Notes on the Inverse Scattering Transform and Solitons

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Among the nonlinear wave equations are very special ones called *integrable equations*. These equations exhibit soliton solutions and have an associated 'nonlinear Fourier transform' $\mathcal{F}_{NL}$ called the *Inverse Scattering Transform* (IST). Some integrable wave equations are;

$$\begin{array}{rrcll} \text{sine-Gordon:} & \partial_t^2\varphi - \partial_x^2\varphi + \sin\varphi & = & 0, & \mathcal{F}_{NL}^{sG} \\ \text{KdV:} & \partial_t\varphi + \partial_x^3\varphi + p\,\varphi^{p-1}\partial_x\varphi & = & 0, & \mathcal{F}_{NL}^{KdV} \\ \text{NLS:} & i\partial_t u + \partial_x^2 u + |u|^2 u & = & 0, & \mathcal{F}_{NL}^{NLS} \end{array}$$

Each of these equations exhibit 'radiation-type' solutions, and solitons. Radiation-type solutions are ones you most often see in wave equations; oscillating functions (in space x and time t) that travel out to infinity and decay in amplitude (these solutions carry away energy to infinity; Strauss mentions this on page 369 when he says, 'a *dispersive tail* which gradually disappears' - dispersion means the function decays in amplitude). The soliton solutions, however, remain localized in space for all time (so their energy remains localized too). Each of these equations have their own special soliton solutions; s-G has both kinks $K(x)$ and breathers $B(x,t)$, KdV and NLS have both the $\text{sech}^2(x)$-type solitons (which we'll write as $S(x)$) as well as breathers. (Breather solitons are solutions that are localized in x and time-periodic in t. They could be either stationary or moving, just like the $S(x)$ type solitons could be stationary or moving, however, in the case of KdV those solitons are always moving; $S(x,t) = f(x-ct), c > 0$.)

Let $\mathcal{S}(\mathbf{R})$ denote the set of smooth functions on $\mathbf{R}$ that decay rapidly along with all their derivatives as $|x| \to \infty$, that is, $|u^{(n)}(x)| < Mx^{-N}$, for any $N > 0$ and for all n (this set is called the *Schwartz* class of functions and is a little larger than the space of test functions $\mathcal{D}(\mathbf{R})$ that we considered when discussing distributions). Each of these IST can be defined for functions $u \in \mathcal{S}$. $\mathcal{F}_{NL}$ is similar to the Fourier transform $\mathcal{F}$ in that $\mathcal{F}_{\mathcal{NL}}(u)$ produces another function in $\mathcal{S}$, which we'll call $g(k)$ (for the Fourier transform, $g(k) = \hat{u}(k)$). However, $\mathcal{F}_{NL}$ also produces a set of (complex) numbers; $\{c_j\}_{j=1}^d$;

$$\mathcal{F}_{NL}(u) \quad = \quad g(k),\ \{c_j\}_{j=1}^d$$

The collection $g(k), \{c_j\}$ is referred to as the <u>scattering data</u> of $u(x)$ (the computation of $\mathcal{F}_{NL}^{KdV}(u)$ is described in Strauss [S][1], Section 14.2). The function $g(k)$ is related to the 'radiation content' of $u(x)$ ($g(k)$ is the amount of radiation of frequency k contained in $u(x)$, very much like what the Fourier transform $\hat{u}(k)$ says about $u(x)$), while the $\{c_j\}$ correspond to the (s-G or NLS or KdV) solitons that $u(x)$ contains. We know that the solitons S can be moving, so that they can be functions of time too, $S(x,t)$, while the breather B is a function of x and t. So

[1] References are given at the end

when we measure the soliton content of $u(x)$ with $\mathcal{F}_{NL}$, it may also include these time-dependent solitons. But $u(x)$ is not time-dependent, so what this means is that $u(x)$ contains moving or time-periodic solitons (Breathers) at time $t = 0$; the $\{c_j\}$ encode both the number and type of solitons in addition to information as to how they are moving at a certain fixed time. (In fact, several of the c_j are required to completely specify one single soliton (the type, it's velocity, where it is centred at $t = 0$, it's frequency in the case of a breather), but let's not bother with writing out all those various parameters.)

The IST has an inverse $\mathcal{F}_{NL}^{-1}$ which 'reconstructs' the function $u(x)$ from its scattering data $g(k), \{c_j\}$;

$$\mathcal{F}_{NL}^{-1}(g(k), \{c_j\}) \;=\; u(x)$$

just like $\mathcal{F}^{-1}$, the inverse Fourier transform, reconstructs $u(x)$ from $\hat{u}(k)$. Here, the c_j correspond to either a type S soliton or a breather B (including information about how they are moving at $t = 0$), and $g(k)$ corresponds to radiation; $\mathcal{F}_{NL}^{-1}(g(k)) = g(x)$, $\mathcal{F}_{NL}^{-1}(c_j) = S(x, 0)$ or $B(x, 0)$. (The rather difficult calculation $\mathcal{F}_{NL}^{-1}$ can be carried out several ways, one of which is the Gelfand-Levitan method Strauss [S] mentions on page 371. See also the references [DEGM], [DJ], [FT], [NMPZ] for discussion about this.)

So, an arbitrary function $u(x) \in \mathcal{S}$ can be thought of as being composed of a certain amount of radiation and solitons, and this decomposition is with respect to which particular IST one is applying; $\mathcal{F}_{NL}^{sG}, \mathcal{F}_{NL}^{KdV}, \mathcal{F}_{NL}^{NLS}$;

$$\begin{aligned}
u(x) &= g^{sG}(x) + \sum_j S_j^{sG}(x,0) + \sum_k B_k^{sG}(x,0), &\quad \text{using } \mathcal{F}_{NL}^{sG}\\
u(x) &= g^{KdV}(x) + \sum_j S_j^{KdV}(x,0) + \sum_k B_k^{KdV}(x,0), &\quad \text{using } \mathcal{F}_{NL}^{KdV}\\
u(x) &= g^{NLS}(x) + \sum_j S_j^{NLS}(x,0) + \sum_k B_k^{NLS}(x,0), &\quad \text{using } \mathcal{F}_{NL}^{NLS}
\end{aligned}$$

The above discussion summarizes one of the important aspects of the IST ($\mathcal{F}_{NL}^{sG}, \mathcal{F}_{NL}^{KdV}, \mathcal{F}_{NL}^{NLS}$); it can be used to decompose arbitrary function $u \in \mathcal{S}$ into a sum of radiation or soliton type functions, and can reconstruct a function from its scattering data (this is entirely analogous to the Fourier transform, except for the soliton part).

The second important aspect of the IST is concerned with the *dynamics* of the nonlinear wave equations, and this is what we turn to next.

Now we think of $u(x)$ as being the initial data for one of these integrable NLW; $u(x) = \varphi(x, 0)$ (for sG we need two initial functions, one for $\varphi(x, 0)$ and one for $\varphi_t(x, 0)$, but for simplicity we will use just one initial function). Let $\varphi(x,t)$ be the solution at time t. Now we can use the IST to measure the soliton content of $\varphi(x,t)$ (think of $v(x) = \varphi(x,t)$ and consider $\mathcal{F}_{NL}(v)$). What is the relation between the scattering data of $\varphi(x,0)$ and the scattering data of $\varphi(x,t)$? In general (that is, if $\varphi(x,t)$ is evolving under arbitrary dynamics (i.e., an arbitrary NLW)), there is *no* relation (or, if anything, a very complicated relation) between the scattering data at time= 0 and the scattering data at time time= t. The 'relation' between scattering data at these two times in mathematical terms are (ordinary) differential equations (in time) that the components $g(k,t), c_j(t)$ satisfy (note that now we indicate the time dependence of the scattering data of $\varphi(x,t)$ because it changes in time (along with $\varphi(x,t)$).

In principle, one could determine the differential equations for the scattering data of $\varphi(x,t)$ from the NLW satisfied by $\varphi(x,t)$. Then, one could solve these (ordinary) differential equations and then reconstruct $\varphi(x,t)$ at a later time using $\mathcal{F}_{NL}^{-1}$ (note that this is exactly what we did when we solved the linear wave and heat equations using the Fourier transform). But, unless φ is evolving under one of the integrable NLW and you are using the $\mathcal{F}_{NL}$ corresponding to that equation, the differential equations for the evolution of the scattering data is very complicated (and in practice intractable).

This is where the second remarkable fact of integrable equations comes in. If $\varphi(x,t)$ is evolving under, say, the sine-Gordon dynamics (that is, if $\varphi(x,t)$ is a solution of the sG equation), then the sG scattering data for $\varphi(x,t)$ evolves in a very simple - in fact <u>linear</u> - way! You can see what these equations are for the scattering data for KdV in Strauss [S] page 370 (Theorem 1).

We have two important consequences of this (remarkable) fact;

(1) One can determine the solution $\varphi(x,t)$ of an integrable wave equation at any time by computing the scattering data $g(k,0), \{c_j(0)\}$ of the initial condition $\varphi(x,0)$ using $\mathcal{F}_{NL}$, then solve the equations for the scattering data to determine the scattering data $g(k,t), \{c_j(t)\}$ at the later time, and finally reconstruct the solution $\varphi(x,t)$ at this time via $\mathcal{F}_{NL}^{-1}$

(2) The dynamics of the scattering data are simple (linear) and *uncoupled*. 'Uncoupled' means that the various components of the initial scattering data (radiation and solitons) do not interact as the solution evolves. This is reflected in the differential equations describing the evolution of the scattering data; the equation describing, say, the evolution of a particular breather component $c_j^B(t)$ of the solution does <u>not</u> contain any other scattering component (eg. radiation, kinks, or other breathers). That means we can solve each ode for each component separately. Consequently, *the number of each type of soliton present in the initial data does not change as the solution evolves.* Furthermore, the way one of the solitons evolves (eg., how it moves) is unaffected by the presence of any other soliton (or radiation) present in the solution (because it's ode does not contain anything about those other components).

In particular, (2) shows that solitons interact *elastically*; they will pass through each other with out affecting each other. This is what we saw in the videos of solutions of these integrable equations (see also Example 2 in Strauss [S] page 372). Note that even if the nonintegrable ϕ^4 wave equation had a transform $\mathcal{F}_{NL}$ that could resolve $u(x)$ into φ^4 kinks and radiation, the differential equations describing the evolution of this scattering data would have to be coupled because we saw that two colliding kinks in fact turn into radiation! (we would have to solve all the ode for the scattering data simultaneously).

The above description also indicates that as time proceeds, the solution of an integrable equation will resolve itself into distinct solitons and radiation (the same solitons and radiation that are present in the initial data). We witnessed this in the video about the solution of the sine-Gordon equation (`http://www.sfu.ca/~rpyke/m418/solitons` and then go to the video "Solution of the sine-Gordon equation with arbitrary initial data"). Strauss [S] also mentions this in his Example 1 on page 372.

Schematically, we have the following comparison between the Fourier transform and the Inverse Scattering transform;

Fourier Transform for linear wave equation $L(u) = 0$:

$$\begin{array}{lccccl}
 & \text{initial data } \ \varphi(x,0) = \varphi(x) & \xrightarrow{\mathcal{F}} & \hat{\varphi}(k) & & \\
\text{linear WE} & \downarrow & & \downarrow & & \text{linear ODE} \\
 & \varphi(x,t) & \xleftarrow{\mathcal{F}^{-1}} & \hat{\varphi}(k,t) & &
\end{array}$$

Inverse Scattering Transform for nonlinear WE:

$$\begin{array}{lcccl}
 & \text{initial data } \ \varphi(x,0) = \varphi(x) & \xrightarrow{\mathcal{F}_{NL}} & g(k),\ \{c_j\}_{j=1}^{d} & \\
\text{nonlinear WE} & \downarrow & & \downarrow & \begin{array}{c}\text{linear}\\ \text{ODE}\end{array} \\
 & \varphi(x,t) & \xleftarrow{\mathcal{F}_{NL}^{-1}} & g(k,t),\ \{c_j(t)\}_{j=1}^{d} &
\end{array}$$

It is a current topic of research in mathematics why there are so few nonlinear wave equations with soliton solutions, and in particular why there are so few breather soliton solutions (the s-G and KdV breathers are the only known breather solitons). The IST can provide insight into this as follows.

If one uses, say, $\mathcal{F}_{NL}^{sG}$ to study the (s-G) soliton and radiation content of a solution $\varphi(x,t)$ of an equation that is only slightly different than the sine-Gordon equation (for example, $\partial_t^2\varphi - \partial_x^2\varphi + \sin(\varphi) + \varepsilon\varphi^2 = 0$ for some small $\varepsilon > 0$), then the ode describing the evolution of the scattering data are now coupled and so the solitons interact (and, it seems, always disappear into radiation as $t \to \infty$). In general, it seems that for any *nonintegrable* nonlinear wave equation, the equations describing the evolution of solitary waves and radiation are coupled, and in fact this coupling is responsible for the non-soliton nature of the solitary waves (i.e., that they interact and decay).

For an interesting discussion about the interaction of nonintegrable solitons and perturbations of integrable equations, including numerical experiments, start with the papers [CP] and the book [DEGM]. More sophisticated discussion can be found in [FT].

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