# Noether's Theorem

Emmy Noether: 1882 - 1935

- Number fields
- Calculus of variations
- Noncommutative algebras

University of Erlangen 1907, mathematics

1915-1933 University of Göttingen (David Hilbert, Felix Klein)

Plenary address, International Congress of Mathematicians, 1932

Bryn Mawr College, Pennsylvania, 1933.

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**INVARIANT VARIATIONAL PROBLEMS**

(For F. Klein, on the occasion of the fiftieth anniversary of his doctorate)

by **Emmy Noether** in Göttingen

Presented by F. Klein at the session of 26 July 1918*

We consider variational problems which are invariant[A] under a continuous group (in the sense of Lie); the consequences that are implied for the associated differential equations find their most general expression in the theorems formulated in §1, which are proven in the subsequent sections. For those differential equations that arise from variational problems, the statements that can be formulated are much more precise than for the arbitrary differential equations that are invariant under a group, which are the subject of Lie's researches. What follows thus depends upon a combination of the methods of the formal calculus of variations and of Lie's theory of groups. For certain groups and variational problems this combination is not new; I shall mention Hamel and Herglotz for certain finite groups, Lorentz and his students (for example, Fokker), Weyl and Klein for certain infinite groups.[1] In particular, Klein's second note and the following developments were mutually influential, and for this reason I take the liberty of referring to the final remarks in Klein's note.

# Outline:

- History of the calculus of variations: the Principle of Least Action
- The Calculus of Variations in a nutshell, some applications
- Action functionals and Euler-Lagrange equations
- Noether’s Theorem; Transformation groups, symmetries, conservation laws
- Some ideas on the proof
- Some examples of conservations laws derived from symmetries

# Origins of the Variational Calculus: The Principle of Least Action in Physics

• Plato ($\sim$ 400 B.C.): *the architect of the world created the cosmos by reducing the primal state of chaos into order...*

• Aristotle ($\sim$ 350 B.C.): motion follows the 'simplest' patterns; *the minimum movement is the swiftest...*

• Hero of Alexandria ($\sim$ 125 B.C.): showed that when a ray of light is reflected by a mirror, the path actually taken from the light source to the observer's eye is shorter than any other possible path so reflected.

• William of Ockham ($\sim$ 1325): *It is futile to employ many principles when it is possible to employ fewer.* (i.e., apply the razor...)

• Newton ($\sim$ 1680): *We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.*

• Leibniz ($\sim$ 1680): ours is the best of all possible worlds, exhibiting *the greatest simplicity in its premises and the greatest wealth in its phenomena.*

• Malbrange (philosopher, $\sim$ 1675): the *economy of nature.*

• Fermat ($\sim$ 1650): The principle of least time in optics (reflection or refraction).

• Maupertuis ($\sim$ 1744): *Nature changes in such as way as to make the action the least possible.* (collisions of particles, refraction)

• John Bernoulli ($\sim$ 1696): brachistochrone problem.

• Euler and Lagrange ($\sim$ 1750): formal development of calculus of variations.

• Hamilton ($\sim$ 1850): Hamiltonian formulation of Lagrange's variational problem, and the canonical equations of mechanics.

# The Calculus of Variations:
Variational Derivative, Euler-Lagrange Equations

Functional $S \;:\; X \;\to\; \mathbf{R}$, $\quad X$ a space of functions $\varphi$ on $\mathbf{R^m}$
Action density $\mathcal{S}(u, \mathbf{p}) : \;\mathbf{R} \times \mathbf{R}^m \;\to\; \mathbf{R}$

$$S[\varphi] \;=\; \int_\Omega \mathcal{S}(\varphi(x), \nabla\varphi(x))\, d^m x, \qquad \Omega \subset\subset \mathbf{R}^m$$

Derivative of $S$ at $\varphi$ in the direction $\psi$;

$$\begin{aligned} S'[\varphi](\psi) \;&=\; \frac{d}{d\varepsilon} S[\varphi + \varepsilon\psi]|_{\varepsilon=0} \\ &=\; \int_\Omega \frac{d}{d\varepsilon} \mathcal{S}(\varphi + \varepsilon\psi,\; \nabla\varphi + \varepsilon\nabla\psi)|_{\varepsilon=0}\, d^m x \\ &=\; \int_\Omega \left(\mathcal{S}_u \psi + \mathcal{S}_{\mathbf{p}} \cdot \nabla\psi\right)\, d^m x \\ &=\; \int_\Omega \left(\mathcal{S}_u - \nabla \cdot \mathcal{S}_{\mathbf{p}}\right) \psi\, d^m x \end{aligned}$$

since

$$\mathcal{S}_{\mathbf{p}} \cdot \nabla\psi = \nabla \cdot (\mathcal{S}_{\mathbf{p}}\psi) - (\nabla \cdot \mathcal{S}_{\mathbf{p}})\psi; \qquad \text{integration by parts}$$

and

$$\int_\Omega \nabla\cdot(\mathcal{S}_{\mathbf{p}}\psi)\, d^m x \;=\; \int_{\partial\Omega} \mathcal{S}_{\mathbf{p}}\psi\, ds \;=\; 0; \quad \text{divergence theorem and } \psi = 0 \;\text{ on }\; \partial\Omega$$

If $S'[\varphi] = 0$, i.e., $S'[\varphi](\psi) = 0$ for all (test) functions $\psi$, then $\varphi$ must satisfy the **Euler-Lagrange** equations;

$$\mathcal{S}_u(\varphi, \nabla\varphi) - \nabla \cdot \mathcal{S}_{\mathbf{p}}(\varphi, \nabla\varphi) = 0 \;=\; \frac{\partial\mathcal{S}}{\partial\varphi} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial\mathcal{S}}{\partial\varphi_i}, \qquad \partial\varphi_i = \frac{\partial\varphi}{\partial x_i}$$

<u>Classical mechanics</u>: Particle position $\mathbf{q}(t)$, $\;\mathcal{S}(\mathbf{q}(t), \dot{\mathbf{q}}(t))$. Euler - Lagrange equations are;

$$\frac{\partial\mathcal{S}}{\partial\mathbf{q}} \;-\; \frac{d}{dt}\left(\frac{\partial\mathcal{S}}{\partial\dot{\mathbf{q}}}\right)$$

**Action functional**: $S[\varphi] \; = \; \int_\Omega \mathcal{S}(\varphi(x), \nabla\varphi(x))\, d^m x$

**Euler-Lagrange (E-L) equations**: $\dfrac{\partial \mathcal{S}}{\partial \varphi} - \sum_{i=1}^{m} \dfrac{\partial}{\partial x_i} \dfrac{\partial \mathcal{S}}{\partial \varphi_i}$

| **Action density $\mathcal{S}$** | **E-L equations** | Commonly known as... |
|---|---|---|
| $\frac{1}{2} m \lvert \dot{\mathbf{q}} \rvert^2 - V(\mathbf{q})$ | $\ddot{\mathbf{q}} = -\nabla V(\mathbf{q})$ | Newton's equation |
| $\lvert \nabla \varphi \rvert^2$ | $\Delta \varphi = 0$ | Laplace equation |
| $-\frac{1}{2}(\partial_t^2 \varphi)^2 + \frac{1}{2} \lvert \nabla \varphi \rvert^2 + F(\varphi)$ | $\partial_t^2 \varphi - \Delta \varphi + F^{'}(\varphi) = 0$ | nonlinear wave equation |
| $-\frac{\hbar}{2} \mathrm{Im} \psi \dot{\bar{\psi}} - \frac{\hbar^2}{2m} \lvert \nabla \psi \rvert^2 + V \lvert \psi \rvert^2$ | $i\hbar \partial_t \psi = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \psi$ | Schrödinger equation |
| $\frac{1}{2} \left( \lvert \dot{A} \rvert^2 - \lvert \nabla \times A \rvert^2 \right)$ | $\Box A = 0, \quad \nabla \cdot A = 0$ | Maxwell equations |
| $g^{ij} R_{ij}$ | $\sqrt{-g} \left( R_{ij} - \frac{1}{2} g_{ij} R \right) = 0$ | Einstein equation |

# Noether's Theorem

Action functional; $S: \; X \; \to \; \mathbf{R}$, $\qquad S[\varphi] \; = \; \int_\Omega \mathcal{S}(\varphi(x), \nabla_m \varphi(x))\, d^m x;$

One parameter group of transformations; $T_\lambda : \; X \; \to \; X, \quad \lambda \in \mathbf{R}$

$T_\lambda$ is a **symmetry group** of $S$ if $S[\varphi_\lambda] = S[\varphi]$ for all $\varphi, \lambda$ where $\varphi_\lambda \equiv T_\lambda \varphi$.
($\Longrightarrow T_\lambda$ is a symmetry of the Euler-Lagrange equations, but not conversely)

i.e., If $\tilde{\varphi}(\tilde{x}, \tilde{t}) = T_\lambda(\varphi)$ and $\tilde{\Omega} = T_\lambda(\Omega)$ then

$$S[\tilde{\varphi}] \; = \; \int_{\tilde{\Omega}} \mathcal{S}\left(\tilde{\varphi}(\tilde{x}), \tilde{\nabla}\tilde{\varphi}(\tilde{x})\right)\, d^m \tilde{x} \quad = \quad \int_\Omega \left(\mathcal{S}(\varphi(x), \nabla\varphi(x)\right)\, d^m x \; = \; S[\varphi]$$

Examples of symmetries: translations in space and time, rotations in space, Lorentz transformations (rotations in space and time), etc.

**Noether:** If $T_\lambda$ is a symmetry of $S$, then the solutions to the associated Euler-Lagrange equations satisfy a conservation law, i.e. there is some function $E(t)$, $(E(t) = \int \mathcal{E}(\varphi, \nabla\varphi)\, dx)$, such that $\frac{d}{dt}E(t) = 0$.
(Here we are distinguishing one coordinate as time $t$ and the rest as spatial variables, so we have $\mathbf{R}^m = \mathbf{R} \times \mathbf{R}^{m-1}$ and $d^m x = dt\, d^{m-1}x$ in the above formulae.)

**Point particles** $q_i(t)$: $\quad m_i \ddot{q}_i(t) \;=\; -\nabla V(q(t))$

| **Symmetry** | **Conserved quantity** (i.e., $\frac{d}{dt}E(t) = 0$) |
|---|---|
| translation in time | energy $E(t) = \frac{1}{2}\sum_i m_i \lvert \dot{\mathbf{q}}_i\rvert^2(t) + V(\mathbf{q}(t))$ |
| translation in space | linear momentum $\mathbf{P}(t) = \sum_i m\dot{\mathbf{q}}_i(t)$ |
| rotation in space | angular momentum $\mathbf{L}(t) \;=\; \sum_i \mathbf{q_i}(t) \times m_i \dot{\mathbf{q}}_i(t)$ |

$$\text{Virial Theorem: } \frac{1}{2}\sum_i \overline{m\dot{q}_i^2(t)} \;=\; K\,\overline{V(q(t))}$$

**Wave equation** $\varphi(x,t)$: $\quad \partial_t^2\varphi - \Delta\varphi + f(\varphi) = 0; \quad f(u) = V^{'}(u)$

$$\partial_t^2 \equiv \frac{\partial^2}{\partial_t^2}, \quad \Delta \equiv \sum_i \frac{\partial^2}{\partial_{x_i}^2}$$

| **Symmetry** | **Conserved quantity** |
|---|---|
| translation in time | energy $E(t) = \int_{\mathbf{R^n}} \left\{\frac{1}{2}\Big((\partial_t\varphi(x,t))^2 + \lvert\nabla\varphi(x,t)\rvert^2\Big) + V(\varphi(x,t))\right\} d^n x$ <br> $= \int_{\mathbf{R^n}} e(\varphi(x,t))\, d^n x$ |
| translation in space | linear momentum $\mathbf{P}(\mathbf{t}) = \int_{\mathbf{R^n}} \partial_t\varphi(x,t)\nabla\varphi(x,t)\, d^n x$ |
| rotation in space (about $\mathbf{r}$) | angular momentum $\mathbf{L}(\mathbf{t}) \;=\; \int_{\mathbf{R^n}} \Big(\mathbf{r} \wedge \nabla\varphi(x,t)\Big)\partial_t\varphi(x,t)\, d^n x$ |
| Lorentz boost direction $\mathbf{r}$ | $\Lambda_{\mathbf{r}}(t) \;=\; \int_{\mathbf{R^n}} \{e(\varphi(x,t))\mathbf{r} - t(\mathbf{r}\cdot\nabla\varphi(x,t))\partial_t\varphi(x,t)\}\, d^n x$ |

$$\text{Virial Theorem: } \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T}\int_{\mathbf{R^n}} \left\{-\frac{1}{2}(\partial_t\varphi(x,t)^2 + \frac{1}{2}\lvert\nabla\varphi(x,t)\rvert^2 + V(\varphi(x,t))\right\} d^n x\, dt \;=\; 0$$

## The idea of Noether's proof

Action functional; $S: \ X \ \to \ \mathbf{R}, \qquad S[\varphi] \ = \ \int_\Omega \mathcal{S}(\varphi(x), \nabla_m \varphi(x))\, d^m x;$

One parameter group of transformations; $T_\lambda: \ X \ \to \ X, \quad \lambda \in \mathbf{R}$

$T_\lambda$ is a **symmetry group** of $S$ if $S[\varphi_\lambda] = S[\varphi]$ for all $\varphi, \lambda$ where $\varphi_\lambda \equiv T_\lambda \varphi$.
($\Longrightarrow T_\lambda$ is a symmetry of E-L, but not conversely)

For any transformation group $T_\lambda$, by the Chain Rule,

$$\frac{d}{d\lambda} S[\varphi_\lambda]\Big|_{\lambda=0} \ = \ S^{'}[\varphi](A\varphi), \qquad \text{where } \ A = \frac{d}{d\lambda} T_\lambda\Big|_{\lambda=0}, \quad \varphi = \varphi_{\lambda=0}$$

**Conservation law** for $\varphi(x,t)$: $\quad Div(e, \mathbf{p}) = \nabla_m \cdot (e, \mathbf{p}) = \partial_t e + \nabla \cdot \mathbf{p} = 0,$
where $e, \mathbf{p}$ are functions of $\varphi(x,t)$. Note $\nabla_m = (\partial_t, \nabla)$.
This is because if we define $E(t) = \int_{\Omega_x} e(\varphi(x,t), \nabla\varphi(x,t))\, dx$, then integrating the equation $\partial_t e = -\nabla \cdot \mathbf{p}$ over all space $\Omega_x$ we obtain $\frac{d}{dt} E(t) = \int_{\Omega_x} \partial_t e\, dx = -\int_{\Omega_x} \nabla \cdot \mathbf{p}\, dx = 0$ since the last integral vanishes (Divergence Theorem and the hypothesis that $\varphi$ vanishes on the boundary of $\Omega_x$). We are also singling out one variable as time $t$ so that $\Omega = [-T, T] \times \Omega_x$ (often $\Omega_x = \mathbf{R}^n$).

Recall; $\quad S^{'}[\varphi](A\varphi) = \int_\Omega \left(\mathcal{S}_u(A\varphi) + \mathcal{S}_{\mathbf{p}} \cdot \nabla_m(A\varphi)\right)\, d^m x$

**Noether:** *If $T_\lambda$ is a symmetry of $\mathcal{S}$, then for <u>any</u> $\varphi$ ;*

$$(\mathcal{S}_u(A\varphi) + \mathcal{S}_{\mathbf{p}} \cdot \nabla_m(A\varphi)) = Div(\mathcal{Q}), \qquad \text{where } \mathcal{Q} \text{ is explicitly given from } A$$

*So,* $\ (\mathcal{S}_u(A\varphi) + \mathcal{S}_{\mathbf{p}} \cdot \nabla_m(A\varphi)) - Div(\mathcal{Q}) \ = \ 0.$

*Now, we can write this as;*

$$\left(\mathcal{S}_u - \nabla_m \cdot \mathcal{S}_{\mathbf{p}}\right)(A\varphi) + Div\left(\mathcal{S}_{\mathbf{p}}(A\varphi) - \mathcal{Q}\right) \ = \ 0$$

(integration by parts; $\mathcal{S}_{\mathbf{p}} \cdot \nabla_m(A\varphi) = \nabla_m \cdot (\mathcal{S}_{\mathbf{p}}(A\varphi)) - (\nabla_m \cdot \mathcal{S}_{\mathbf{p}})(A\varphi)$)

*If in addition $\varphi$ is also a critical point of $\mathcal{S}$, then the first term on the left vanishes and we obtain the conservation law;*

$$Div\left(\mathcal{S}_{\mathbf{p}}(A\varphi) - \mathcal{Q}\right) \ = \ 0$$

## Some References

These notes: http://www.sfu.ca/∼rpyke/presentations.html

*Notes on Symmetries, Conservation Laws, and Virial Relations*; http://www.sfu.ca/∼rpyke/research/index.html

*Calculus of Variations*, I.M. Gelfand, S.V. Fomin.

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*The Calculus of Variations*, C. Carathéodory, 1935.