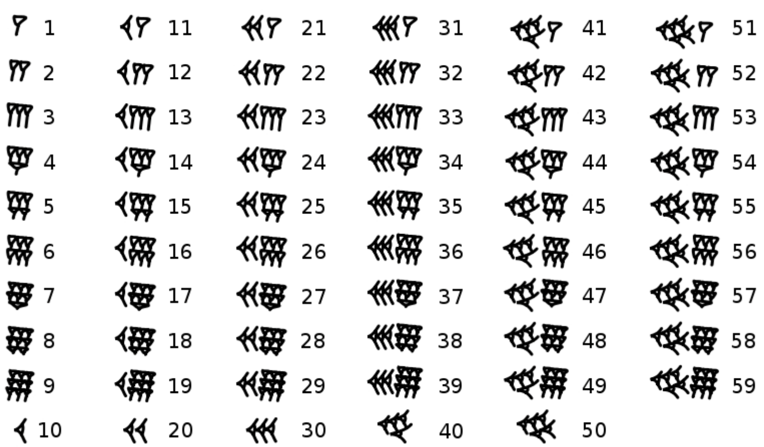
A Brief History of Mathematics

- Egypt; 3000B.C.
  - Positional number system, base 10
  - Addition, multiplication, division. Fractions.
  - Complicated formalism; limited algebra.
  - Only perfect squares (no irrational numbers).
  - Area of circle; $(8D/9)^2 \rightarrow \pi = 3.1605$. Volume of pyramid.

Value	1	10	100	1,000	10,000	100,000	1 million, or many
Hieroglyph	𓏺	𓎆	𓍢	𓆼	𓂭	𓆐 or 𓆏	𓁨
Description	Single stroke	Heel bone	Coil of rope	Water lily (also called Lotus)	Bent Finger	Tadpole or Frog	Man with both hands raised, perhaps Heh.[1]

A Brief History of Mathematics

- Babylon; 1700-300B.C.
  - Positional number system (base 60; sexagesimal)
  - Addition, multiplication, division. Fractions.
  - Solved systems of equations with many unknowns
  - No negative numbers. No geometry.
  - Squares, cubes, square roots, cube roots
  - Solve quadratic equations (but no quadratic formula)
  - Uses: Building, planning, selling, astronomy (later)



A Brief History of Mathematics

- Greece; 600B.C. – 600A.D.   Papyrus created!
  - Pythagoras; mathematics as abstract concepts, properties of numbers, irrationality of $\sqrt{2}$, Pythagorean Theorem $a^2+b^2=c^2$, geometric areas
  - Zeno paradoxes; infinite sum of numbers is finite!
  - Constructions with ruler and compass; ‘Squaring the circle’, ‘Doubling the cube’, ‘Trisecting the angle’
  - Plato; plane and solid geometry

A Brief History of Mathematics

- Greece; 600B.C. – 600A.D.

Aristotle; mathematics and the physical world (astronomy, geography, mechanics), mathematical formalism (definitions, axioms, proofs via construction)

- Euclid; *Elements* – 13 books. Geometry, algebra, theory of numbers (prime and composite numbers, irrationals), method of exhaustion (calculus!), Euclid's Algorithm for finding greatest common divisor, proof that there are infinitely many prime numbers, **Fundamental Theorem of Arithmetic** (all integers can be written as a product of prime numbers)
- Apollonius; conic sections
- Archimedes; surface area and volume, centre of gravity, hydrostatics
- Hipparchus and Ptolemy; Trigonometry (circle has 360°, sin, cos, tan; $\sin^2 + \cos^2 = 1$), the *Almagest* (astronomy; spherical trigonometry).
- Diophantus; introduction of symbolism in algebra, solves polynomial equations

Some mathematical facts known to the ancient Greeks

- There are infinitely many prime numbers:

(prime P: only factors of P are 1 and P)

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  (prime P: only factors of P are 1 and P)

  - Suppose not. So there is a largest prime; $\hat{P}$.

    Let $M = 2 \bullet 3 \bullet 5 \bullet 7 \bullet 11 \bullet 13 \bullet 17 \bullet \bullet \bullet \bullet \hat{P}$ (product of all primes)

    Note that none of these primes can divide M+1 (remainder is 1).

    But $M+1 = q_1 q_2 q_3 \bullet\bullet\bullet qn$, product of primes, by the <u>Fundamental Theorem of Arithmetic</u>. What are these primes q? So there must be more primes than the ones factoring M.

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We can assume that ‘a’ and ‘b’ have no common factors.

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And so $(2k)^2 = 2b^2 \rightarrow 2k^2 = b^2 \rightarrow$ b is also even!

But this <u>contradicts</u> the assumption that 'a' and 'b' have no common factors.

A few important problems in the development of mathematics

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Solving polynomial equations (roots of equations)

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Solving polynomial equations (roots of equations)

– Linear; $ax + b = 0 \rightarrow \boxed{x = -b/a \quad (a \neq 0)}$

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– Quadratic; ‘easy’ case $ax^2 + b = 0 \rightarrow \boxed{x = \pm \sqrt{-b/a}}$

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Solving polynomial equations (roots of equations)

– Quadratic; 'easy' case $ax^2 + b = 0 \rightarrow x = \pm\sqrt{-b/a}$

general case; $ax^2 + bx + c = 0$;

→ complete the square;

$$\boxed{x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}}$$

quadratic formula; ***formula*** *for the solution*

Known since ancient times

A few important problems in the development of mathematics

Solving other polynomial equations

– Cubic ; $x^3 + bx^2 + cx + d = 0$
– Quartic; $x^4 + ax^3 + bx^2 + cx + d = 0$

General solutions discovered around 1550;
a **formula** that gives you the solutions
in terms of a,b,c,d and works for **all** such
polynomials using **only roots**

$$\sqrt{\cdots}, \quad \sqrt[3]{\cdots}, \quad \sqrt[4]{\cdots}$$

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Solving polynomial equations

– Quintic; $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$

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Solving polynomial equations

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Is there a formula, with only a,b,c,d,e in it, that gives you the solutions (roots) using only square roots, cube roots, fourth and fifth roots?

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Solving polynomial equations

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Many tries. Suspected not possible in 1700's.

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Liouville announces some reasons why; 1843.

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<u>Solving polynomial equations</u>

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Many tries. Suspected not possible in 1700's.

Liouville announces some reasons why; 1843.

Galois solves problem around same time →

ushers in new ideas into algebra; <u>*Galois Theory*</u>

Now we know <u>***why***</u> for quintic (and higher) polynomials there is <u>no formula</u> for the roots and that works for *all* polynomials

A few important problems in the development of mathematics

<u>The development of calculus</u> (1600’s)

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<u>The development of calculus</u> (1600's)

Motivated by 4 problems;

1. Instantaneous velocity of accelerating object
2. Slope of a curve (slope of tangent line)
3. Maximum and minimum of functions
4. Length of (non-straight) curves (e.g., circumference of an ellipse?

A few important problems in the development of mathematics

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3. Maximum and minimum of functions
4. Length of (non-straight) curves (e.g., circumference of an ellipse? $L = 4a\int_0^{\pi/2}\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2\phi\;d\phi}$)

A few important problems in the development of mathematics

<u>The development of calculus</u> 1600's

Using calculus, Newton explained (in the *Principia*);

- why tides occur
- why the shapes of planetary orbits are conic sections (ellipses, parabolas, and hyperbolas)
- Kepler's 3 Laws of planetary motion
- shape of a rotating body of fluid
- etc, etc, etc

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- etc, etc, etc

and then …..

A few important problems in the development of mathematics

<u>The development of calculus</u> 1600’s

The discovery of Neptune on paper! (1846)

(Celestial Mechanics)

(Uranus ‘accidentally’ discovered by telescope; William Herschel 1781)

A few important problems in the development of mathematics

The notion of **infinity** 20th Century

$$\infty \quad \text{or} \quad \aleph_0$$

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"Aleph"

Georg Cantor; mathematician 1845 - 1918

A few important problems in the development of mathematics

The notion of **infinity** 20th Century

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How many integers are there?

∞ “=“ {1,2,3,….} The ‘usual’ infinity is the whole set of **natural numbers** (counting numbers) .

A few important problems in the development of mathematics

<u>The notion of **infinity**</u> 20^{th} Century

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How many integers are there?

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Remarkably, this is the same 'size' as <u>all</u> the integers (positive and negative), and the same 'size' as all the rational numbers!

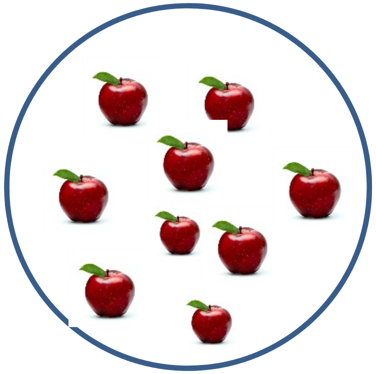
- How do we count?

- How do we count?

  By ‘labeling’ the objects with counting numbers!

- How do we count?

apples



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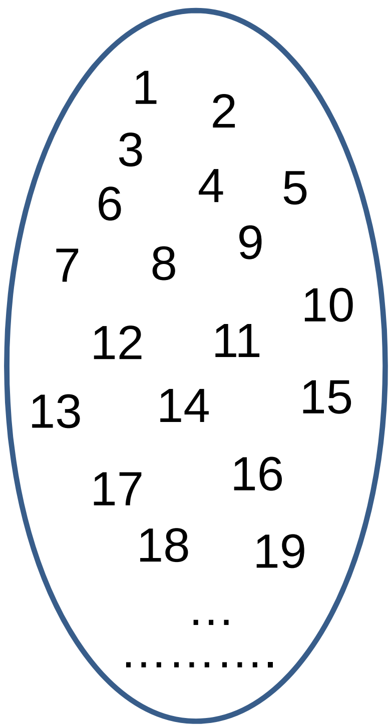
natural numbers



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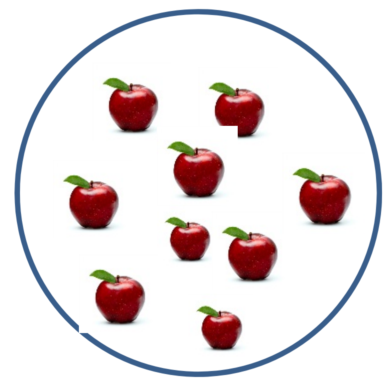

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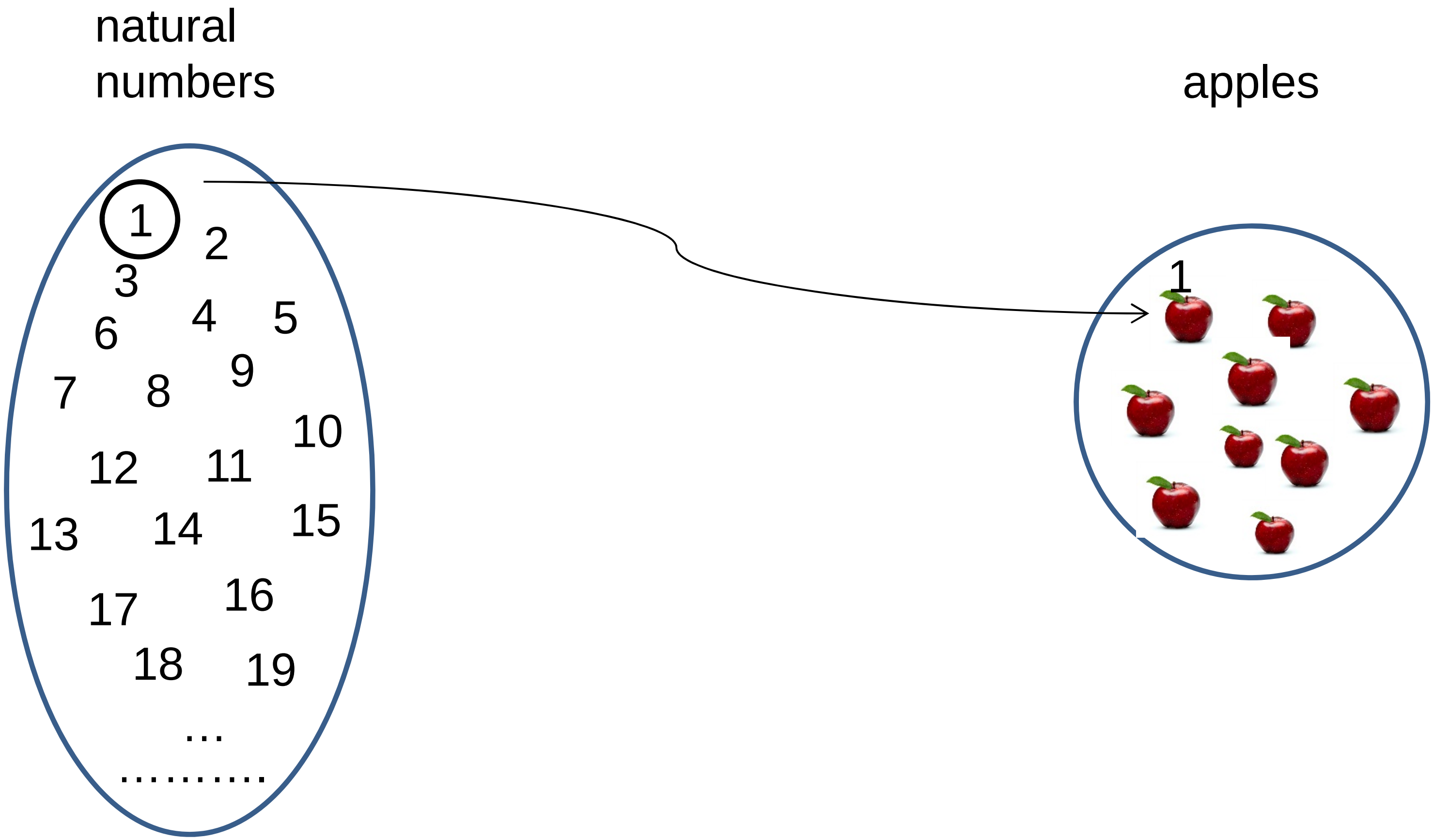

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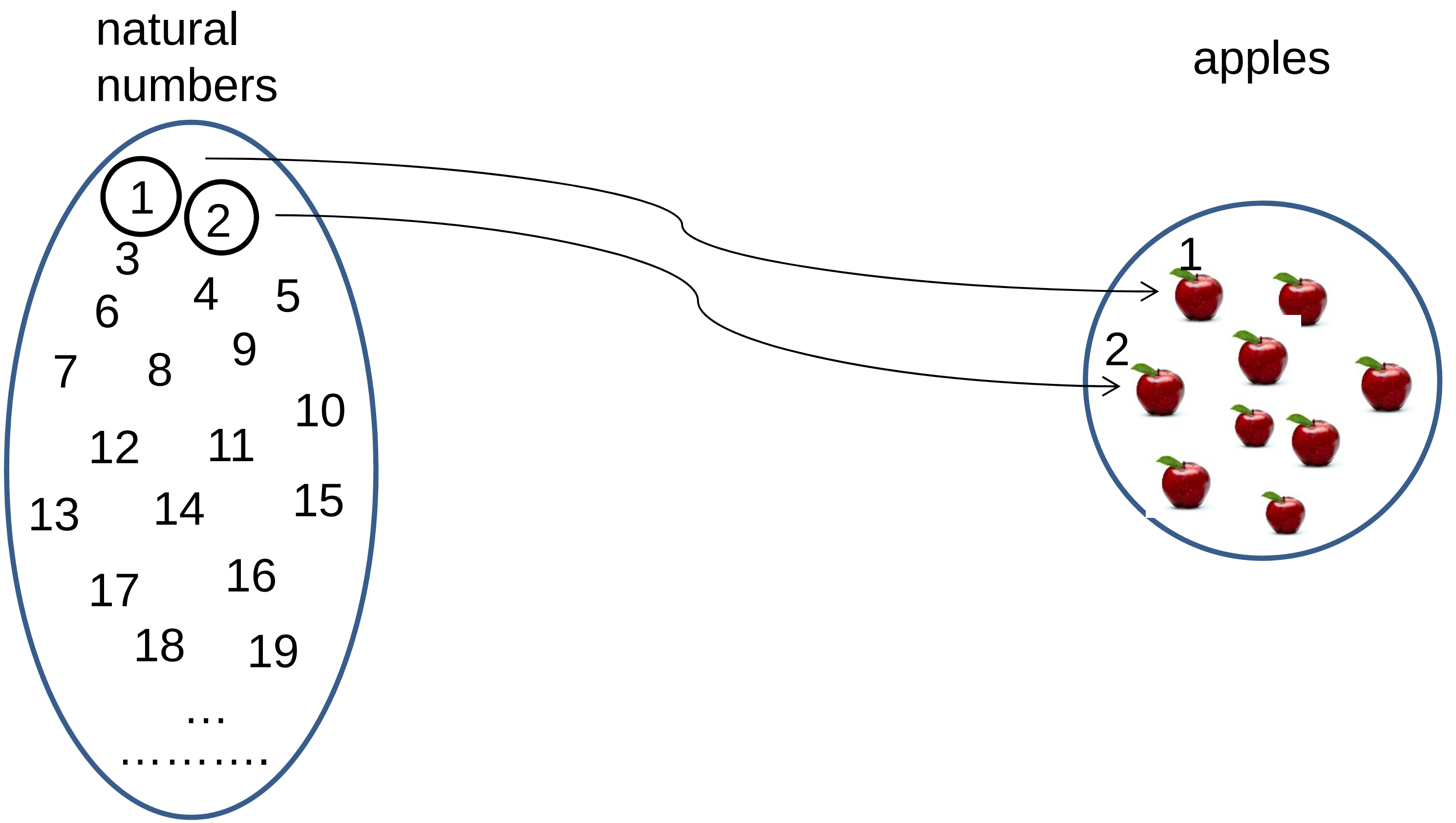

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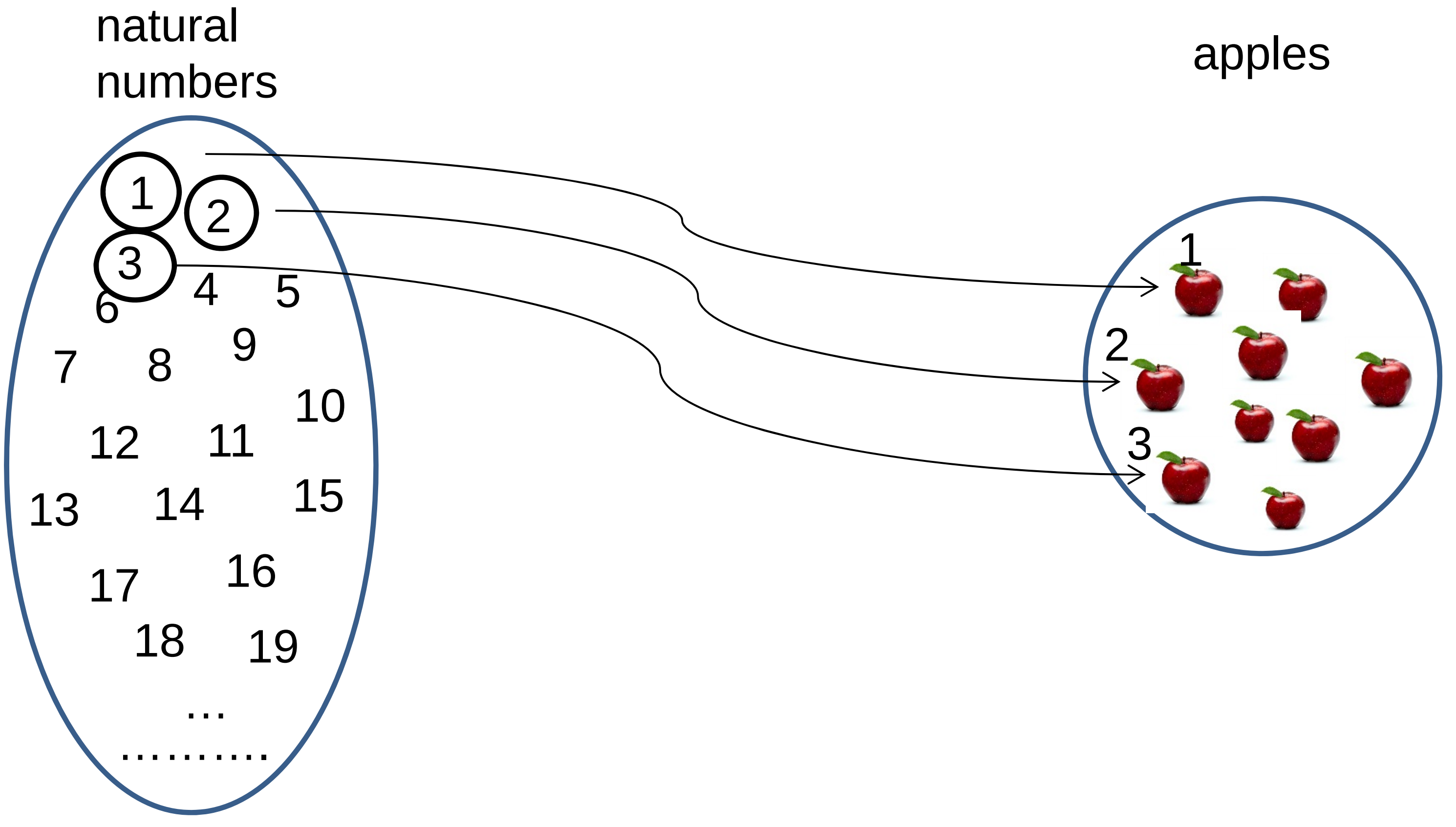

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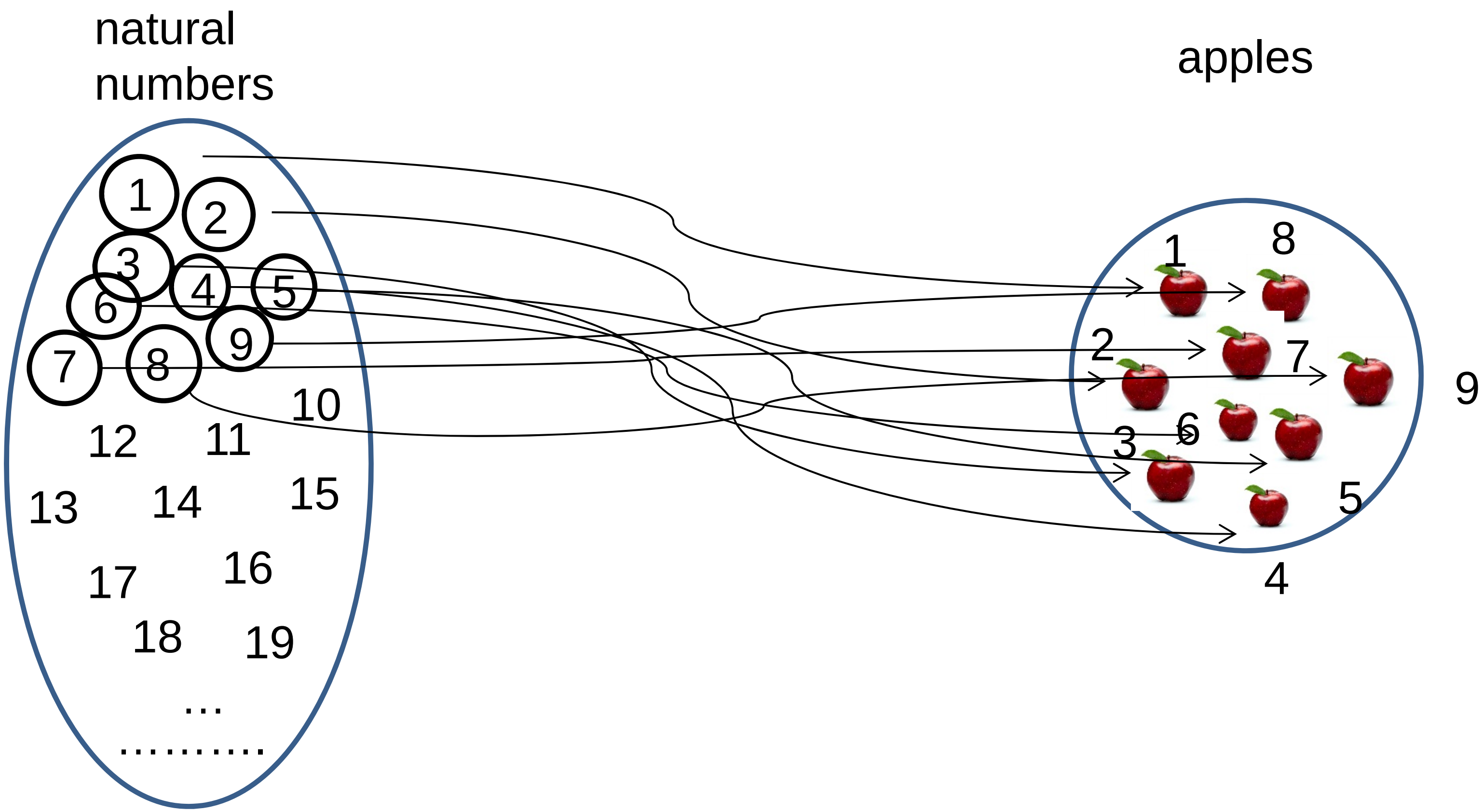

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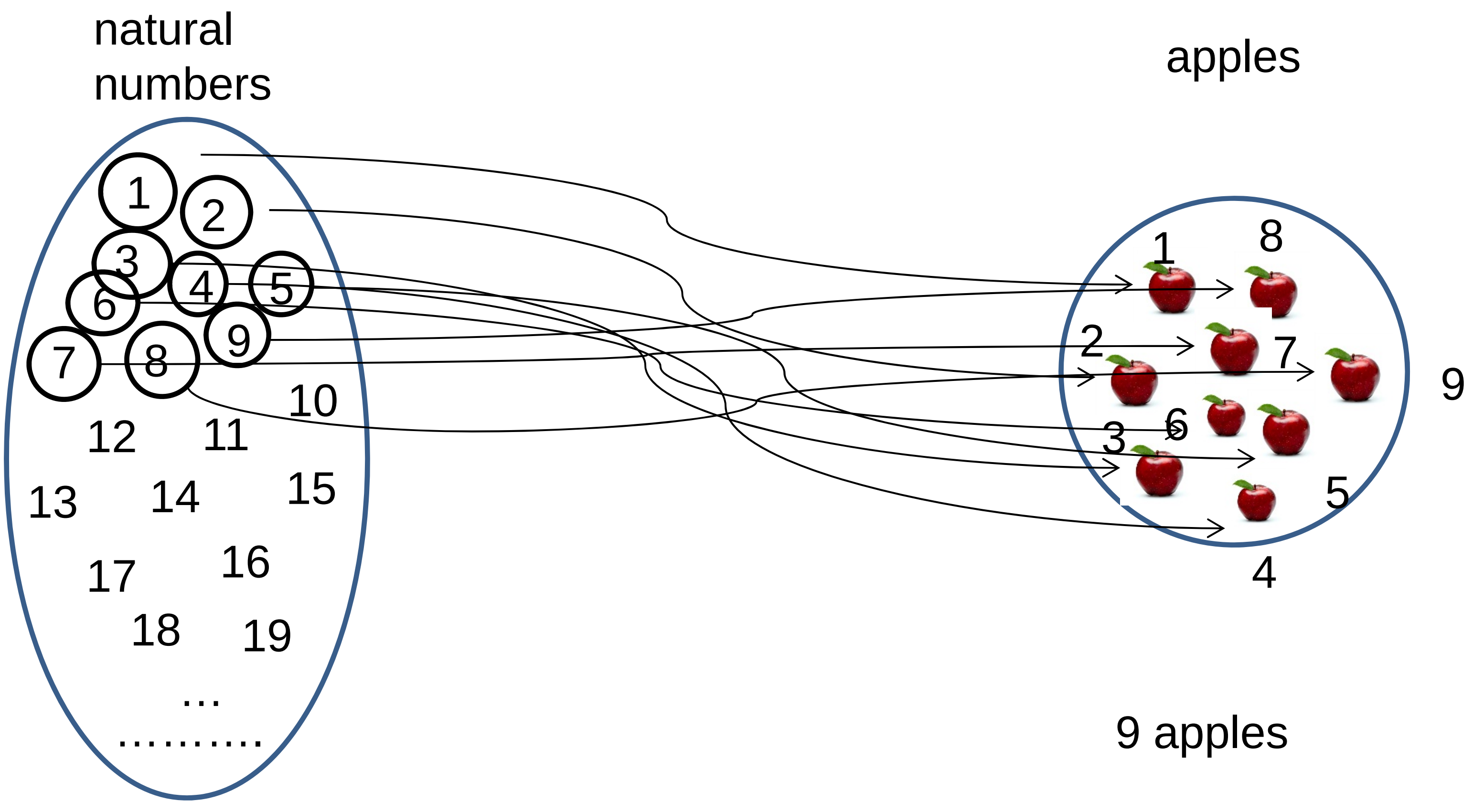

Set of integers: $\mathcal{Z} = \{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \}$

<u>There are (only) ∞ many integers;</u>

Set of integers: $\mathcal{Z}$ = { . . . -3, -2, -1, 0, 1, 2, 3, . . . }

<u>There are (only) ∞ many integers;</u>

Counting numbers (natural numbers);

1 2 3 4 5 6 7 8 9 10 . . . .

Integers;

. . . -3 -2 -1 0 1 2 3 . . .

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$$\begin{array}{cccccccccccccc} & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\ \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \ldots \end{array}$$

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$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad . \; . \; . \; .$$

$$. \; . \; . \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad . \; . \; .$$

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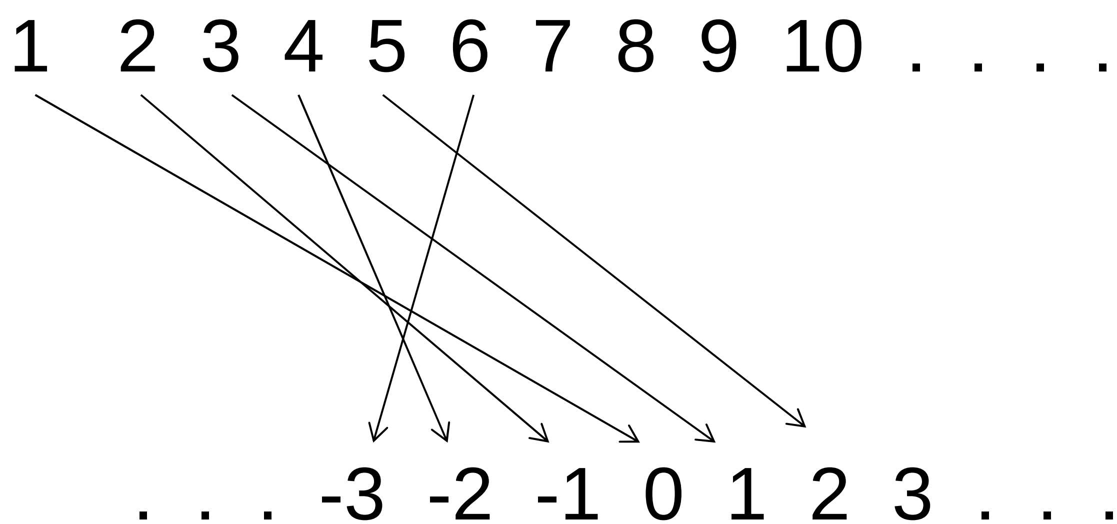

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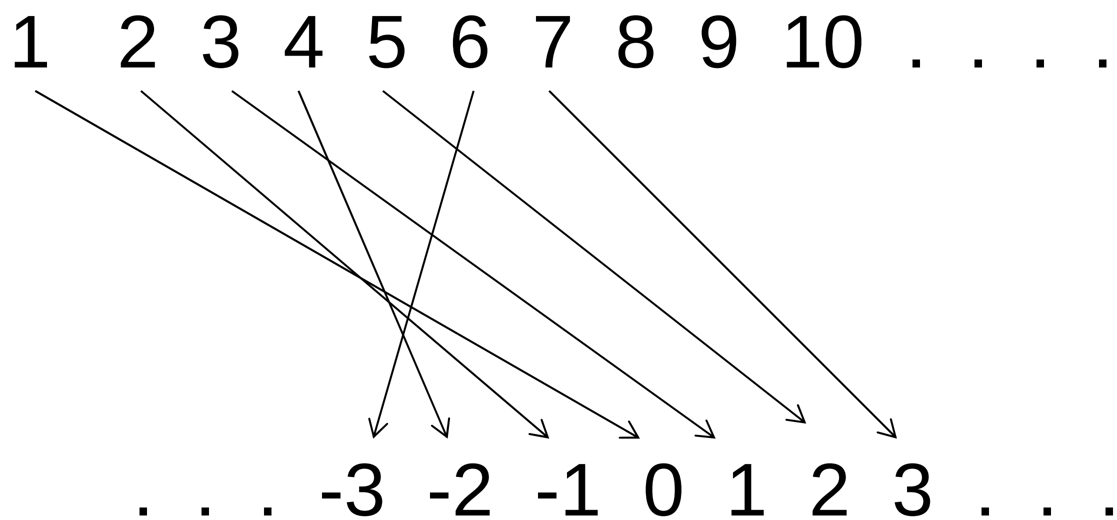

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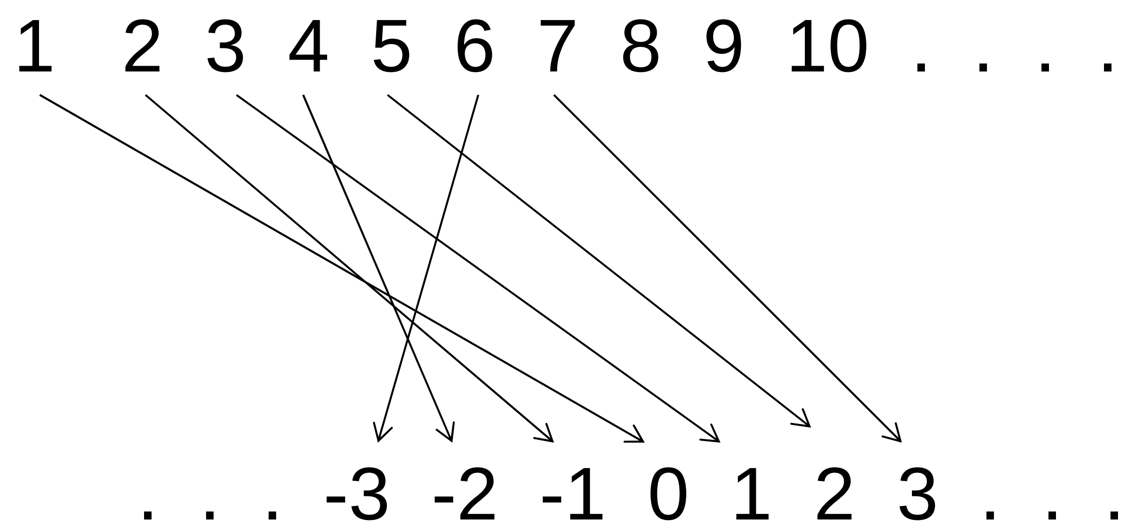

We can count all the integers this way!

A few important problems in the development of mathematics

The notion of infinity 20th Century

Algebra with infinity;

A few important problems in the development of mathematics

The notion of infinity 20th Century

Algebra with infinity;

- $\infty + 1 = \infty$, $\infty + 100 = \infty$, etc
- $\infty \bullet \infty = \infty$
- $\infty^{\infty} = \infty$

A few important problems in the development of mathematics

The notion of infinity 20th Century

Algebra with infinity;

- $\infty + 1 = \infty$, $\infty + 100 = \infty$, etc
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Is there anything bigger than ∞ ?

A few important problems in the development of mathematics

<u>The notion of infinity</u> 20th Century

There are (only) ∞ many rational numbers . . .

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Is there anything larger ? ………..

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The notion of infinity 20th Century

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Decimal representation of a number in [0, 1];

0. d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8 d_9 ••• each d is 0,1,2,…8, or 9

e.g., 0.2154960012…….. Here, d_1=2, d_2=1, d_3=5, etc.

A few important problems in the development of mathematics

The notion of infinity 20^{th} Century

How many *real numbers* are there?? Let's count;

1 $0.\ d_1^1\ d_2^1\ d_3^1\ d_4^1\ d_5^1\ d_6^1\ d_7^1\ d_8^1\ d_9^1 \bullet\bullet\bullet$

2 $0.\ d_1^2\ d_2^2\ d_3^2\ d_4^2\ d_5^2\ d_6^2\ d_7^2\ d_8^2\ d_9^2 \bullet\bullet\bullet$

3 $0.\ d_1^3\ d_2^3\ d_3^3\ d_4^3\ d_5^3\ d_6^3\ d_7^3\ d_8^3\ d_9^3 \bullet\bullet\bullet$

•

•

•

(decimal representation of numbers in [0, 1])

A few important problems in the development of mathematics

The notion of infinity 20th Century

1 $0.\ d_1^1\ d_2^1\ d_3^1\ d_4^1\ d_5^1\ d_6^1\ d_7^1\ d_8^1\ d_9^1 \cdots$

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3 $0.\ d_1^3\ d_2^3\ d_3^3\ d_4^3\ d_5^3\ d_6^3\ d_7^3\ d_8^3\ d_9^3 \cdots$

•

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A few important problems in the development of mathematics

The notion of infinity 20th Century

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2 $0.\, d_1^2\, d_2^2\, d_3^2\, d_4^2\, d_5^2\, d_6^2\, d_7^2\, d_8^2\, d_9^2 \cdots$

3 $0.\, d_1^3\, d_2^3\, d_3^3\, d_4^3\, d_5^3\, d_6^3\, d_7^3\, d_8^3\, d_9^3 \cdots$

•

•

Here's a number;

$x = 0.\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_8\, x_9 \cdots$ where

$x_1 \neq d_1^1,\ x_2 \neq d_2^2,\ x_3 \neq d_3^3,\ x_4 \neq d_4^4,\ x_5 \neq d_5^5,\ \cdots$

A few important problems in the development of mathematics

<u>The notion of infinity</u> 20th Century

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$x_1 \neq d_1^1,\ x_2 \neq d_2^2,\ x_3 \neq d_3^3,\ x_4 \neq d_4^4,\ x_5 \neq d_5^5,\ \bullet\bullet\bullet$

Check: <u>This x is not in our list above!!</u>

A few important problems in the development of mathematics

1 $0.\, d_1^1\, d_2^1\, d_3^1\, d_4^1\, d_5^1\, d_6^1\, d_7^1\, d_8^1\, d_9^1 \cdots$

2 $0.\, d_1^2\, d_2^2\, d_3^2\, d_4^2\, d_5^2\, d_6^2\, d_7^2\, d_8^2\, d_9^2 \cdots$

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$x_1 \neq d_1^1$, $x_2 \neq d_2^2$, $x_3 \neq d_3^3$, $x_4 \neq d_4^4$, $x_5 \neq d_5^5$, $\cdots$ (note there are *many* such choices for x)

x is not in our list!!

So there are more real numbers than ∞

A few important problems in the development of mathematics

1 $0.\ d_1^1\ d_2^1\ d_3^1\ d_4^1\ d_5^1\ d_6^1\ d_7^1\ d_8^1\ d_9^1 \bullet\bullet\bullet$

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3 $0.\ d_1^3\ d_2^3\ d_3^3\ d_4^3\ d_5^3\ d_6^3\ d_7^3\ d_8^3\ d_9^3 \bullet\bullet\bullet$

•

$x = 0.\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9 \bullet\bullet\bullet$ where

$x_1 \neq d_1^1$, $x_2 \neq d_2^2$, $x_3 \neq d_3^3$, $x_4 \neq d_4^4$, $x_5 \neq d_5^5$, $\bullet\bullet\bullet$ (note there are *<u>many</u>* such choices for x)

x is not in our list!!

So there are more real numbers than ∞

How many real numbers are there? . . . .

A few important problems in the development of mathematics

<u>The notion of infinity</u> 20th Century

The **Continuum Hypothesis** (1900);

The size of the real numbers is the 'next' infinity after ∞;

$$|\mathcal{Z}| = \infty = \boxed{\aleph_0 < |\mathcal{R}|} = \aleph_1 < \aleph_2 < \cdots$$

A few important problems in the development of mathematics

The notion of infinity 20th Century

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Proof?

A few important problems in the development of mathematics

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Proof? In fact, *this statement cannot be proved to be true nor can it be proved to be false*! (1963) In other words, assuming it is true or assuming it is false will not get you into trouble ~ (See Gödel's *Incompleteness Theorem*, 1931)

Some important questions in modern mathematics

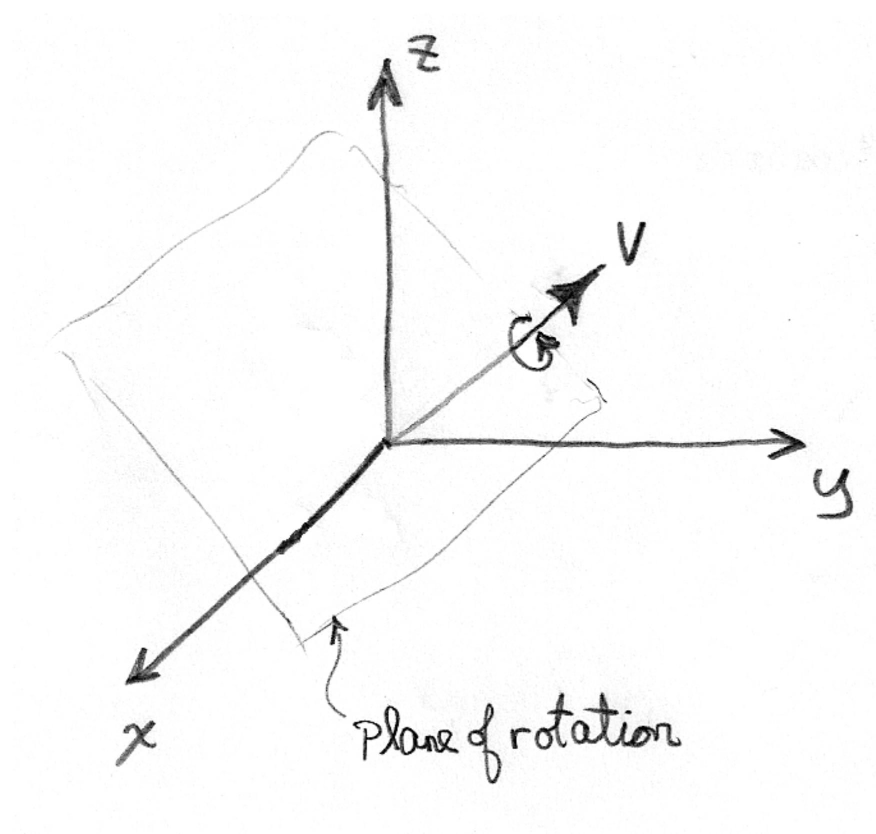
- How well can an irrational number be approximated by rational numbers?

(there are different ‘types’ of irrational numbers)

How ‘close’ to an irrational number can you get using only rational numbers whose denominators are no larger than b? (a/b - type rational numbers)

Some important questions in modern mathematics

- Can every rotation be obtained by rotating around (only) the x, y and z axes?



Some important questions in modern mathematics

- Is the solar system stable? Will the planets continue to orbit the sun in regular patterns forever or will they someday collide?

Some questions in industry where mathematics is used

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- Vehicle emission (pollution control)
- How to allocate Intensive Care beds at a hospital to minimize patient waiting times?
- How effective are carbon trading schemes in reducing greenhouse gasses?
- Deciding the best (government) policy for encouraging solar power development
- Why are the tides at the Bay of Fundy so large? (16m) *resonance…..*

Areas of modern mathematics

- Algebra
- Analysis (aka calculus)
- Topology
- Mathematical Logic
- Numerical analysis (using computers)
- Discrete mathematics

http://www.sfu.ca/~rpyke/presentations.html