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MODELLING HETEROGENEITY AND AN OPEN-MINDEDNESS SOCIAL NORM IN OPINION DYNAMICS

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Abstract. We study *heterogeneous* interactions in a time-continuous bounded confidence model for opinion formation. The key new modelling aspects are to distinguish between open-minded and closed-minded behaviour and to include an open-mindedness social norm. The investigations focus on the equilibria supported by the proposed new model; particular attention is given to a novel class of equilibria consisting of multiple *connected* opinion clusters, which does not occur in the absence of heterogeneity. Various rigorous stability results concerning these equilibria are established. We also incorporate the effect of media in the model and study its implications for opinion formation.

1. Introduction.

The mathematical modelling of opinion formation within a group of interacting agents has recently been attracting considerable interest. Various models for opinion formation have been proposed, covering a diverse range of issues including consensus formation or the polarization of opinions [25, 36, 42], the emergence of extremism [13], the evolution of political organizations [5], and media influence [8]. Such models may be viewed as falling within the more general study of self-organized dynamics [10] and aggregation phenomena [43], which has also seen a surge of interest in the last decade: Due to inter-individual interactions among the members of a group, self-organization may occur in a physical space (insect swarms, fish schools, robots) or, more abstractly, in an opinion space.

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The modelling of opinion dynamics predominantly seeks to represent different aspects of social influence [4]. This is manifested in the general tendency of most individuals to change their own opinion to align with those of their peers, and has been studied extensively since the classical work of Festinger on social comparison and cognitive dissonance [17, 18]. A standard mathematical framework for modelling opinion formation considers a number N of individuals (agents) and an opinion modelled as a real number. In the notation of the present paper, an individual i has an opinion x_i, with x_i taking values in the interval $[-1, 1]$. The end values -1 and 1 are regarded as *extreme* opinions. Social interactions between individuals are typically represented by an adjacency matrix $A = (a_{ij})_{1 \leq i,j \leq N}$, where the generic entry a_{ij} in the matrix describes the social influence exerted *on* individual i *by* individual j [19]. In the most general form, the matrix A depends on time and is not necessarily symmetric.

A model for opinion formation is a set of rules for opinions x_i to evolve over time. Such rules can be *time-discrete*, resulting in difference equations [13, 19, 25] or *time-continuous*, expressed as differential equations [42]. In both types of models, an individual's opinion is driven by a weighted average of the opinions of those other agents with whom he/she interacts. Depending on the model, and in particular on the evolution of the time-dependent interaction matrix A, the dynamics could result in full consensus (all x_i approach the same common value) or in the formation of several clusters (groups) of individuals sharing the same opinion value.

An individual's response to peer opinions is in general not homogeneous; instead, there is an enhanced tendency of individuals to interact with and be influenced by others with similar opinions (homophily). Mathematically, this can be modelled by permitting no social influence between individuals i and j when their opinions x_i and x_j differ by more than a fixed amount (which we denote by ϵ), that is, by requiring the corresponding interaction weight a_{ij} to vanish. Models that incorporate this feature are referred to as *bounded confidence* models [12, 25, 36, 42, 23]. Compared to *global* models of opinion formation [1, 14], which promote consensus formation, bounded confidence models are able to demonstrate a more diverse set of equilibria.

The model considered in this paper is a time-continuous social influence model which generalizes the Motsch-Tadmor model [11, 41, 42] described in Section 2 below. The generalization is done in two stages. First, we focus on the *heterogeneous* response within a population [31, 33] and consider the psychological mechanism behind its occurrence. Second, we include media in the model and investigate its effects on the opinions of the population. The two generalizations are elaborated below.

We model heterogeneity by recognizing different approaches to communication, while acknowledging that in a social interaction with peers, an individual acts simultaneously as both communicator and communicatee. Among communicatees, we distinguish between open- and closed-minded individuals; while as communicators, agents are assumed to act either by encouraging open discussion or by avoiding it [51]. Open-minded communicatees who also act openly as communicators are simply referred in the paper as *open-minded* individuals; regardless of their opinion, they are assumed to interact with everyone else within their interaction range ϵ.

The remainder of the population is assumed to interact in a cold, discussion-averse manner when acting as communicators, and also to be disposed to behave in a closed-minded manner as communicatees when their opinions become too extreme. Specifically, we introduce a critical threshold X_c in the opinion space, and regard opinions in the interval $[-X_c, X_c]$ as being "moderate". Individuals who have a closed-mindedness disposition but who hold sufficiently moderate opinions, are assumed to respond open-mindedly as communicatees. On the other hand, those having non-moderate opinions behave closed-mindedly, and blindly approach their respective group's extreme opinion (at 1 or -1). However, there is a caveat: in one of the main novelties of our model, we optionally allow for the inclusion of an *open-mindedness social norm*. There is empirical evidence that by acting openly and warmly as communicators, individuals can induce communicatees to be more receptive and thereby facilitate open-minded discussion that allows for efficient communication about and potential reconciliation of conflicting views [40, 49, 50, 51]. We model this effect by having extremist, otherwise closed-minded individuals behave open-mindedly when—and only when—they interact with communicators that approach discussion openly, namely those which in our model are represented by the open-minded individuals.

A key focus of our work concerns the diversity and stability of equilibria obtained with the proposed heterogeneous opinion formation model. In previous *homogeneous* bounded confidence models [25, 12, 42], the dynamic evolution leads either to a consensus at an averaged opinion value or to several *isolated* opinion groups. Specifically, in a multi-cluster equilibrium, all opinions within each group asymptotically approach a common value [30], but the groups are disconnected from each other (the opinions of individuals in different clusters are separated by a distance larger than the bound of confidence ϵ). As a consequence, such equilibria are neutrally stable; for instance, a perturbation consisting of a small uniform translation of all opinions would not die out.

By contrast, we demonstrate that the *heterogeneous* model proposed in this paper, incorporating both open- and closed-minded agents, supports additional types of equilibria, in particular *connected* multi-cluster groups, extreme polarization and consensus at an extreme opinion. We illustrate numerically various qualitatively different equilibria that emerge from our model, and provide bifurcation studies showing their dependence on some important parameters. Most significantly, it turns out that the extreme consensus and polarization equilibria are *asymptotically stable*, as are the multi-cluster states provided they are suitably connected; we prove this result analytically using a graph-theoretical framework. Such stability is not often found in such aggregation or opinion dynamics models; in the present case it is due to the attraction experienced by (non-moderate) closed-minded individuals towards the extremes at ± 1, which act as sinks in the dynamics.

We note that various authors [20, 32, 34, 35] have recently considered heterogeneity in bounded confidence models using a different approach. The common strategy in these papers is to distinguish between open- and closed-minded agents by assigning different bounds of confidence to individuals, with the main goal of studying how heterogeneity affects the likelihood of consensus formation in society. In [35], for instance, it is noted that heterogeneity substantially enriches the complexity of the dynamics and in particular, perhaps surprisingly, enhances the chances of reaching a consensus. We point out that our approach of modelling heterogeneity is different, and also that our focus is on connected steady states and not solely on pure consensus. Note that while some *partially connected* equilibria have been obtained using other heterogeneous models [35], the discussion there does not dwell at all on this aspect.

The effect of attraction to (possibly) extreme views has also received attention. In one recent such study, Hegselmann and Krause [26] incorporated radical individuals or charismatic leaders into their original time-discrete model [25]. Such individuals, who by assumption maintain constant opinion values, send signals to the rest of the population (referred to as normals in [26]), signals which may or may not drive normals to their radical view. Among the differences between their model and ours is that we assume two radical opinions (placed at the extremes, 1 and -1), not just one as in [26]. More importantly, in our model the attraction exerted by these extreme opinions acts only on a subset of a population, namely on closed-minded agents with non-moderate views. The remainder of the population experiences an attraction to the extremes only indirectly, via interactions with individuals in the susceptible subset.

The time-discrete bounded confidence model of Deffuant *et al.* [12] has also been extended [13, 38] by incorporating extremists (at two extremes) as well as moderates; this generalization can also give rise to central multi-cluster and polarization states. In this model, which evolves by randomly-chosen pairwise interactions, when the bounds of confidence are fixed and sufficiently large, a statistically stationary state can arise in which the moderates' opinions fail to equilibrate [38]; this may be interpreted as an alternative form of "connectedness" maintained by ongoing opinion fluctuations.

Our second major extension to the Motsch-Tadmor model [42] is to incorporate the effect of media influences on the opinions in a population. It is important to take into account that, as for their reactions to peer opinions, individuals' responses to media exposure are heterogeneous. As noted in [31, 33], while some individuals become more moderate when exposed to *cross-cutting media* (which expresses a different partisan perspective from their own), a majority of individuals experiencing media influences with such contrary partisan biases tend either to not change their opinion, or to react by becoming more extreme in their own beliefs.

Existing mathematical models that consider the effects of media focus primarily on the interplay between consensus-seeking behaviour and the media [8, 39], and thus seem to overlook the effect of closed-mindedness or consensus-resisting behaviour in the presence of media. The model proposed in this work does consider these effects, systematically extending the above-mentioned ideas on open- and closed-mindedness to allow for media influences: We discuss the role of media in this framework and, in particular, the effect of social norms of open-mindedness in facilitating consensus or polarization. These questions are particularly intriguing since the perception of news media is often influenced by a social phenomenon known as the hostile media effect [52].

The outline of the paper is as follows. Section 2 introduces the heterogeneous model for opinion formation proposed in this paper. Section 3 focusses on the equilibria of this model, including a numerical illustration of various qualitatively different states accessible in our model, a bifurcation study with respect to several parameters, and a rigorous analysis of the stability of connected equilibria. In Section 4 we introduce media into our model, including both some numerical examples and some rigorous stability results for the extended system.

2. **Formulation of the model.**

The modeling of group opinion dynamics using differential equations has grown in popularity over the last few years [42, 16, 8, 39, 43, 7, 2]. Our work is based is an ordinary differential equation (ODE) model recently proposed and investigated by Motsch and Tadmor [41, 42].

Consider N individuals with opinions x_i, $i = 1, \dots, N$, which we assume to take real values in the interval $[-1, 1]$, where the end values -1 and 1 represent the *extreme* opinions; this introduction of signed opinions allows for a group classification based on the sign of x_i. Following [41, 42], a basic model describing the time evolution of these opinions subject only to interpersonal interactions is a system of ordinary differential equations (ODEs) of the form

$$\frac{dx_i}{dt} = \alpha \sum_{j \neq i} a_{ij}(x_j - x_i), \qquad i = 1, ..., N, \tag{1}$$

where

$$a_{ij} = \phi(|x_j - x_i|)/\phi_i, \qquad \text{with} \qquad \phi_i = \sum_{j \neq i} \phi(|x_j - x_i|). \tag{2}$$

Here, $\alpha > 0$ represents an interaction strength, while $\phi \geq 0$ is an *influence* function acting on the relative difference of opinions $|x_j - x_i|$. We observe that the interaction coefficients a_{ij} have been normalized such that

$$\sum_{j \neq i} a_{ij} = 1, \quad i = 1, \dots, N, \tag{3}$$

and that the adjacency matrix (a_{ij}) is in general not symmetric.

One of the central ideas in [42] is that *heterophily*, the tendency to interact more strongly with those who have rather different opinions than with those holding more similar views (the opposite of *homophily*), may play a key role in the formation of opinion consensus and of group clusters in general. Mathematically, such tendencies in social interactions may be encapsulated in the choice of influence function ϕ. If ϕ has global support, $\phi(r) > 0$ for all $r \geq 0$, then the dynamics (1)–(2) always approach a consensus state [42]; that is, there is an opinion x_∞ so that $x_i(t) \to x_\infty$ as $t \to \infty$ for all i. However, if ϕ has compact support ϵ, then disjoint opinion clusters can form displaying local consensus [30]. The observation of Motsch and Tadmor [42] was that for a given ϵ, consensus formation is more likely if ϕ is a step function increasing over part of its support, so that individuals interact more weakly with others of very similar opinions than with those holding somewhat more distant views—this models heterophilious dynamics—than if ϕ is just the simple indicator function supported on $[0, \epsilon]$.

These observations concerning the effect of heterophily [42] foreshadow a general principle that consensus is more likely to occur when individuals are more open to being influenced by opinions somewhat different from their own. In the present paper we restrict our attention to non-increasing influence functions ϕ—specifically, we use the characteristic function $\phi = \mathbf{1}_{[0,\epsilon]}$—and focus instead on the distinction between *open-* and *closed-minded individuals*; in particular, we investigate the effect of an open-mindedness social norm, by which open-minded individuals could promote consensus even in the presence of closed-minded individuals.

In formulating our model, we consider different approaches to communication, recognizing that in each interaction, an individual acts simultaneously to transmit opinions and ideas as the *communicator*, and to receive them as the *communicatee* [6, 9]. As a communicatee, an individual can be *open-minded*, willing to listen to and be influenced by others' ideas; or on the contrary, be *closed-minded* and impervious to the opinions held by others. On the other hand, for communicators one can broadly distinguish between those who are warm and open in their communication, and thus more likely to be considered reliable and to influence others, versus those whose communication style is colder and more averse to open discussion [51].

In our model we consider two classes of agents. The first class consists in individuals who act warmly and openly as communicators and also behave open-mindedly as communicatees. We denote by O the set of such agents, whom we refer to as *open-minded individuals.* The remaining individuals, comprising the set C, are assumed to act coldly and in a discussion-avoiding manner in their capacity as communicators[1].

However, when functioning as communicatees, the response of individuals in C in our model depends on their opinion values. To incorporate the observation that closed-minded behaviour is usually only possible amongst extremist individuals [46], we introduce a critical threshold X_c, such that agents in C with *moderate* opinions falling between $-X_c$ and X_c are assumed to behave in an open-minded fashion as communicatees. On the other hand, individuals in C with *extreme* opinions are assumed to act closed-mindedly with respect to any other agent in C, in a manner that reinforces their extreme-seeking tendencies. With respect to their interactions with agents in O, we consider two possibilities: In one version of our model we assume that such extreme individuals always react similarly closed-mindedly, independent of with whom they are communicating. Our alternative modelling assumption is to introduce an *open-mindedness social norm*, in the light of evidence that interactions with individuals having warm and empathetic communication styles may induce others to react more open-mindedly [40, 49, 50, 51]; this is characterized by an open-minded response (as communicatees) of the extreme individuals in C when they interact with agents in O.

Throughout this paper the term *non-closed-minded* refers both to all individuals in O as well as to those individuals in C whose opinions are not sufficiently extreme as to induce closed-minded behaviour (that is, individuals in C with opinions in the range $[-X_c, X_c]$); while a *closed-minded individual* is by definition a member of C with opinion x_i satisfying $|x_i| > X_c$. The transition between open-minded and closed-minded behaviour is usually presumably continuous in practice, but for modelling simplicity we approximate this behavioural transition as an on-off process with a discrete threshold. We denote the total number of open-minded individuals by m; it follows that the number of individuals in C is $N - m$.

The general form of the opinion model is

$$\frac{dx_i}{dt} = f_i, \qquad i = 1, \dots, N, \tag{4}$$

where f_i represents the social force from peer pressure, whose form depends on whether an individual is behaving open- or closed-mindedly. We consider two versions of the model, depending on whether or not we include a social norm of open-mindedness. A schematic diagram showing interactions between individuals in our model is shown in Figure 1.

Model without open-mindedness social norm. In the absence of an open-mindedness social norm, we assume that closed-minded individuals are attracted to the nearest extremist views, regardless of with whom they are interacting [15, 27,

[1]With an abuse of notation we also let O and C be the sets of *indices* $i \in \{1, \dots, N\}$ that correspond to such individuals. No confusion should arise from this dual interpretation.

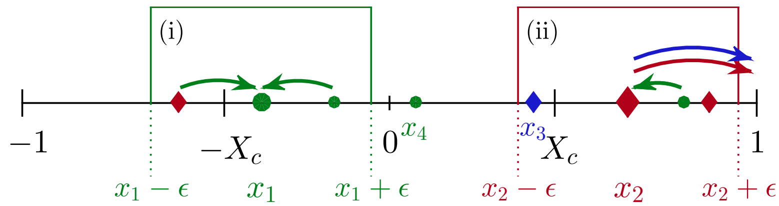


FIGURE 1. Schematic diagram of opinion interactions along the opinion continuum $[-1, 1]$ in the presence of a social norm of open-mindedness (note that opinion space is shown on the vertical axis in all other figures). Green circles represent warm and open-minded individuals (in O), while diamonds correspond to individuals in C, those having colder and less open communication styles. (i) The interactions on the left show the influences acting *on* the open-minded agent at x_1, which experiences attractive social forces (green arrows) from both open- and closed-minded individuals with opinions within the bound of confidence ϵ; the influence function $\phi(|x - x_1|) = \mathbf{1}_{[x_1-\epsilon, x_1+\epsilon]}$ is shown in green. (ii) The social forces acting on the closed-minded individual with non-moderate opinion x_2 (large red diamond; note $|x_2| > X_c$), in the presence of a social norm of open-mindedness, are shown on the right. Interactions with other members of C, both moderate (blue diamond at x_3) and extremist (red diamond), reinforce the opinion x_2 and drive it closer to the nearest extreme at 1 (blue and red arrows). However, given an open-mindedness social norm (model (6)), the interaction with an agent in O within its bound of confidence (green circle near x_2) induces an open-minded response and attractive interaction (green arrow) according to the influence function shown in red. In the absence of a social norm of open-mindedness (model (5)), this interaction would instead also drive the individual at x_2 to the extreme at 1.
Further notes: The individual with opinion x_3 (blue diamond) is in C, but has a sufficiently moderate opinion ($|x_3| < X_c$) to respond in an open-minded manner. In particular, it interacts with and is attracted to the agent at x_2 (arrow not shown); however, due to its intrinsically colder communication style it fails to attract the closed-minded individual at x_2, so that the influence is not reciprocal. Lastly, the distance of the opinion of the open-minded individual at x_4 to each of the opinions x_1, x_2 and x_3 is greater than the bound of confidence ϵ, so it does not interact with any of these three other individuals.

45]. The model reads:

$$f_i = \begin{cases} \alpha_2(N-1)\hat{a}_i\,(\mathrm{sgn}(x_i) - x_i) & \text{if } |x_i| > X_c \text{ and } i \in C, \\ \alpha_1 \sum\limits_{j \neq i} a_{ij}(x_j - x_i) & \text{otherwise.} \end{cases} \tag{5}$$

Here α_1 and α_2 are positive parameters capturing the magnitudes of interpersonal influences, while the interaction coefficients a_{ij} for non-closed-minded individuals are given by (2) as in the Motsch-Tadmor model [42]. In addition, we introduce coefficients $\hat{a}_i$ governing the interactions of closed-minded individuals with others. We note here that throughout the paper the "hat" is used for coefficients and parameters that relate to closed-minded individuals.

The psychological forces shaping $\hat{a}_i$ are different from those shaping a_{ij}. That is, by comparison with the schematic diagram in Figure 1, the closed-minded individual at x_2 would not be attractively influenced by the nearby open-minded agent (right-hand green arrow), but would rather be driven to the extreme as in all of their other interactions.

One might typically expect that the rate at which a closed-minded individual i approaches the extreme when interacting with the jth individual might depend on the sign of x_j, representing the perceived group categorization of the jth individual [46]. However, for simplicity we shall assume that the $\hat{a}_i$ in (5) are constant, $\hat{a}_i = 1/(N-1)$, independent of x_j. This is equivalent to postulating that the two psychological responses driving closed-minded individuals to seek their group's extreme opinion (depending on whether they are interacting with similarly-minded agents or with those holding opposing views) have comparable strength, that is, closed-minded individuals effectively see all others as similarly predisposed; see also Remark 2.2.

Model with open-mindedness social norm. Empirical studies [50, 49, 51, 40] have identified a strong link between open and warm behaviour and the enhancement of mutual understanding between individuals. It was found in these studies that an open and warm approach to communication induced curiosity, which in turn led to a search for more information. Participants confronted with opposing views in open, direct discussions started to doubt and feel uncertain about their own position. They showed more interest in learning, and felt motivated to ask questions and to search for more arguments [51]. The result was an enhancement of shared understanding on the issue in question. Throughout the paper we refer to this approach to communication using the notion of an open-mindedness social norm [40].

We modify our model (4)-(5) to incorporate an open-mindedness social norm by letting the extremist closed-minded individuals act in an open-minded manner, but only when interacting with open-minded individuals. In this case, the model is given by (4) with f_i defined by

$$f_i = \begin{cases} \alpha_2 (N-m-1)\hat{a}_i \left(\operatorname{sgn}(x_i) - x_i\right) + \alpha_3 \sum\limits_{j \in O} \hat{a}_{ij}(x_j - x_i) & \text{if } |x_i| > X_c \text{ and } i \in C, \\ \alpha_1 \sum\limits_{j \neq i} a_{ij}(x_j - x_i) & \text{otherwise.} \end{cases} \tag{6}$$

In (6), the interactions of non-closed-minded individuals with all others are as before, with interaction coefficients a_{ij} as in (2), where we fix the influence function $\phi = \mathbf{1}_{[0,\epsilon]}$.

The new features of this model, by comparison with (5), arise when we consider the behaviour of closed-minded agents. Thus suppose that the ith individual is in C and holds a non-moderate opinion, $|x_i| > X_c$; and we consider the effect on it due to its interaction with the jth individual:

If $j \in C$—that is, as a communicator the jth agent is assumed to act coldly and in a discussion-averse manner—then such an interaction merely reinforces the closed-minded and extremist tendencies of the ith individual and contributes to its attraction to the nearest extreme, at a rate governed by a normalized interaction coefficient $\hat{a}_i$ weighted by $\alpha_2 > 0$, as in (5); note that there are now $N - m - 1$ such interactions. The influence of the extremes ± 1 on closed-minded individuals is subject to the kernel $\hat{\phi}$ with support $\hat{\epsilon}$, where (as for the inter-individual influence function ϕ) we shall take $\hat{\phi} = \mathbf{1}_{[0,\hat{\epsilon}]}$. As in the model (5), we assume that the effect of this interaction on x_i is independent of x_j, provided $j \in C$; this is motivated by observations that closed-minded individuals consolidate their extreme opinions as a result of interactions with peers, regardless of whether the latter hold similar or dissimilar views [45].

If on the other hand $j \in O$, so that the interlocutor is one who communicates warmly and openly, then—in the presence of a social norm of open-mindedness—our extremist ith individual becomes receptive and is induced to respond open-mindedly, with strength $\alpha_3 > 0$ and an interaction coefficient $\hat{a}_{ij}$ playing the same role as a_{ij} encountered previously.

These interactions are represented schematically in Figure 1. Expressing the above description quantitatively, we thus model $\hat{a}_i$ and $\hat{a}_{ij}$ by

$$\hat{a}_i = \hat{\phi}(|\operatorname{sgn}(x_i) - x_i|)/\hat{\phi}_i \qquad \text{and} \qquad \hat{a}_{ij} = \phi(|x_j - x_i|)/\hat{\phi}_i. \tag{7}$$

The difference between $\hat{a}_{ij}$ and a_{ij} appears in the normalization condition: whenever the ith individual is closed-minded, we require

$$(N - m - 1)\hat{a}_i + \sum_{j \in O} \hat{a}_{ij} = 1, \tag{8}$$

so that the normalization factor $\hat{\phi}_i$ is given by

$$\hat{\phi}_i = (N - m - 1)\hat{\phi}(|\operatorname{sgn}(x_i) - x_i|) + \sum_{j \in O} \phi(|x_j - x_i|). \tag{9}$$

For simplicity we shall take $\hat{\phi}(|\operatorname{sgn}(x_i) - x_i|) = 1$ in all of our simulations unless explicitly stated otherwise. This means that the support $\hat{\epsilon}$ of $\hat{\phi}$ must be large enough to encompass all closed-minded individuals (those $i \in C$ with $|x_i| > X_c$); for this it is sufficient that $\hat{\epsilon} \geq 1 - X_c$. For theoretical purposes, though, this assumption is not necessary. If this condition does not hold ($\hat{\epsilon} < 1 - X_c$), then closed-minded individuals with relatively moderate opinions satisfying $|x_i| \in (X_c, 1 - \hat{\epsilon})$ would have $\hat{a}_i = 0$, so that their opinion dynamics would be governed entirely by the open-mindedness social norm.

We end this section with some remarks:

Remark 2.1. Since a_{ij} and $\hat{a}_{ij}$ are modelled using the same inter-individual influence function ϕ and differ only in their normalization (see (2), (7) and (9)), we have the following property for interactions between a closed-minded ith individual ($i \in C$ with $|x_i| > X_c$) and an open-minded jth individual ($j \in O$): While $\hat{a}_{ij}$ (modelling the influence of the open-minded individual j on the closed-minded individual i) and a_{ji} (the effect of the ith on the jth individual) are in general not equal, they *vanish simultaneously* (or not). This observation is important in establishing rigorous results on the stability of equilibria using the connectivity of the interaction matrix (see Section 3.4).

Remark 2.2. In the absence of open-minded individuals ($m = 0$, that is, O is empty), there can clearly be no active social norm of open-mindedness, so that

(7)–(9) become

$$\hat{a}_i = \hat{\phi}(|\operatorname{sgn}(x_i) - x_i|)/\hat{\phi}_i \qquad \text{with} \qquad \hat{\phi}_i = (N-1)\hat{\phi}(|\operatorname{sgn}(x_i) - x_i|). \tag{10}$$

This is simply $\hat{a}_i = 1/(N-1)$, so that the dynamics of closed-minded individuals governed by (6) reduce to (5). In fact, (10) could have been taken as the modelling assumption for $\hat{a}_i$ in model (5).

3. **Analysis and simulations.**

The introduction of distinct open-minded and closed-minded individuals in a Motsch-Tadmor-type opinion dynamics model considerably broadens the range of possible long-time behaviours. To illustrate the qualitatively different equilibria, we show the results of numerical simulations of model (4), with f_i given by (5) or (6), performed using the 4th order Adams-Bashforth method [29].

In our computations, we have two ways of initializing the time evolution of the model. The first, referred below as a *type NS* (non-symmetric) initial condition, distributes the opinions $x_1, \ldots, x_N$ uniformly in $(-1, 1)$, and then *randomly* assigns which m of the N indices represent the open-minded individuals. The second, *type S* (symmetric) initial condition is also an equispaced distribution of the N opinions in $(-1, 1)$, but in this case the m open-minded individuals are assigned *symmetrically and uniformly*. The $N - m$ non-open-minded individuals in symmetric data are assembled in $m + 1$ groups, each group having the same number of agents. More specifically, counting individuals in increasing order of their opinion value, there is a first group of $\frac{N-m}{m+1}$ non-open-minded agents, followed by an open-minded individual, then a second group of $\frac{N-m}{m+1}$ non-open-minded agents, followed by an open-minded, etc. Note that, for a given total population N, this symmetry requirement places a constraint on the possible m values.

In all of our simulations we use a total population $N = 80$; in this case symmetric placements of the open-minded agents (type S data) can occur only for $m = 0, 8, 26, 54, 72$ or 80. The values of the interaction strengths α_1, α_2 and α_3 are set at 1 throughout. All numerical results correspond to influence functions ϕ and $\hat{\phi}$ in the form of indicator functions with supports ϵ and $\hat{\epsilon}$, respectively. The stability results are also formulated for ϕ and $\hat{\phi}$ in this form; the same proof methodology would apply for any influence functions that are piecewise constant.

3.1. **Time evolution and equilibria.**

Figure 2 shows the time evolution for various different initial data and parameters, demonstrating several qualitatively different equilibrium states. For all these plots, we have simulated the model (4) with (6), corresponding to a social norm of open-mindedness, with $N = 80$ individuals. Plots (b), (c) and (e) of Figure 2 were obtained with type NS (non-symmetric) initial conditions, while (a), (d) and (f) correspond to type S (symmetric) initialization. Dashed lines indicate (open-minded) individuals in O, while solid lines represent (non-open-minded) individuals in C; some of the equilibria shown contain both types of agents.

As noted in the Introduction, in addition to consensus and isolated multi-cluster equilibria, our model captures *connected* multi-cluster states, which do not appear to have been observed with previous homogeneous bounded confidence models.

- If all individuals are open-minded ($m = N$), then our model reduces to (1)–(2), the model of Motsch and Tadmor [42]. In that case, the opinions can either coalesce on a single value x_∞, yielding consensus formation—this is guaranteed if the support ϕ is sufficiently large—or there is convergence to

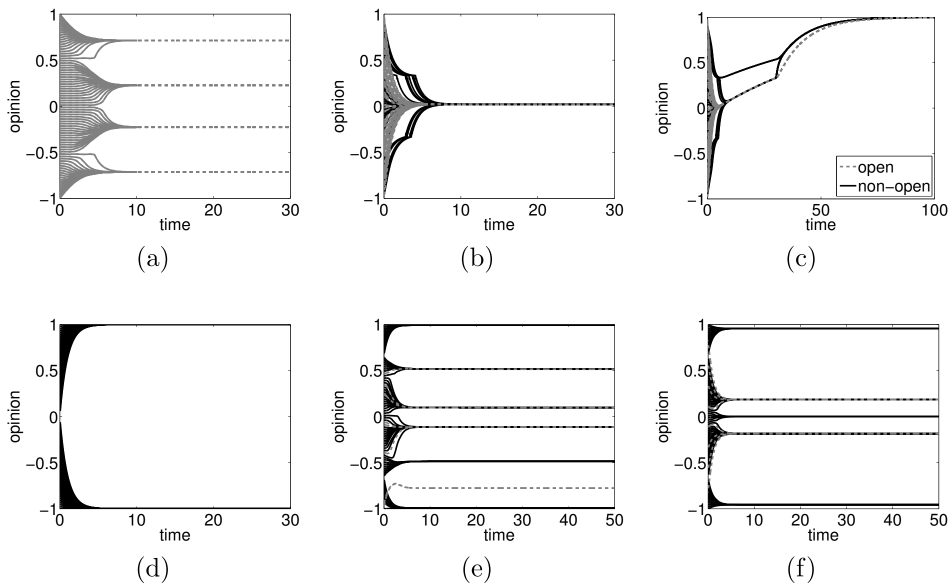


FIGURE 2. Evolution to equilibria of solutions to model (4) with (6) with $N = 80$ for some choices of the parameters m (number of open-minded agents), X_c (critical threshold for extreme-seeking dynamics) and ϵ (bound of confidence), and with both type NS (non-symmetric) and type S (symmetric) initial conditions. The plots illustrate various qualitatively different outcomes as discussed in the text: (a) $m = 80$ (X_c irrelevant), $\epsilon = 0.2$, type S; (b) $m = 54$, $X_c = 1/3$, $\epsilon = 1.5010$, type NS; (c) $m = 54$, $X_c = 1/3$, $\epsilon = 0.7909$, type NS; (d) $m = 0$, $X_c = 0$ $\epsilon = 2$, type S; (e) $m = 8$, $X_c = 2/3$, $\epsilon = 0.5222$, type NS; (f) $m = 8$, $X_c = 2/3$, $\epsilon = 1.0020$, type S.

a long-time state consisting of multiple isolated opinion clusters [30]. In the latter case, the clusters are disconnected (see Definition 3.1 in Section 3.4), in the sense that their separation in opinion space is greater than the radius ϵ of the support of the influence function ϕ; and such separated or consensus equilibria can only be neutrally stable, since their location depends on the initial opinion distribution.

Figure 2(a) shows one of our simulations for this case of purely open-minded individuals, confirming the emergence of neutrally stable *isolated opinion clusters*.

- As shown in Figure 2(b), for sufficiently large ϵ there can be convergence to the previously-observed intermediate consensus [42] even in the presence of individuals with a propensity for closed-mindedness (that is, the set C is non-empty, or equivalently $m < N$), provided that there is a social norm of open-mindedness, and there are sufficiently many open-minded agents to overcome the attraction of closed-minded individuals to the extremes.
- More interestingly, our model (4) with (6) also allows for the creation of an *extremist consensus*; such behaviour requires closed-minded agents and thus cannot occur in (1)–(2). As seen in Figure 2(c), convergence of all the

opinions in a population to one of the extremes at ± 1 can occur even in the presence of a relatively large number of open-minded individuals, provided the initial opinion distribution is asymmetric about 0. The possibility of extremist consensus is consistent with the psychological literature, where it has been shown to arise in group scenarios in which individuals have some sort of group-identifying feature associated with their belief [45, 15]; in such cases, the consensus that is finally attained tends to be more extreme than the initial mean opinion of the group.

- Figure 2(d) demonstrates a common consequence of closed-mindedness, namely *extreme polarization*, in which the population splits into two groups each settling at an extreme opinion; this corresponding to situations containing two hostilely opposed groups who are unwilling to listen to each other. In the simulation shown in (d), there are no open-minded individuals ($m = 0$), but extreme polarization can occur also in their presence, especially with (almost) symmetric initial conditions.
- The versatility of our model becomes particularly apparent in the remaining two plots. In Figure 2(e), the equilibrium consists of seven opinion clusters distributed non-symmetrically in a connected configuration (none is isolated from the others); that is, the separation between adjacent clusters is less than ϵ. In the computation shown, some of the clusters contain only open-minded individuals, some only closed-minded, and some are mixed; there are clusters both at extremes and at intermediate, more moderate opinions. In Section 3.4 we show that for our model, certain connected equilibria are asymptotically stable (Theorems 3.6 and 3.7); in particular, this holds for the equilibrium configuration in plot (e).
- Figure 2(f) shows another example of a connected, asymptotically stable equilibrium configuration; in this simulation, with Type S initial data, opinions are distributed symmetrically in five clusters, with the middle ($x = 0$) and extreme ($|x| \approx 1$) clusters containing no open-minded individuals. We feel that opinion equilibria such as those in Figure 2(e) and (f) are more realistic compared to the *isolated* opinion groups exclusively captured by previous bounded confidence models. Indeed, in such equilibria, there are several different groups of individuals, each with a distinct belief on the particular issue being considered, but at the same time within communication range of at least one other group, and thus able to interact with and influence the opinions of others.

3.2. **Remarks on equilibria and their stability.** As seen in the examples shown in Figure 2, in all our simulations the initial opinion distribution converges to a stationary state; that is, $x_i(t) \to x_i^*$ as $t \to \infty$ for each $i = 1, \dots, N$. Furthermore, the limiting opinions are clustered, taking on only K distinct values, where typically $K \ll N$ (for instance, $N = 80$ and $K \leq 7$ for the runs shown in Figure 2). That is, for any i and j we have either $x_i^* = x_j^*$ or $|x_i^* - x_j^*| \geq \delta$, where we let δ denote the *minimum equilibrium separation between clusters*. Note that two agents i and j (eventually) belong to the same cluster if $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$.

Neutrally stable clusters. For the original Motsch-Tadmor model [42], Jabin and Motsch [30] have recently made the above ideas precise; in particular, they rigorously proved such convergence to clusters with $\delta \geq \epsilon$. That is, if only open-minded (or more generally non-closed-minded) individuals are present, then the minimum cluster separation is at least as large as the radius ϵ of the support of the influence function ϕ: individuals in distinct opinion clusters do not interact at equilibrium. In the graph-theoretic terms introduced in Section 3.4 below, these clusters are isolated or *disconnected*.

Now it is readily seen that an attracting equilibrium consisting of such dynamically disconnected clusters is *neutrally stable* but cannot be asymptotically stable, due to the existence of non-decaying perturbations. Most straightforwardly, since the dynamics (1)–(2) are translation-invariant, a perturbation of each x_i^* in the equilibrium configuration by a sufficiently small constant amount η will simply lead to a translation of all the cluster locations by the same amount η; adding η to all the $x_i(0)$ will similarly translate the final state. More generally, perturbing all the $x_i(t)$ in the same direction must lead to a shift in the steady state. Note that if $\delta > \epsilon$, since the equilibrium clusters are disconnected a sufficiently small perturbation of size $\eta < (\delta - \epsilon)$ to the opinions in one cluster will have no effect on other clusters, so that perturbations can also disturb individual cluster locations independently. The consensus state in Figure 2(b) and the 4-cluster disconnected state in Figure 2(a) are examples of such neutrally stable equilibria.

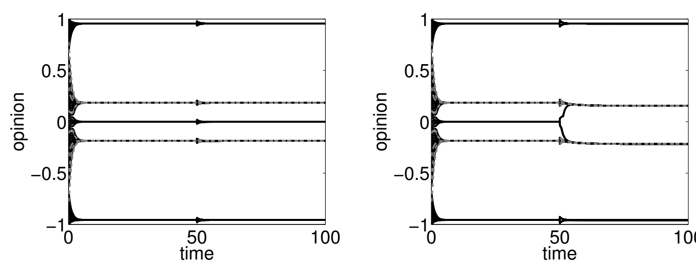
Linearly stable clusters. By contrast, in our model the presence of closed-minded individuals ($i \in C$ with $|x_i| > X_c$) can dramatically affect the stability, and permit *linearly (asymptotically) stable* equilibria to exist in many configurations. The origin of this stabilization is the absorbing effect exerted by the opinion extremes at ± 1 on closed-minded individuals, which are drawn to the extremes whenever they interact with other agents in C or even (in the absence of a social norm of open-mindedness) in O; mathematically this effect manifests itself in the eigenvalues of the linearization about the equilibrium, as is shown in Section 3.4 below.

The simplest types of such asymptotically stable equilibria are the extremist consensus and polarization states, seen in Figures 2(c) and (d), respectively. In each of these cases, all individuals in C become closed-minded, and are attracted to one or both extremes, depending on the initial data; any open-minded individuals present are then influenced by the closed-minded agents and themselves approach the extremes (with a lag, apparent in Figure 2(c)). Note that a symmetric initial opinion distribution must lead to a symmetric (polarized) final state, as in Figure 2(d); only sufficiently asymmetric distributions can give rise to extremist consensus, as in (c).

In addition and possibly most interestingly, our model supports the existence of asymptotically stable, connected multi-cluster states, as shown in Figures 2(e) and (f). Such states can have a variety of properties: for instance, they may, but do not need to, include clusters at the extreme opinions at ± 1 (compare (e) with (f)); and some of the clusters may include only open-minded individuals (such as that with $x^* \approx -0.8$ in (e)) or no open-minded individuals at all (the clusters at $x^* = \pm 1$ and $x^* \approx -0.5$ in (e); the middle and outside clusters in (f)). In general such states are numerous and diverse, and we have not attempted to classify them fully; some aspects of their parameter-dependence are explored in Section 3.3 below.

The common feature of these asymptotically stable K-cluster states, though, is that they are *connected*; that is, the separation in opinion space between any pair of adjacent clusters at equilibrium is less than ϵ (which implies $\delta < \epsilon$). Furthermore, the two clusters nearest to ± 1 must consist only of closed-minded individuals, for whom the attraction experienced towards the opinions of other agents is equal and opposite to the pull of the extremes. This permits the stable equilibrium to exist, via a balance (of course dependent on the initial opinion distribution) between the two extremes, with the opinions of individuals in intermediate clusters stabilized by interactions with others of both higher and lower opinion values. Observe that if such a connected multi-cluster state contains open-minded individuals ($m > 0$), it must necessarily consist of at least three clusters ($K \geq 3$).

Note that the main theorems from Section 3.4 guarantee only linear stability, that is, local stability to sufficiently small perturbations. We illustrate this numerically on the 5-cluster equilibrium solution of Figure 2(f) (which satisfies the assumptions of Theorem 3.7): We perform a time evolution of the initial data of Figure 2(f), allowing it to settle down to the 5-cluster state, which is then perturbed at time $t = 50$; the perturbed opinion configuration is then further evolved to $t = 100$. Given a fixed $\eta > 0$, for each $i = 1, \ldots, N$ the perturbation applied to $x_i(50)$ is $\pm\eta$, where $+$ and $-$ are chosen at random with equal probabilities. Some representative η-dependent outcomes are shown in Figure 3: Sufficiently small perturbations decay to the previous fixed point, as seen in Figure 3(a) with $\eta = 0.02$. However, larger disturbances may drive the system into the basin of attraction of another equilibrium, as shown in Figure 3(b), in which following a perturbation of size $\eta = 0.03$ the opinion distribution approaches a qualitatively different 4-cluster configuration.



(a) (b)

FIGURE 3. Numerical illustration of local stability: At time $t = 50$ a perturbation of size η is applied to the 5-cluster solution in Figure 2(f); the perturbed system is then evolved to $t = 100$. (a) For $\eta = 0.02$, the solution returns to the original 5-cluster configuration; (b) with $\eta = 0.03$, the system approaches a 4-cluster state.

We conclude this discussion on connected clusters by pointing out some similarities between our results and those in [35]. Unlike the approach in this paper, Lorenz modelled heterogeneity by assigning different bounds of confidence to individuals, to distinguish between open and closed-minded agents. We refer in particular to Figure 8 in [35]. In this Figure, the upper plot shows an extremist consensus (as also captured by our model; see Figure 2(c)), while the lower one displays a partially connected cluster equilibrium, in the following sense: the equilibrium configuration has three clusters, the one in the centre consisting of only open-minded agents and the two on the sides containing exclusively closed-minded individuals. The distance between the open-minded group's opinion and the two closed-minded clusters is less than the bound of confidence of the open-minded, meaning that the open-minded agents communicate with the closed-minded at equilibrium. On the other hand, neither of the two closed-minded clusters has the open-minded cluster within its own bound of confidence, so the closed-minded clusters are isolated in that sense. We call this configuration *partially connected*, to distinguish it from the fully connected equilibria that we find in our model. Note that Lorenz [35] does not elaborate further on this new type of equilibrium, aside from mentioning it briefly.

Explicit construction of a connected equilibrium opinion distribution. It is instructive to verify the existence of multi-cluster equilibria satisfying the hypotheses of Theorem 3.7 (with a social norm of open-mindedness) through explicit calculation. The steady state constructed below is similar to the long-time state depicted in Figure 2(f):

We postulate an equilibrium consisting of $K = 5$ opinion clusters located symmetrically at $-z$, $-y$, 0, y, and z, where $0 < y < z < 1$. There are N agents in total for some sufficiently large N (we need at least $N \geq 4$), of whom $m > 0$ are open-minded (in O) and $q > 0$ are closed-minded (in C with opinion magnitudes exceeding X_c), with N, m and q assumed even. The critical threshold X_c and the interaction ranges ϵ and $\hat{\epsilon}$ are chosen later to impose suitable constraints on y and z; the (positive) interaction strengths α_1, α_2 and α_3 are arbitrary, and for simplicity we choose them to be equal, $\alpha_1 = \alpha_2 = \alpha_3$, consistent with our numerics.

The most extreme clusters at $\pm z$ cannot contain open-minded individuals, while the q closed-minded individuals are equally distributed between opinion values z and $-z$; this requires $0 < X_c < z$. Each of the two clusters at $-y$ and y contains m_1 open-minded individuals, and the cluster at 0 has $m_0 = m - 2m_1$ open-minded individuals and $N - m - q$ individuals in C whose opinion values are below the threshold X_c and therefore act open-mindedly.

Choosing $\hat{\epsilon} \geq 1 - X_c$, the opinion separation between the closed-minded agents at $\pm z$ and their respective extremes at ± 1 lies within the support $\hat{\epsilon}$ of $\hat{\phi}$, so that $\hat{a}_i > 0$ for all $i = 1, \ldots, q$. Furthermore, we assume that ϵ is such that the individuals with opinion 0 interact with all others, while those in the cluster with opinion y influence and are influenced by those at z, 0 and $-y$ but not at $-z$; consequently the closed-minded individuals with opinion z are connected to those with opinions y and 0, but not to those at $-y$ and $-z$. This implies the constraints $\epsilon > z$, $\epsilon > 2y$ and $\epsilon < y + z < 2z$.

Due to symmetry, the equilibrium criteria for such a configuration reduce to just two equations, the balance conditions for the clusters at z and y (the influences on the $N - q - 2m_1$ centrist individuals with opinion 0 automatically cancel by symmetry):

$$(N - m - 1)(1 - z) + (m_1(y - z) - m_0 z) = 0,$$
$$-m_1(2y) - m_0 y - (N - m - q)y + \frac{q}{2}(z - y) = 0,$$

which using $m_0 = m - 2m_1$ is equivalent to

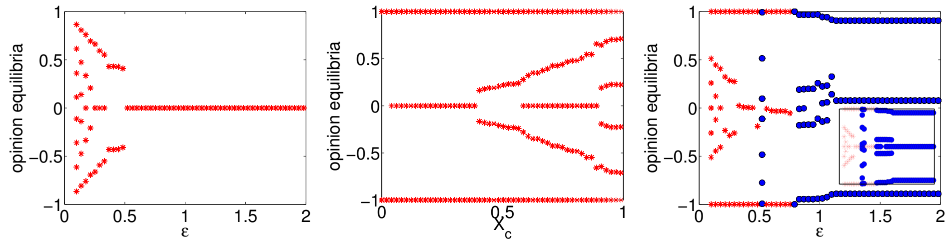
$$-m_1 y + (N - m_1 - 1)z = (N - m - 1), \tag{11}$$
$$-\left(N - \frac{q}{2}\right) y + \frac{q}{2} z = 0. \tag{12}$$

It may readily be verified that for $m_1 + q/2 \leq (m + q)/2 \leq N/2$, this linear system in y and z has a unique solution; since $m > 0$, we have $q < N$ so that by (12) we have $y < z$, and hence by (11) $z < 1$. Thus for any even number m of open-minded individuals, $0 < m < N$, there is a solution of (11)–(12) satisfying $0 < y < z < 1$ for any even number q of closed-minded individuals satisfying $0 < q \leq N - m$ and any $0 < m_1 \leq m/2$. Note that in general, if $m_0 = m - 2m_1 > 0$ and/or $m + q < N$, this is a 5-cluster solution; if neither of these conditions holds, then there is no cluster with opinion 0, and this calculation describes a 4-cluster configuration reminiscent of that in Figure 3(b).

3.3. **Bifurcation phenomena.** The equilibrium configuration of opinion clusters in our model clearly depends on the various parameters as well as the initial opinion distribution. A full investigation of the parameter space and/or the dependence on initial data is well beyond the scope of the present study. However, to gain some insight into the global structure of the solutions of our model, we explore the influence of variations in the open-mindedness threshold X_c, the interaction range ϵ and the number m of open-minded individuals on the locations of equilibria and qualitative changes thereof. For all the computations, as in Section 3.1 we set the number of agents at $N = 80$ and the interaction strengths $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and use a fixed initial condition of either type NS (non-symmetric distribution of open-minded individuals) or type S (symmetric distribution). Some representative bifurcation plots are presented and discussed below, showing the equilibrium locations of opinion clusters, obtained by performing sufficiently long time integrations to permit convergence to the stationary state (as in Figure 2), as functions of the parameters. In addition, for each parameter value we indicate the stability of the steady state, which as discussed in Section 3.2 above and proved in Section 3.4 below, depends on whether or not all opinion clusters are connected to one or both attracting opinion extremes: we identify neutrally stable states—exhibiting moderate consensus or disconnected opinion clusters—using red stars, while linearly (asymptotically) stable states—having either only closed-minded individuals at the extremes, or fully connected clusters—are denoted with blue filled circles.

Parameter-dependence of equilibrium cluster count. The simplest configuration to consider is that in which all individuals are open-minded, that is, $m = N = 80$, in which case our models reduce to the Motsch-Tadmor model (1)–(2). Having fixed the influence function as $\phi = \mathbf{1}_{[0,\epsilon]}$, and since in the absence of closed-minded individuals the threshold X_c becomes irrelevant, the only parameter that remains to be investigated, for given initial conditions, is the interaction range ϵ. Figure 4(a) shows a typical cluster distribution for uniformly spaced (symmetric) initial opinions. Since the equilibrium separation of clusters must be at least ϵ, for small ϵ there can be multiple disconnected clusters, while increasing ϵ leads to consensus; the critical value for consensus formation, which occurs here beyond $\epsilon \approx 0.5$, depends on the initial data. Qualitatively similar bifurcation diagrams were obtained by Lorenz [36] for the discrete-time models of Deffuant *et al.* [12] and Hegselmann and Krause [25]. We note in Figure 4(a) that, as ϵ decreases, the central cluster bifurcates into two clusters at $\epsilon \approx 0.5$. By further decreasing ϵ, the central cluster reappears, and persists for a while, before bifurcating again into two clusters. The pattern then seems to repeat, generating more and more clusters with decreasing ϵ. It has been suggested in [36] that the bifurcation pattern repeats itself on time intervals that scale with $1/\epsilon$, though no clear conclusion in this sense can be drawn from Lorenz's studies, or from ours.

Figure 4(b) represents the opposite extreme $m = 0$, in which no individuals are open-minded. For the relatively small interaction range $\epsilon = 0.2$ chosen here, a connected, linearly stable state would need to consist of at least 10 clusters, and would presumably have a small basin of attraction; we have not observed convergence to such an equilibrium for typical equally spaced or random uniformly distributed



(a) (b) (c)

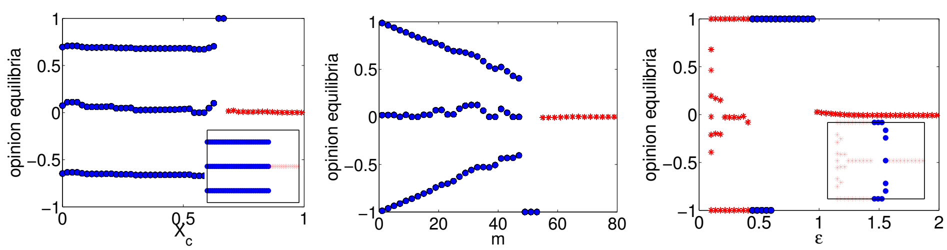
FIGURE 4. Dependence of the number and stability of cluster equilibria on X_c (threshold for extreme-seeking dynamics) and ϵ (bound of confidence): (a) $m = 80$ (all open-minded), type S initial data; (b) $m = 0$ (none open-minded), $\epsilon = 0.2$, type S data; (c) $m = 8$, $X_c = 2/3$, type NS data (type S data in insert). Cluster locations for linearly stable equilibria are represented by blue filled circles, while neutrally stable equilibria are denoted by red stars.

initial conditions in our simulations. Instead, the observed cluster distributions are disconnected and thus neutrally stable. For this fixed ϵ, we plot the clusters as functions of the critical threshold for extremism X_c. Note that for large X_c, most individuals in C will act open-mindedly; but that for all $X_c < 1$, individuals with sufficiently large x_i become closed-minded and are attracted to the extremes at ± 1. We see in Figure 4(b) that for X_c near 1, there are four moderate opinion clusters together with the two at the extremes. As X_c decreases, the opinions of an increasing proportion of individuals begin or are drawn beyond the threshold for closed-minded behaviour and are consequently pulled to the extremes, with only those that remain in $[-X_c, X_c]$ acting open-mindedly, so that the number of moderate clusters monotonically decreases accordingly. Eventually, below a critical X_c value (about 0.4 for the parameters in the figure), apart from the extremes there remains only the cluster at opinion 0, which is present for any $X_c > 0$ provided the initial opinion distributions are symmetric and sufficiently dense.

An intermediate situation is shown in Figure 4(c), in which only 10% of the population is open-minded ($m = 8$), but the critical threshold for closed-mindedness is relatively high, $X_c = 2/3$. For small interaction range ϵ, the situation is similar to that in (b), with convergence typically to disconnected opinion clusters and the closed-minded individuals attracted fully to the extremes; the number of clusters decreases as ϵ increases. However, stable connected multi-cluster states can also exist for such parameter values—albeit presumably with relatively small basins of attraction—and we see that for $\epsilon \approx 0.5$ there is convergence to a linearly stable asymmetric 7-cluster configuration; the time evolution for these parameter values and initial data is that shown in Figure 2(e). We suspect that such connected clusters may occur more readily for symmetric data; indeed, as seen in the insert to Figure 4(c), there are two values of ϵ near 0.5 for which type S initial conditions approach linearly stable states consisting of 9-cluster and 6-cluster configurations, respectively.

As seen in the figure, there is a critical value of the interaction range (occurring at $\epsilon \approx 0.8$ here) beyond which the largest gap between clusters is always below ϵ, so that convergence is to a connected multi-cluster state; observe that since in the presence of open-minded individuals there must be at least one moderate cluster, this critical value of ϵ is bounded above by 1. Interestingly, the initial bifurcation is typically to a 5-cluster state with three moderate clusters (or sometimes to a 4-cluster state), while there is a further bifurcation value (here at $\epsilon \approx 1.1$) beyond which there remains only a single moderate cluster with opinion near 0. Observe that for these large ϵ, the most extreme opinions are bounded away from 1 (in absolute value), as they experience attraction to the moderate cluster as well as to the extremes.

Extremist consensus. A consensus of extremism has been observed in societies (such as Nazi Germany in the 1930s, for instance) and investigated with previous bounded confidence opinion models [13, 26, 35, 38]. In Figure 5 we explore the emergence of an extremist consensus within our model (4)-(6) for representative parameter values. Within each plot, the same non-symmetric (type NS) opinion configuration is used to initialize all the computations, and as before the long-time cluster locations are shown with their stability. Note that an unbalanced distribution of initial opinions is essential for reaching extremist consensus; as verified in the inserts of plots (a) and (c), symmetric (type S) arrangements of open-minded individuals do not lead to such equilibria.



(a) (b) (c)

FIGURE 5. Extremist consensus arising as bifurcations in parameter space. Within each plot, the same non-symmetric (type NS) initial conditions are used to generate equilibria for all parameter values. The inserts show equilibria obtained from symmetric initial data (type S), where no extremist consensus can arise. (a) Dependence on X_c with $m = 26$, $\epsilon = 1.2$; (b) dependence on m with $X_c = 1/3$, $\epsilon = 2$; (c) dependence on ϵ with $m = 54$, $X_c = 1/3$. Symbols are as in Figure 4.

In Figure 5(a) and (b), extremist consensus is observed to emerge from 3-cluster equilibria through bifurcations in X_c and m, respectively (recall that a higher threshold for closed-mindedness X_c implies that individuals in C are relatively more likely to act open-mindedly). The extremist consensus then holds for a relatively small range of parameters, before another bifurcation brings the equilibrium into a moderate opinion consensus. The results suggest that a certain balance between the proportion of open-minded individuals in a group and the opinion threshold X_c may allow a minority of closed-minded individuals to drag the group opinion towards their extreme. A theoretical explanation for this phenomenon was also investigated by Galam and Moscovici [21] using a discrete model for opinions.

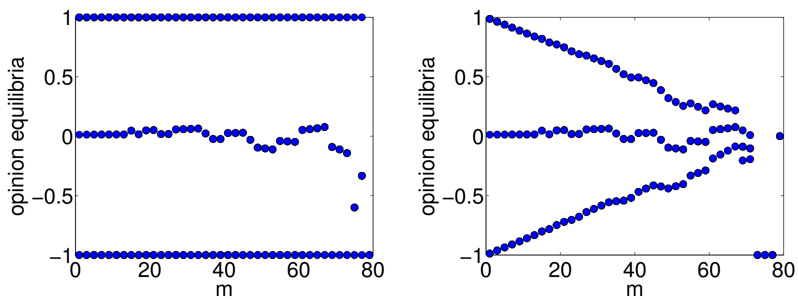
Figure 5(c) shows extremist consensus emerging from a polarized equilibrium, through a bifurcation in the range ϵ of the influence function. At bifurcation, the relatively large number of open-minded individuals ($m = 54$), as they approach an extreme opinion, can drag individuals from one extreme to the other. However, further increase of the influence range brings the extremist consensus into a moderate opinion one.

Sensitivity to the total population size N. We have investigated the sensitivity of the results presented in Figure 5 with respect to the size N of the total population. Specifically, we performed a variety of numerical experiments with population sizes that ranged from $N = 40$ to $N = 520$, with an increment of 40. First, we investigated the robustness of the three qualitatively different equilibria shown in Figure 5(a), that is, the 3-cluster equilibrium, the extreme consensus and the consensus at a moderate opinion value. For this purpose we selected three different values of X_c ($X_c = 1/3$, $X_c = 2/3$ and $X_c = 3/4$) that correspond, respectively, to these three different equilibria, and ran simulations with various randomly chosen non-symmetric initial conditions and different population sizes. For consistency with Figure 5(a), in all these simulations the bound of confidence ϵ and the fraction m/N of open-minded individuals were fixed at $\epsilon = 1.2$ and $m/N = 26/80$, respectively. The 3-cluster equilibrium for $X_c = 1/3$ and the consensus at a moderate opinion for $X_c = 3/4$ proved to be very robust with respect to different population sizes $N = 40, 80, \ldots, 520$, appearing in each of the 30 different initializations for each N. The simulations with $X_c = 2/3$ did not show a similar robustness, however; this is unsurprising in view of the fact that, as seen in Figure 5(a), $X_c = 2/3$ is close to the bifurcation value at which extremist consensus emerges from a 3-cluster equilibrium. In our 60 different runs for each N, both extreme consensus states (as in Figure 5(a)) and 3-cluster equilibria occurred in a significant fraction of cases. For most values of N we observed extreme consensus in more than 70% of the simulations, while we have not seen clear evidence that the probability of extreme consensus depends monotonically on the population size.

In a second set of runs, we tested the sensitivity to N of two equilibria observed in Figure 5(c): the polarized equilibrium at $\epsilon = 0.5$ and the consensus at moderate opinion at $\epsilon = 1.5$. We ran simulations with different non-symmetric initializations, for $N = 40, 80, \ldots, 520$, with $X_c = 1/3$ and the fraction $m/N = 54/80$ fixed. We found that the polarized equilibrium ($\epsilon = 0.5$) is very robust; all 30 different initial conditions we tested resulted in such an equilibrium, regardless of the population size. The numerical simulations for the other choice of the bound of confidence, $\epsilon = 1.5$, were less definitive, however; while the 60 different runs for each N always resulted in a 1-cluster equilibrium, this cluster lay either at a moderate value (as in Figure 5(c)) or at an extreme. It appeared that for most population sizes, an extreme consensus arose in more than 50% of the simulations. Our results for this set of experiments suggest that the likelihood of an extreme consensus increases with the size of the population, but with a slow rate of convergence. Given the important potential sociological implications of such a result, we plan to investigate this issue further.

Effect of a social norm of open-mindedness on consensus formation. Figure 6 demonstrates the relationship (for a specific set of parameters) between the proportion of open-minded individuals in a population and the extent to which an open-mindedness social norm enhances agreement within a population. We set

$\epsilon = 2$, which implies that all individuals communicate with each other; and $X_c = 0$, meaning that all individuals in C act closed-mindedly. The plots correspond to type NS (non-symmetric) initial data.



(a) (b)

FIGURE 6. Effect of a social norm of open-mindedness on opinion convergence. The initial data is of type NS (non-symmetric); parameters are set at $N = 80$, $\epsilon = 2$ and $X_c = 0$, and symbols are as in Figure 4. Increasing the proportion m/N of open-minded individuals has a negligible effect on the opinion distribution when there is no social norm of open-mindedness (plot (a)), but increases agreement significantly when a social norm of open-mindedness is present (plot (b)).

Figure 6(a) shows results for the case when no social norm of open-mindedness is present, while Figure 6(b) corresponds to the model incorporating an open-mindedness social norm. The most common equilibrium configuration in the two plots consist of three clusters, where the outside clusters contain exclusively closed-minded individuals, while the centre cluster has a mixed population. As m varies however, the locations of three clusters behave essentially different in the two plots.

When no social norm of open-mindedness is present, the closed-minded adopt extreme opinions and the qualitative configuration of the three-cluster equilibrium changes very little with increasing m. For small numbers of closed-minded individuals ($m \gtrsim 75$), the asymmetry of the initial data plays a significant role, as observed in the central opinion cluster approaching one of the extremes. For the last data point (one closed-minded individual), the entire population reaches the extreme opinion of that closed-minded agent. We conclude that increasing the proportion of open-minded individuals in the model with no social norm of open-mindedness has little effect on consensus.

Figure 6(b) demonstrates how including an open-mindedness social norm has the potential to enhance agreement. We observe there that, by increasing the number of open-minded individuals, the two outside clusters drift monotonically toward the central opinion cluster, meaning that opinions in the population become less diverse. This trend is consistent for values of $m \lesssim 71$. Beyond that we note the emergence of an extremist consensus. This is an effect of the asymmetric initial placement of the open-minded individuals, similar to what has been observed in Figure 5. This effect is particularly strong when only a small number of closed-minded agents is present. As long as there is at least one closed-minded individual, the consensus will be dragged to an extreme (unless it occurs exactly at 0, as for the last data point).

3.4. **Stability analysis.** In this section we provide the details and proofs for the stability results referred to in Section 3.2 above.

A key concept used in the stability analysis is the (graph) connectivity of a group of individuals $i = 1, \dots, N$, or equivalently, of their opinion states $\{x_i\}_{i=1,\dots,N}$. In our context, connectivity is defined in terms of the interaction coefficients: One can associate a *directed* graph to the opinions of a group of individuals, where the nodes of the graph are the opinions and the directed edges (or arrows) are *ordered* pairs of opinions connected by a nonzero interaction coefficient. Given this framework we can talk about the connectivity of the group, and the following definition is adopted [42]:

Definition 3.1. An individual i is said to be *connected*[2] to an individual j if there exists a path linking i and j. A path, in this instance, is defined as a chain of nonzero interaction coefficients which starts at i and ends at j. If any two individuals in a population are mutually connected to each other (in both directions), the population is said to be *connected*.

For instance, in model (6) a closed-minded individual i may connect to a non-closed-minded individual j either directly, provided $\hat{a}_{ij} \neq 0$, or indirectly, through a path that reaches j. This path may consist of one or more individuals (for instance a path $\hat{a}_{ik}, a_{kj}$ involving an open-minded individual k such that both $\hat{a}_{ik}$ and a_{kj} are nonzero).

Remark 3.2. In graph-theoretic terminology, a population is connected according to Definition 3.1 if the directed graph associated to its opinion state is *strongly connected*. This property can be equivalently characterized in terms of the interaction matrix. Indeed, it is a well known result [53] that a matrix is *irreducible* if and only if its associated directed graph is strongly connected. Consequently, a population is connected if and only if its interaction matrix is irreducible.

The following assumptions are made to prove the asymptotic stability results. First, the influence functions ϕ and $\hat{\phi}$ are taken to be indicator functions with arbitrary supports ϵ and $\hat{\epsilon}$ respectively. Second, we assume the following for the equilibria:

Assumption 3.3. *The equilibrium opinion state* $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_N^*)$ *satisfies*

1. $|x_i^*| \neq X_c$ *for all individuals* i in C;
2. $|x_i^* - x_j^*| \neq \epsilon$ *for all non-closed-minded individuals* i *and all individuals* j.

The next two assumptions apply to model (6) only*:*

3. $|x_i^* - x_j^*| \neq \epsilon$ *for all closed-minded individuals* i *and all open-minded individuals* j*;*
4. $|\operatorname{sgn}(x_i^*) - x_i^*| \neq \hat{\epsilon}$ *for all closed-minded individuals* i*.*

Assumption 1 simply excludes the ambiguous case when an opinion of an individual in C is at the threshold. Assumptions 2-4 guarantee that all coefficients $\hat{a}_i$, $\hat{a}_{ij}$ and a_{ij} in models (5) and (6) are well-defined at equilibrium, as ϕ and $\hat{\phi}$ are never evaluated at their points of discontinuity. Moreover, due to continuity, it follows

[2] In a strict mathematical sense, connectivity between two individuals i and j should be interpreted as connectivity between their opinions x_i and x_j.

that there is an entire neighbourhood of $\mathbf{x}^*$ where $\hat{a}_i$, $\hat{a}_{ij}$ and a_{ij} are well-defined, and in fact they are all constants in this neighbourhood (since ϕ and $\hat{\phi}$ are assumed to be indicator functions). Below we use $\hat{a}_i^*$, $\hat{a}_{ij}^*$ and a_{ij}^* to denote the values of $\hat{a}_i$, $\hat{a}_{ij}$ and a_{ij}, respectively, at equilibrium.

Remark 3.4. We noted above that *all* coefficients $\hat{a}_i$, $\hat{a}_{ij}$ and a_{ij} are constant in a neighbourhood of an equilibrium $\mathbf{x}^*$ that satisfies Assumption 3.3. The size of this neighbourhood can be calculated exactly for a given equilibrium $\mathbf{x}^*$; it corresponds to the largest perturbation of $\mathbf{x}^*$ that would cause Assumption 3.3 to fail. For as long as the solution remains in this neighbourhood of an equilibrium, the dynamics is linear, and hence systems (4)-(5) and (4)-(6) are *locally exactly linear* near such equilibria. As a consequence, linear (asymptotic) stability in this case is equivalent to *nonlinear* (asymptotic) stability. Our analysis below deals with linear theory, but all results prove in fact the stronger nonlinear stability of the equilibria.

The stability analysis requires us to distinguish between the closed-minded individuals (individuals in C with opinion values that exceed X_c in magnitude) and the others. Thus suppose that there are $q \leq N - m$ individuals in the population who act closed-mindedly; relabelling if necessary, we assume that these individuals have indices $i = 1, \ldots, q$. In both models (5) and (6), the interaction coefficients between the non-closed-minded individuals are denoted by a_{ij}, with $q + 1 \leq i, j \leq N$. Two coefficients a_{ij} and a_{ji} are not equal in general, but they are simultaneously zero or nonzero.

The stability proofs for the models without a social norm of open-mindedness, with interactions defined by (5), and with such a social norm, described by (6), need to be treated separately, due to the differences in the interaction coefficients corresponding to the q closed-minded individuals. In model (5), a closed-minded individual's opinion is not influenced by anyone else's opinion. However, the opinion of a closed-minded individual $i \in \{1, \ldots, q\}$ affects the opinion of a non-closed-minded individual $j \in \{q+1, \ldots, N\}$ provided the corresponding interaction coefficient a_{ji} is nonzero. Model (5) is *asymmetric* in that sense. Part of this asymmetry transfers to (6) as well; indeed, by the same reasoning as above, closed-minded individuals are not influenced by non-closed-minded individuals in C, but could affect their opinions in turn. Model (6) is symmetric, however, with respect to interactions between closed-minded and open-minded individuals—see Remark 2.1.

Stability for model without open-mindedness social norm. We start by investigating the linear stability of an equilibrium solution $\mathbf{x}^*$ for (4)-(5), which reduces to studying the eigenvalues of its linear approximation.

Given that the $\hat{a}_i$ and a_{ij} are all constants near $\mathbf{x}^*$, the Jacobian matrix at equilibrium for (5) has the form (recall that $(N-1)\hat{a}_i = 1$)

$$J_1 = \begin{pmatrix} -\alpha_2 I & 0 \\ B & A \end{pmatrix}, \tag{13}$$

where I is the identity matrix of size $q \times q$, B is a $(N-q) \times q$ matrix of entries $\alpha_1 a_{ij}^*$ $(q+1 \leq i \leq N,\ 1 \leq j \leq q)$ and A is a $(N-q) \times (N-q)$ square matrix given by

$$A = \begin{pmatrix} -\alpha_1 & \underset{[i<j]}{\alpha_1 a^*_{ij}} & \dots \\ \underset{[i>j]}{\alpha_1 a^*_{ij}} & \ddots & \vdots \\ \vdots & \dots & -\alpha_1 \end{pmatrix}. \tag{14}$$

The indices i and j of the entries a^*_{ij} in A correspond to non-closed-minded individuals, i.e., $i, j \in \{q+1, \dots, N\}$; note that we used the normalization (3) of coefficients to simplify the diagonal entries of A.

The following classical result in linear algebra is an essential tool in the proof of our stability result:

Theorem 3.5. *If a matrix is irreducible and weakly diagonally dominant, with strict diagonal dominance holding for at least one row, then it is non-singular* [47, Theorem 2]*.*

We now state and prove the stability theorem for model (5).

Theorem 3.6. *Let* $\mathbf{x}^*$ *be an equilibrium solution of* (4)-(5) *that satisfies Assumption 3.3. Then* $\mathbf{x}^*$ *is asymptotically stable provided that, at equilibrium, all non-closed-minded individuals are connected, and in addition at least one non-closed-minded individual is connected to a closed-minded individual.*

Proof. We have to show that the eigenvalues of J_1 from (13) have negative real part. Given that $\alpha_2 > 0$, this amounts to showing that the real parts of the eigenvalues of A are negative. To this end, note that A has negative diagonal entries and is weakly diagonally dominant (by the normalization (3) and the non-negativity of the interaction coefficients a_{ij}). Hence, by the Gershgorin circle theorem [22, pg 320], all its eigenvalues lie in the left half of the complex plane, with the Gershgorin discs overlapping the imaginary axis only at the origin.

To conclude the proof we have to rule out the possibility that one (or more) of these eigenvalues is 0. We show this using Theorem 3.5. Note that, aside from the diagonal entries and the multiplicative constant α_1, the matrix A is the interaction matrix of the non-closed-minded individuals at equilibrium. Since by assumption, all non-closed-minded individuals are connected at equilibrium, from Remark 3.2 we conclude that the matrix A is irreducible.

Also by assumption, there exists at least one non-closed-minded individual connected to a closed-minded individual, which means that at least one of the entries $\alpha_1 a^*_{ij}$ in B is nonzero; that is, $a^*_{ij} \neq 0$ for some indices $i \in \{q+1, \dots, N\}$ and $j \in \{1, \dots, q\}$. Then, by the normalization condition (3), the corresponding row of A must be *strictly* diagonally dominant. Hence A satisfies the hypotheses of Theorem 3.5, and consequently is non-singular. This excludes the possibility of A having a zero eigenvalue; so all the eigenvalues of J_1 have strictly negative real part, and the equilibrium $\mathbf{x}^*$ is linearly (asymptotically) stable. □

Stability for model with open-mindedness social norm. The study of the stability of equilibria for the model (4) with (6) follows along similar lines. Notation is as before; in particular, for an equilibrium opinion state $\mathbf{x}^*$, without loss of generality we let the indices $\{1, \dots, q\}$ correspond to the individuals in C with opinions that exceed X_c in absolute value.

Now for linear stability to be possible, at least one of these q closed-minded individuals must be connected to an open-minded individual; if not, then all the moderate clusters containing non-closed-minded agents would be disconnected from the extremes and hence necessarily be neutrally stable, as discussed in Section 3.2. Hence among the closed-minded individuals $\{1, \ldots, q\}$ we distinguish two subgroups: we let the labels $\{1, \ldots, r\}$ refer to those r closed-minded individuals ($0 \leq r < q$) who do not interact with any other individual, while individuals $\{r+1, \ldots, q\}$ are connected with at least one open-minded agent, so that for each $i \in \{r+1, \ldots, q\}$, $\hat{a}_{ij}^* \neq 0$ for some $j \in O$. Note that the results established below also apply (trivially) in the special case when $r = 0$. The remaining $N - q$ non-closed-minded individuals are then labelled so that the $N-q-m$ non-closed-minded individuals in C are listed first, with labels $\{q+1, \ldots, N-m\}$, while the last m indices $\{N-m+1, \ldots, N\}$ correspond to the m open-minded individuals in O.

Recall that under Assumption 3.3, all coefficients $\hat{a}_i$, $\hat{a}_{ij}$, and a_{ij} are constants in a neighbourhood of $\mathbf{x}^*$. Hence the Jacobian matrix at equilibrium for (6) has the form

$$J_2 = \begin{pmatrix} -\alpha_2(N-m-1)D_1 & 0 \\ 0 & E \end{pmatrix}, \tag{15}$$

with D_1 an $r \times r$ diagonal matrix with entries $\hat{a}_i^*$, $i = 1, \ldots, r$, corresponding to the first group of closed-minded individuals.

The $(N-r) \times (N-r)$ matrix E contains the entries in the Jacobian matrix corresponding to the remaining $q-r$ closed-minded and the $N-q$ non-closed-minded individuals; via (6) one finds that E has the form

$$E = \begin{pmatrix} D_2 & F \\ B & A \end{pmatrix}, \tag{16}$$

where D_2 is a diagonal matrix of size $(q-r) \times (q-r)$, with entries

$$d_i = -\alpha_2(N-m-1)\hat{a}_i^* - \alpha_3 \sum_{j \in O} \hat{a}_{ij}^*, \qquad i = r+1, \ldots, q. \tag{17}$$

The matrix F, of size $(q-r) \times (N-q)$, has two main blocks:

$$F = \begin{pmatrix} 0 & | & \alpha_3(\hat{a}_{ij}^*)_{j \in O} \end{pmatrix}. \tag{18}$$

The zero block has $N-q-m$ columns and is due to the lack of explicit interaction between closed-minded individuals and the non-closed-minded agents in C. The other block has entries $\alpha_3\hat{a}_{ij}^*$, and represents the interactions between closed-minded individuals $i \in \{r+1, \ldots, q\}$ and open-minded individuals $j \in \{N-m+1, \ldots, N\}$.

Finally, the matrices A and B in (16) have a similar expression as A and B from (13). Specifically, B has size $(N-q) \times (q-r)$ with entries $\alpha_1 a_{ij}^*$, and A is given by (14).

The stability theorem for our model (4) with (6) now reads:

Theorem 3.7. *Let $\mathbf{x}^*$ be an equilibrium solution of* (4) *with* (6) *that satisfies Assumption 3.3. Assume additionally that $\hat{a}_i^* > 0$ for at least one $i \in \{r+1, \ldots, q\}$. Then $\mathbf{x}^*$ is asymptotically stable provided that, at equilibrium, all non-closed-minded individuals are connected, and also that at least one closed-minded individual is connected to an open-minded individual.*

Proof. We follow the strategy from the proof of Theorem 3.6. First note that the closed-minded individuals $1, \ldots, r$ do not interact with any other individual and consequently, their opinions lie at one of the extremes ± 1. Hence, $\hat{a}_i^* > 0$ for all $i \in \{1, \ldots, r\}$ and since $\alpha_2 > 0$, D_1 is a diagonal matrix with positive entries. We conclude from here that the eigenvalues of J_2 given by (15) all have negative real parts provided the same is true of the eigenvalues of E. We prove the latter fact following an argument similar to that used to show that matrix A in the proof of Theorem 3.6 has eigenvalues with negative real parts.

Matrix E has negative diagonal entries and is weakly diagonally dominant. The weak diagonal dominance for the first $q - r$ rows follows from (17) and (18), since $\alpha_2 > 0$, $\alpha_3 > 0$ and all interaction coefficients are non-negative. For the rows of E that involve the rows of B and A, the weak diagonal dominance follows from the normalization condition (3). Hence, by the Gershgorin circle theorem, all the eigenvalues of E have either strictly negative real parts or fall at the origin.

We rule out the possibility of E having zero eigenvalues using Theorem 3.5. Due to the assumption that at least one $\hat{a}_i^* > 0$ with $i \in \{r + 1, \ldots, q\}$, one of the first $q - r$ rows of E is in fact strongly diagonally dominant. The irreducibility of E follows by Remark 3.2, as the assumptions made on the connectivity at equilibrium render the directed graph associated to E strongly connected. Hence, by Theorem 3.5, E is non-singular, and this completes the proof. □

4. **Opinion dynamics model in the presence of media.** In the following we further extend the Motsch-Tadmor [42] model to incorporate the effects of media on opinion formation. Various attempts to model media-influenced opinion models have appeared in the recent literature [39, 48, 8, 2, 37, 44]. In particular, Pineda and Buendia [44] have recently considered media effects in the discrete-time bounded confidence models of Deffuant *et al.* [12] and Hegselmann and Krause [25]. Their approach is to assume that individuals interact with the media with a certain probability, or otherwise interact with other individuals in their confidence range; within this framework, the effectiveness of the media is studied in terms of the sizes of the bounds of confidence and the media intensity [44].

We introduce media via a natural extension of the basic model (4), modelling a media effect similarly to the inter-individual interaction term f_i. For simplicity we first present the case where there is a single source of media influence. Each media source is assumed to have an intrinsic perspective or *bias* μ measured on the same scale as individuals' opinions, that is, $\mu \in [-1, 1]$. The model proposed and investigated in this section is then given by

$$\frac{dx_i}{dt} = f_i + g(x_i, \mu), \qquad i = 1, \ldots, N, \tag{19}$$

where f_i represents the inter-individual interactions (as given by (5) or (6)) and $g(x_i, \mu)$ models the effect of media on individual i.

In modelling $g(x_i, \mu)$, we assume that individuals respond to media influences in an analogous way to how they respond to other individuals' opinions; this assumption that media-individual interactions are similar to inter-individual interactions is supported by empirical studies [33, 3, 45, 15, 27, 3]. In particular, we distinguish again between open-minded and closed-minded individuals, which presumably have qualitatively different responses to media. Following the notation of previous sections, we thus assume that g takes the following form:

$$g(x_i, \mu) = \begin{cases} \alpha_4 \hat{b}_i (\operatorname{sgn}(x_i) - x_i) & \text{if } |x_i| > X_c \text{ and } i \in C, \\ \alpha_5 b_i (\mu - x_i) & \text{otherwise.} \end{cases} \tag{20}$$

Here α_4 and α_5 are constants representing the strength of the media source, and the partisan dependence of the media influence is captured by the coefficients b_i and $\hat{b}_i$, whose formulation within our model is given below.

The analogies between (20) and (5) (or (6)) are immediate. We assume in (20) that the closed-minded individuals approach the extremes upon exposure to media, regardless of the actual bias of the media. This feature is similar to the closed-minded individuals' response in (5). Non-closed-minded individuals do respond to the media, however, and their opinions are steered toward the media bias μ.

To model coefficients b_i and $\hat{b}_i$, we define two *media-influence* functions ϕ_m and $\hat{\phi}_m$, the analogues of the functions ϕ and $\hat{\phi}$ used in previous sections. Again, for simplicity both ϕ_m and $\hat{\phi}_m$ are taken to be simple indicator functions, with supports denoted by ϵ_m and $\hat{\epsilon}_m$, respectively; here the magnitude of ϵ_m indicates the extent of the media influence on non-closed-minded individuals, while $\hat{\epsilon}_m$ describes the range of influence of the extreme opinions, as triggered by the presence of media. The coefficients b_i and $\hat{b}_i$ in our model are then assumed to be given by

$$b_i = \phi_m(|\mu - x_i|) \qquad \text{and} \qquad \hat{b}_i = \hat{\phi}_m(|\operatorname{sgn}(x_i) - x_i|). \tag{21}$$

More general media models. Model (19) can be generalized in various ways. First, one could consider more media sources, in which case the media term g should account for the effects of all such sources. Suppose that there are P media sources with biases μ_p and strengths α_{4p} and α_{5p}, where $p = 1, \ldots, P$. The extension of (20) to multiple media sources is straightforward:

$$g(x_i, \boldsymbol{\mu}) = \begin{cases} \sum_p \alpha_{4p}\, \hat{b}_{ip}\, (\operatorname{sgn}(x_i) - x_i) & \text{if } |x_i| > X_c \text{ and } i \in C, \\ \sum_p \alpha_{5p}\, b_{ip}\, (\mu_p - x_i) & \text{otherwise,} \end{cases} \tag{22}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_P)$, and the coefficients $\hat{b}_{ip}$, b_{ip} are just the extensions of $\hat{b}_i$, b_i from (21), appropriately normalized, to take into account the potential diversity of media influences.

A second possible generalization is to allow the media bias(es) to vary in time, potentially driven by the opinion distribution, to take into account changing media perspectives due to economic and competitive pressures [24]. A general framework to describe the time evolution of the bias of a single media source in model (20) may be given by

$$\frac{d\mu}{dt} = \lambda(t, \mu, \mathbf{x}), \tag{23}$$

where $\mathbf{x} = (x_1, \ldots, x_N)$; however, obtaining an appropriate form of the rate function λ is presumably a challenging modelling problem in itself, and we have not attempted this.

In the remainder of this paper we focus on the restricted case of a single media source of constant bias μ; some preliminary investigations of the more general case may be found in [28].

4.1. Numerical results.

In all simulations of opinion dynamics with one constant media source, we assume that both ϕ_m and $\hat{\phi}_m$ are indicator functions with identical supports to ϕ and $\hat{\phi}$, respectively (that is, $\epsilon_m = \epsilon$, and $\hat{\epsilon}_m$ large enough so that $\hat{\phi}_m(|\operatorname{sgn}(x_i) - x_i|) = 1$, for all closed-minded individuals i).

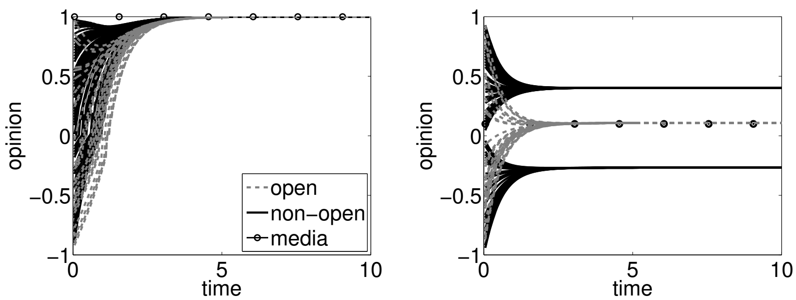
Representative simulations with media. Figure 7 shows the time evolution of the model (19)-(20) with (6) in two cases: (a) media bias located at an extreme ($\mu = 1$), and (b) media bias located at a moderate value ($\mu = 0.1$). In both simulations, there are $m = 26$ open-minded individuals and the influence range is $\epsilon = 1.2$, and the initial conditions are non-symmetric (Type NS—see Section 3.1). Also, in both cases, $X_c = 0$, meaning that *all* non-open minded individuals with nonzero opinions act closed-mindedly, and hence are not *directly* affected by the location of the media—see modelling assumption (20). Due to the presence of an open-mindedness social norm, however, the non-open-minded agents are *indirectly* affected by the media, through their interactions with the media-influenced open-minded agents.

In Figure 7(a) all individuals approach the extreme media bias. This occurs both because the open-minded individuals are attracted to the media and drag the non-open-minded individuals along with them, and due to interactions among closed-minded individuals which cause them to approach the extremes. For more moderate media, as in Figure 7(b), open-minded agents again approach the media bias, but this time the non-open-minded agents do not follow entirely, but instead form two distinct opinion groups. Note that in this 3-cluster equilibrium configuration, with the media bias μ lying between opinion clusters, again the two clusters with the most positive and negative opinions can contain only closed-minded individuals, who experience a balance between the attraction towards the opinions of others and towards the extremes; open-minded agents, who are drawn to others' opinions and to the media but not to the extremes, must lie in the intermediate cluster. On the other hand, the cluster at $\mu > 0$ cannot contain closed-minded agents; if it did, even in the presence of a social norm of open-mindedness, such individuals would not experience any attraction to open-minded individuals in the same cluster, so that the only net contribution to their social force f_i would be their attraction to the extremist views at $+1$, contradicting the assumption of a steady state.

As shown in Theorem 4.3 below, the equilibria in both (a) and (b) are asymptotically stable; indeed, the presence of media enhances the potential for stability by providing an additional external attractive force for the dynamics. As in the non-media model, the connectivity of the configuration plays a major role in stability; in particular, note that in (b) the two groups of non-open minded individuals are within the interaction range ϵ of both the open-minded individuals at the media bias, and of each other.

The effect of the media bias on equilibria. We perform a bifurcation study, similar to that done for Figure 4 but now focussing on the effect of the parameter μ that describes the media bias. The goal is to monitor, as μ varies, the number and location of the opinion clusters that form at equilibrium, using both Type NS (non-symmetric) and Type S (symmetric) initial conditions. We fix the threshold X_c for closed-mindedness at $X_c = 0$, meaning that all individuals in C act closed-mindedly. The bound of confidence is set at $\epsilon = 1.2$.

Figure 8 shows the equilibrium cluster locations using three values for the number of open-minded individuals m ($m = 0, 26, 54$ with $N = 80$), and a Type NS initial condition. The inserts contain the corresponding runs for Type S initial data, where the open-minded are symmetrically distributed; of course for Figure 8(a), in the absence of open-minded individuals ($m = 0$), these two plots would be the same. The green dashed line, which is simply the graph of the identity map, indicates the location of the media μ.



(a) (b)

FIGURE 7. Evolution to equilibria of the opinion model with media (19)-(20) with (6) with Type NS (non-symmetric) initial data, and parameter values $m = 26$ (number of open-minded agents), $X_c = 0$ (threshold for extreme-seeking dynamics) and $\epsilon = 1.2$ (bound of confidence). The circles indicate the media location μ. (a) $\mu = 1$: all individuals approach the extremist media bias. (b) $\mu = 0.1$: open-minded individuals converge to the media bias, while the opinions of non-open-minded individuals form two clusters balanced between attraction to open-minded agents and to the extremes.

Figure 8(a) corresponds to a population consisting only of closed-minded individuals; predictably, the equilibrium consists of two opinion clusters located at the extremes, regardless of the location of the media. On the other hand, Figures 8(b) and (c) demonstrate a stronger persuasive effect of the media as the number of open minded individuals increases. Note in particular figure (b), showing significantly different outcomes for the two types of initialization: The insert, which corresponds to a symmetric placement of the open-minded individuals, shows the formation of a group that deviates from the media bias; while a non-symmetric initial distribution of open-minded agents somehow seems to enhance the effect of the media (for $\mu = 0.1$ and $\mu = 1$, the full dynamic evolution with this initial condition corresponds to those illustrated in Figure 7). Finally, in figure (c), a relatively large number of open-minded individuals ($m = 54$) results in a strong effect of the media, with little differences noticed between the two types of initialization.

As seen in Figures 8(b) and (c), there are two typical configurations of such opinion dynamics models with media and a social norm of open-mindedness in the absence of symmetry. One is a 3-cluster configuration, occurring for relatively modest media, as discussed already in Figure 7(b). In this case the open-minded individuals form the intermediate cluster located at or near the media bias, experiencing a balance between an attraction to the closed-minded individuals at higher and lower opinion values, and to the media. Simultaneously, there are two clusters of closed-minded individuals: one at a positive opinion between μ and $+1$, balanced between the attractive effects of the extremist state at $+1$ and the open-minded individuals; and one at a negative opinion between -1 and μ, whose opinion values are pulled down by the lower extreme and up by the open-minded agents. The second case consists of only two opinion clusters, both located between the media location and an extreme: the open-minded individuals are closer to the media bias, and balanced between the media and the closed-minded agents; while the closed-minded individuals equilibrate between the open-minded agents and an extreme. In Figures 8(b) and (c), with μ positive and increasing, the transition between these two configurations occurs when the location of the lower closed-minded cluster in the 3-cluster case passes $x = 0$ (this occurs for $\mu \approx 0.5$ in (b) and $\mu \approx 0.2$ in (c)); once that happens, all closed-minded individuals are attracted to the upper extremist state at $+1$, and the equilibrium becomes a 2-cluster configuration. A degenerate case of this configuration is that shown in Figure 7(a), in which the media bias $\mu = 1$ coincides with an extreme and the two clusters merge.

We conclude by noting that all equilibria illustrated in Figure 8 are asymptotically stable, as they satisfy the connectivity properties required by the analytical results in Section 4.2.

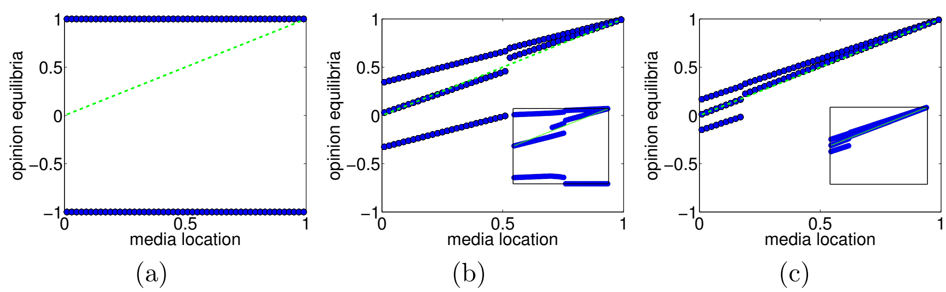


Figure 8. Bifurcation with respect to the location of the media source. The filled circles represent the locations of equilibrium clusters, while the dashed line indicates the media bias; parameter values are $X_c = 0$ and $\epsilon = 1.2$, with (a) $m = 0$, (b) $m = 26$, and (c) $m = 54$. In the absence of open-minded individuals (a), the media has no effect, as all closed-minded individuals approach extremist views. Plots (b) and (c) correspond to non-symmetric (type NS) initial data, with inserts that show the results obtained from symmetric (type S) initializations. Asymmetry seems to enhance the effect of the media, in particular when the number of open-minded individuals is relatively low ($m = 26$, plot (b)).

4.2. **Stability of equilibria.** The stability of equilibria for the opinion formation model in the presence of media, (19) with (5) or (6) coupled to the media term (20), can be studied similarly to that for the model without media (Section 3.4). We present the analysis briefly, highlighting the changes from Section 3.4.

Consider an equilibrium opinion state $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_N^*)$ of (19). We suppose that all assumptions made in Section 3.4 hold here as well; specifically, $\mathbf{x}^*$ satisfies Assumption 3.3, and the influence functions ϕ and $\hat{\phi}$ are simple indicator functions with supports ϵ and $\hat{\epsilon}$, respectively.

We assume, similarly, that the media-influence functions ϕ_m and $\hat{\phi}_m$ are also simple indicator functions, with supports ϵ_m and $\hat{\epsilon}_m$, respectively. Additionally, we require

Assumption 4.1. *The equilibrium state* $\mathbf{x}^*$ *satisfies:*

1. $|\mu - x_i^*| \neq \epsilon_m$, *for all non-closed-minded individuals* i*;*

2. $|\operatorname{sgn}(x_i^*) - x_i^*| \neq \hat{\epsilon}_m$, *for all closed-minded individuals* i.

Assumption 4.1 guarantees that the coefficients $\hat{b}_i$ and b_i in (20) are well-defined at equilibrium, by ensuring that $\hat{\phi}_m$ and ϕ_m are not evaluated at a point of discontinuity. Again, by continuity, there is in fact an entire neighbourhood of $\mathbf{x}^*$ where all $\hat{b}_i$ and b_i are constants.

Extending the terminology introduced in Definition 3.1, we say that a non-closed-minded individual i is *connected to the media* if its interaction coefficient with the media is nonzero, that is, $b_i \neq 0$.

As in Section 3.4, we study stability via the eigenvalues of the linear approximation. The terms in the Jacobian that correspond to the inter-individual interactions modelled by f_i are given by previous computations (see (13) and (15)). Since all coefficients $\hat{b}_i$ and b_i are constants near $\mathbf{x}^*$, the media term (20) only contributes to the diagonal entries of the Jacobian for (19). Specifically, (20) contributes $-\alpha_4\hat{b}_i^*$ to each diagonal term corresponding to closed-minded individuals, and $-\alpha_5 b_i^*$ to the diagonal entries that correspond to non-closed-minded individuals. (As before, an asterisk indicates that the respective coefficients are evaluated at the equilibrium $\mathbf{x}^*$).

Stability for model with media and no social norm of open-mindedness. We use the setup from Section 3.4, and label by $1, \dots, q$ the closed-minded individuals. If no social norm of open-mindedness is present, so that f_i is given by (5), the Jacobian matrix of (19) is given by

$$J_{1m} = \begin{pmatrix} D_m & 0 \\ B & A_m \end{pmatrix}, \tag{24}$$

where D_m is the $q \times q$ diagonal matrix with entries $-\alpha_2 - \alpha_4\hat{b}_i^*$, B is the same $(N-q) \times q$ matrix with entries $\alpha_1 a_{ij}^*$ as in (13), and A_m is a $(N-q) \times (N-q)$ square matrix given by

$$A_m = \begin{pmatrix} -\alpha_1 - \alpha_5 b_{q+1}^* & \underset{[i<j]}{\alpha_1 a_{ij}^*} & \cdots \\ \underset{[i>j]}{\alpha_1 a_{ij}^*} & \ddots & \vdots \\ \vdots & \dots & -\alpha_1 - \alpha_5 b_N^* \end{pmatrix}. \tag{25}$$

Here the indices i and j of the entries in A_m correspond to non-closed-minded individuals.

The argument now proceeds as in the proof of Theorem 3.6: By the normalization condition (3), A_m is weakly diagonally dominant, so the eigenvalues of J_{1m} have negative real parts provided A_m is non-singular. If $b_i^* > 0$ for some $i \in \{q+1, \dots, N\}$, strict diagonal dominance holds in that row of A_m. On the other hand, if all $b_i^* = 0$, strict diagonal dominance in a row of A_m can be inferred—as in Section 3.4—by making the additional assumption that at least one entry in the corresponding row of B is nonzero. Finally, provided A_m is also irreducible, Theorem 3.5 can be used to conclude that A_m is non-singular.

Hence for linear stability to hold in this case, one of the connectivity assumptions in Theorem 3.6 can be replaced by the presence of a nontrivial media influence at equilibrium. The stability result is given by the following theorem:

Theorem 4.2. *Let* $\mathbf{x}^*$ *be an equilibrium solution of the model for opinion dynamics with media in the absence of an open-mindedness social norm,* (19) *with* (5) *and*

(20)*, that satisfies Assumptions 3.3 and 4.1. Then* $\mathbf{x}^*$ *is asymptotically stable provided that, at equilibrium, all non-closed-minded individuals are connected, and in addition at least one non-closed-minded individual is connected either to the media or to a closed-minded individual.*

Stability for model with media and a social norm of open-mindedness. As in Section 3.4, closed-minded individuals separate into two groups: a group consisting of the individuals $1, \ldots, r$ who do not connect to any other individual, and a second group of closed-minded individuals who connect to at least one open-minded individual.

The Jacobian for (19) can be written as

$$J_{2m} = \begin{pmatrix} D_{1m} & 0 \\ 0 & E_m \end{pmatrix}, \tag{26}$$

where D_{1m} is a diagonal matrix of size $r \times r$, with entries

$$-\alpha_2(N-m-1)\hat{a}_i^* - \alpha_4 \hat{b}_i^*, \qquad i = 1, \ldots, r,$$

that corresponds to the first group of closed-minded individuals. The matrix E_m has a form similar to E from (15), namely

$$E_m = \begin{pmatrix} D_{2m} & F \\ B & A_m \end{pmatrix}, \tag{27}$$

where the $(q-r) \times (q-r)$ diagonal matrix D_{2m} has entries

$$-\alpha_2(N-m-1)\hat{a}_i^* - \alpha_3 \sum_{j \in O} \hat{a}_{ij}^* - \alpha_4 \hat{b}_i^*, \qquad i = r+1, \ldots, q.$$

The matrices F and B are the same as in (15) (the media term only affects the diagonal entries of the Jacobian), and the matrix A_m is given by (25).

Note that D_{1m} has negative diagonal entries, as individuals $1, \ldots, r$ have extreme opinions ± 1 and hence $\hat{a}_i^*$ and $\hat{b}_i^*$ are strictly positive for all $i \in \{1, \ldots, r\}$. Then the argument follows along the same lines as that for the proof of Theorem 3.7, with E replaced by E_m. In particular, to show that E_m is irreducible it is necessary to make the same connectivity assumption as in Theorem 3.7; that is, unlike in the case of media in the absence of an open-mindedness social norm, here the presence of media cannot be used to replace the connectivity assumption at equilibrium. Strong diagonal dominance holds for a row of E_m either if one of $\hat{a}_i^*$ or $\hat{b}_i^*$ is strictly positive for some $i \in \{r+1, \ldots, q\}$, or if at least one non-closed-minded individual is connected to the media at equilibrium (that is, $b_i^* > 0$ for some $i \in \{q+1, \ldots, N\}$).

The stability result in this case reads:

Theorem 4.3. *Let* $\mathbf{x}^*$ *be an equilibrium solution of the model for opinion dynamics with media in the presence of an open-mindedness social norm,* (19) *with* (6) *and* (20)*, that satisfies Assumptions 3.3 and 4.1. Also assume either that one of* $\hat{a}_i^*$ *or* $\hat{b}_i^*$ *is strictly positive for some* $i \in \{r+1, \ldots, q\}$*, or that at least one non-closed-minded individual is connected to the media at equilibrium. Then* $\mathbf{x}^*$ *is asymptotically stable provided that, at equilibrium, all non-closed-minded individuals are connected, and in addition at least one closed-minded individual is connected to an open-minded individual.*

Theorems 4.2 and 4.3 suggest that media has the potential to increase stability. This is not surprising, given the tendency of the media to attract individuals who have been perturbed away from it. Empirical studies show indeed that the distance from a media source in opinion space is positively correlated with the tendency to move towards said media source upon exposure [33, 3].

5. Discussion and concluding remarks.

We have generalized the continuous-time opinion dynamics model introduced by Motsch and Tadmor [42] to incorporate closed-minded, extreme-seeking individuals who may, however, be induced to act more open-mindedly when they interact with naturally open-minded individuals. Our model yields a richly diverse range of potential equilibrium states, which we have explored analytically and numerically: in addition to moderate consensus and mutually disconnected opinion clusters, we have shown the existence of extremist consensus and extreme polarization states, as well as of connected opinion clusters. Moreover, we have proved that these latter configurations are asymptotically, not just neutrally, stable. We have further extended the model to include media, which are assumed to have different influences on open- and closed-minded individuals, and confirm using analysis and simulation that media introduce an additional attractive and stabilizing effect on the long-time opinion distribution.

Our modelling framework suggests numerous natural extensions. For instance, at present our model contains sharp cutoffs both at the bound of confidence ϵ in the influence function ϕ, and at the critical threshold X_c for extreme-seeking dynamics; it would be useful to explore whether the qualitative and rigorous conclusions are modified by smoothing out the transitions. Among other natural generalizations, a particularly challenging but realistically important one appears to be robustness to the addition of noise; since many equilibria of our model are asymptotically stable, presumably our qualitative conclusions would persist in the presence of sufficiently small random perturbations.

Understanding the influence of media on societal opinion distributions is particularly important, and much more could be done within our modelling framework to explore this influence, by for instance adding multiple and potentially competing media sources with possibly time-varying strengths and/or intrinsic biases, accounting for the hostile media effect, or even as suggested in equation (23), incorporating a feedback loop by letting media bias be driven by the opinion distribution.

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