7.8

6. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r - 2 = 0$, with roots $r_1 = 2$ and $r_{2,3} = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\xi_1 - \xi_3 = 0$$
$$\xi_2 - \xi_3 = 0.$$

A corresponding eigenvector vector is given by $\boldsymbol{\xi}^{(1)} = (1, 1, 1)^T$. Setting $r = -1$, the system of equations is reduced to the *single* equation

$$\xi_1 + \xi_2 + \xi_3 = 0.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(2)} = (1, 0, -1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r = -1$) is $\boldsymbol{\xi}^{(3)} = (0, 1, -1)^T$. Therefore the general solution may be written as

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}e^{-t}.$$

7. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\mathbf{I}) = r^2 + 6r + 9 = 0$. The only root is $r = -3$, which is an eigenvalue of multiplicity *two*. Substituting $r = 3$ into the coefficient matrix, the system reduces to the single equation $\xi_1 - \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}.$$

For a second linearly independent solution, we search for a *generalized eigenvector*. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

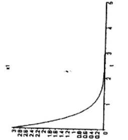
that is, $4\eta_1 - 4\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. It follows that a second solution is given by

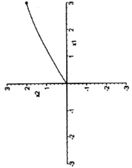
$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-3t} + \begin{pmatrix} k + 1/4 \\ k \end{pmatrix}e^{-3t}$$
$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}e^{-3t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}e^{-3t}\right].$$

(10)





Imposing the initial conditions, we require that

$$c_1 + \frac{1}{4}c_2 = 3$$
$$c_1 = 2,$$

which results in $c_1 = 2$ and $c_2 = 4$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix}te^{-3t}.$$