20. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + \omega^2 Y(s) = \frac{s}{s^2+4}\,.$$

Applying the *initial conditions,*

$$s^2 Y(s) + \omega^2 Y(s) - s = \frac{s}{s^2+4}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{1}{4-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+4}\right].$$

First note that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos\omega t \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4-\omega^2}\cos\omega t - \frac{1}{4-\omega^2}\cos 2t + \cos\omega t \\ &= \frac{5-\omega^2}{4-\omega^2}\cos\omega t - \frac{1}{4-\omega^2}\cos 2t\,. \end{aligned}$$

25. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions,*

$$s^2 Y(s) + Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_0^\infty f(t)\, e^{-st} dt \\ &= \int_0^1 t\, e^{-st} dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\,. \end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+1)} - e^{-s}\frac{s+1}{s^2(s^2+1)}\,.$$

Using *partial fractions,*

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

and

$$\frac{s}{s^2(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

We find, by inspection, that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = t - \sin t\,.$$

Referring to *Line 13*, in Table 6.2.1,

$$\mathcal{L}[u_c(t) f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Let

$$\mathcal{L}[g(t)] = \frac{s+1}{s^2(s^2+1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\,.$$

Then $g(t) = 1 + t - \cos t - \sin t$. It follows, therefore, that

$$\mathcal{L}^{-1}\left[e^{-s}\cdot\frac{s+1}{s^2(s^2+1)}\right] = u_1(t)[1 + (t-1) - \cos(t-1) - \sin(t-1)]$$

Combining the above, the solution of the IVP is

$$y(t) = t - \sin t - u_1(t)[1 + (t-1) - \cos(t-1) - \sin(t-1)]\,.$$