10. We first note that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = t\,e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t\,.$$

Based on the *convolution theorem*,

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] = \frac{1}{2}\int_0^t (t-\tau)e^{-(t-\tau)}\sin 2\tau\,d\tau$$
$$= \frac{1}{2}\int_0^t \tau\,e^{-\tau}\sin(2t-2\tau)\,d\tau\,.$$

15. The transform of the ODE (given the specified initial conditions) is

$$4s^2\,Y(s) + 4s\,Y(s) + 17Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)}{4s^2+4s+17}\,.$$

First write

$$\frac{1}{4s^2+4s+17} = \frac{\frac{1}{4}}{\left(s+\frac{1}{2}\right)^2+4}\,.$$

Based on the elementary properties of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{1}{4s^2+4s+17}\right] = \frac{1}{8}e^{-t/2}\sin 2t\,.$$

Applying the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{1}{8}\int_0^t e^{-(t-\tau)/2}\sin 2(t-\tau)\,g(\tau)\,d\tau\,.$$

7.7

5. The general solution, found in Prob. 3, Section 7.6, is given by

$$\mathbf{x} = c_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t\end{pmatrix} + c_2\begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t\end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$5c_1 = 1$$
$$2c_1 - c_2 = 0\,,$$

resulting in $c_1 = 1/5\,, c_2 = 2/5$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix}\cos t + 2\sin t \\ \sin t\end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$5c_1 = 0$$
$$2c_1 - c_2 = 1\,,$$

resulting in $c_1 = 0\,, c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -5\sin t \\ \cos t - 2\sin t\end{pmatrix}.$$

Therefore the fundamental matrix is

$$\boldsymbol{\Phi}(t) = \begin{pmatrix}\cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t\end{pmatrix}.$$