

6.4

2. Let  $h(t)$  be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 2[s Y(s) - y(0)] + 2 Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 2s Y(s) + 2 Y(s) - 1 = \mathcal{L}[h(t)].$$

The forcing function can be written as  $h(t) = u_\pi(t) - u_{2\pi}(t)$ . Its transform is

$$\mathcal{L}[h(t)] = \frac{e^{-\pi s} - e^{-2\pi s}}{s}.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{1}{s^2 + 2s + 2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2 + 2s + 2)}.$$

First note that

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{1}{2} \frac{1}{s} - \frac{1}{2} \frac{(s+1) + 1}{(s+1)^2 + 1}.$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 2s + 2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2 + 1}\right] = e^{-t} \sin t.$$

Now let

$$G(s) = \frac{1}{s(s^2 + 2s + 2)}.$$

Then

$$\mathcal{L}^{-1}[G(s)] = \frac{1}{2} - \frac{1}{2} e^{-t} \cos t - \frac{1}{2} e^{-t} \sin t.$$

Using Theorem 6.3.1,

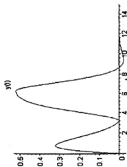
$$\mathcal{L}^{-1}[e^{-cs} G(s)] = \frac{1}{2} u_c(t) - \frac{1}{2} e^{-(t-c)} [\cos(t-c) + \sin(t-c)] u_c(t).$$

Hence the solution of the IVP is

$$y(t) = e^{-t} \sin t + \frac{1}{2} u_\pi(t) - \frac{1}{2} e^{-(t-\pi)} [\cos(t-\pi) + \sin(t-\pi)] u_\pi(t) - \frac{1}{2} u_{2\pi}(t) + \frac{1}{2} e^{-(t-2\pi)} [\cos(t-2\pi) + \sin(t-2\pi)] u_{2\pi}(t).$$

That is,

$$y(t) = e^{-t} \sin t + \frac{1}{2} [u_\pi(t) - u_{2\pi}(t)] + \frac{1}{2} e^{-(t-\pi)} [\cos t + \sin t] u_\pi(t) + \frac{1}{2} e^{-(t-2\pi)} [\cos t + \sin t] u_{2\pi}(t).$$



The solution starts out as free oscillation, due to the initial conditions. The amplitude increases, as long as the forcing is present. Thereafter, the solution rapidly decays.

