DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **34**, Number **12**, December **2014**

doi:10.3934/dcds.2014.34.5325

pp. **5325–5357**

OPTIMAL PARAMETER-DEPENDENT BOUNDS FOR KURAMOTO-SIVASHINSKY-TYPE EQUATIONS

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(Communicated by Tatsien Li)

Abstract. We derive *a priori* estimates on the absorbing ball in L^2 for the stabilized and destabilized Kuramoto-Sivashinsky (KS) equations, and for a sixth-order analog, the Nikolaevskiy equation, and in each case obtain bounds whose parameter dependence is demonstrably optimal. This is done by extending a Lyapunov function construction developed by Bronski and Gambill (*Nonlinearity* **19**, 2023–2039 (2006)) to take into account the dependence on both large and small parameters in the system. In the case of the destabilized KS equation, the rigorous bound $\limsup_{t\to\infty}\|u\| \le K\alpha L^{3/2}$ is sharp in both the large parameter α and the system size L. We also apply our methods to improve previous estimates on a nonlocal variant of the KS equation.

1. Introduction.

The Kuramoto-Sivashinsky (KS) equation [26, 40] in one space dimension,

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \tag{1}$$

with L-periodic boundary conditions, has been intensively studied as a canonical model of complex dynamics and spatiotemporal chaos in spatially extended systems, and of the application of dynamical systems methods to partial differential equations (PDEs) [22, 25]. It is also the prototypical example of a family of related PDEs of the form $u_t + uu_x = \mathcal{L}u$ displaying a wide variety of dynamical and bifurcation properties. In the present work we shall consider the following members of this family: The PDE

$$u_t + uu_x = \left(\varepsilon^2 - (1+\partial_x^2)^2\right) u \;\equiv\; \mathcal{L}_{\mathrm{sKS}} u \tag{2}$$

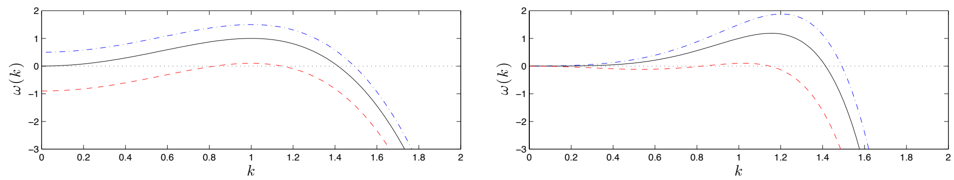
for $0 < \varepsilon^2 < 1$ is the *stabilized* (or *damped*) Kuramoto-Sivashinsky (sKS) equation [31], while for $\varepsilon^2 > 1$ it is the *destabilized* Kuramoto-Sivashinsky (dKS) equation [49]; in the latter case it is convenient to set $\alpha = \varepsilon^2 - 1$ and write the dKS equation in the form

$$u_t + uu_x = \alpha u - 2u_{xx} - u_{xxxx} \;\equiv\; \mathcal{L}_{\mathrm{dKS}} u. \tag{3}$$

(Note that (2) and (3) reduce to the KS equation (1) in the rescaled form

$$u_t + u_{xxxx} + 2u_{xx} + uu_x = 0; \tag{4}$$

2010 *Mathematics Subject Classification.* Primary: 35B45, 35K30; Secondary: 37L30.
Key words and phrases. Kuramoto-Sivashinsky equation, Nikolaevskiy equation, a priori bounds, global attractor, Lyapunov function.



(a) (b)

FIGURE 1. Linear dispersion relation $\omega(k)$ (growth/decay rates of the kth Fourier mode $\hat{u}_k(t)$ about the zero solution) for (a) the (de)stabilized Kuramoto-Sivashinsky equation for $\varepsilon^2 = 0.1$ (sKS equation (2): dashed), $\alpha = \varepsilon^2 - 1 = 0$ (KS equation (4), solid), and $\alpha = 0.5$ (dKS equation (3), dot-dashed); (b) the Nikolaevskiy equation (5) for $\varepsilon^2 = 0.1$ (dashed), $\varepsilon^2 = 1$ (solid), $\varepsilon^2 = 1.5$ (dot-dashed curve).

for $\varepsilon^2 = 1$ and $\alpha = 0$, respectively; in the remainder of this paper we shall work in this scaling.) An analogous sixth-order PDE is the Nikolaevskiy equation [2]

$$u_t + uu_x = -\partial_x^2 \left(\varepsilon^2 - (1 + \partial_x^2)^2\right) u \;\equiv\; \mathcal{L}_{\mathrm{Nik}} u, \tag{5}$$

which has recently been attracting considerable interest in its own right due to its unexpected instability and scaling properties [29, 46, 47].

Since the pioneering work of Nicolaenko *et al.* [33], who proved the existence of an absorbing ball in L^2 for a restricted class of solutions of (1), there has been much work aimed at obtaining and refining *a priori* bounds on the solution of the KS equation (1), as outlined in more detail below. Considerable progress has been made in clarifying the most common and readily generalized of the available bounding approaches, the Lyapunov function method, particularly by Bronski and Gambill [6]. In the present work we apply this bounding approach to the more general PDEs (2)–(5), and show how to generalize the Lyapunov function method, incorporating the dependence on additional parameters, to yield bounds whose *parameter dependence* is sharp. In particular, for the destabilized KS equation (3) with large α, we obtain L^2 bounds with the optimal scaling in both α and L.

1.1. **Background on the KS equation and its generalizations.** The relation between the KS, sKS, dKS and Nikolaevskiy PDEs (2)–(5) and the origins of their complex dynamical behaviors may most readily be appreciated by considering them in their Fourier space formulations: On a one-dimensional L-periodic domain, via the Fourier decomposition $u(x,t) = \sum_k \hat{u}_k(t) \exp(\mathrm{i}kx)$, $k = 2\pi n/L$, $n \in \mathbb{Z}$, equations (2)–(5) take the form

$$\frac{d}{dt}\hat{u}_k = \omega(k)\hat{u}_k - \mathrm{i}\sum_{k'} k'\, \hat{u}_{k'}\, \hat{u}_{k-k'} \ , \tag{6}$$

where the linear dispersion relations $\omega(k)$ in the various cases are shown in Fig. 1. The long-wave limit of the dispersion relation determines the evolution of the spatial mean $\hat{u}_{\mathbf{0}}(t) = L^{-1}\int_{-L/2}^{L/2} u(x',t)\,dx'$, via $d\hat{u}_{\mathbf{0}}/dt = \omega(0)\hat{u}_{\mathbf{0}}$; in particular, solutions which are initially mean zero remain so.

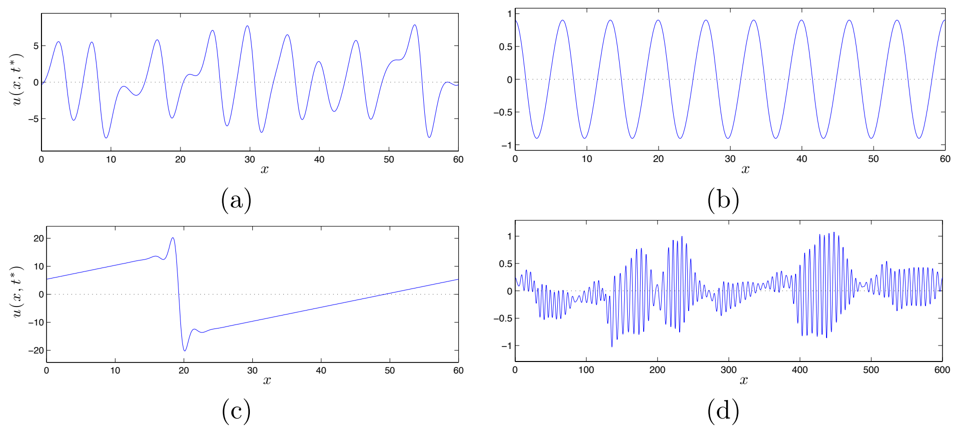


FIGURE 2. Typical snapshots $u(x, t^*)$ of L-periodic solutions (shown for $x \in [0, L]$) on the attractor at fixed times t^* for: (a) the Kuramoto-Sivashinsky equation (4) for $L = 60$; (b) the stabilized (damped) KS equation (2) for $\varepsilon^2 = 0.04$, $L = 60$; (c) the destabilized KS equation (3) for $\alpha = 0.5$ ($\varepsilon^2 = 1.5$), $L = 60$; (d) the Nikolaevskiy equation (5) for $\varepsilon^2 = 0.04$, $L = 600$.

Kuramoto-Sivashinsky equation. For the KS equation (4), since $\omega(k) \equiv \omega_{\mathrm{KS}}(k) = 2k^2 - k^4$, the $\lfloor \sqrt{2}L/2\pi \rfloor$ Fourier modes with low wave number $0 < |k| < \sqrt{2}$ are linearly unstable about the zero solution. In particular, while the dominant features of the intrinsic KS dynamics have characteristic length near 2π, associated with the maximum of $\omega_{\mathrm{KS}}(k)$ at $|k| = 1$ (see Fig. 2(a)), all larger scales in the system are also unstable. The nonlinear uu_x term, which is itself globally energy neutral, stabilizes the system by coupling Fourier modes and thereby facilitating energy transfer from the intrinsically growing large scales to the small-scale modes with $|k| > \sqrt{2}$, which are strongly damped.

Equation (4) is preserved under the Galilean transformation $x \mapsto x - ct$, $u \mapsto u + c$ for $c \in \mathbb{R}$; indeed, the KS equation, which was initially derived in various contexts including thin film flow of viscous fluids down inclines [21], plasma ion wave instabilities [27], phase dynamics in reaction-diffusion systems [26] and instabilities of flame fronts [40], may be considered as a generic model of long-wave instability in the presence of Galilean invariance [29, 31]. Associated with this continuous symmetry, the spatially uniform mode is neutrally stable ($\omega_{\mathrm{KS}}(0) = 0$); the evolution thus preserves the mean $\hat{u}_{\mathbf{0}}(t)$, which we may choose to be zero. Additional symmetries satisfied by (4) (in fact by (2)–(5)) and preserved by the periodic boundary conditions include invariance under space and/or time translation, and the reflection symmetry $x \mapsto -x$, $u \mapsto -u$, under which the subspace of odd solutions is invariant.

Extensive numerical simulations of the KS equation (1) on L-periodic domains suggest that spatial correlations decay exponentially, and that "intensive properties" such as pointwise statistics, especially amplitudes of the solution $u(x, t)$ and its derivatives, are asymptotically L-independent; this is consistent with the view that in the large-L limit, the local dynamics are independent of the system size [50]. Correspondingly, the numerical evidence shows that, for instance, pointwise amplitudes

$|u(x,t)|$, the instantaneous Fourier space exponential decay rate [10] and scaled time-averaged Fourier power spectrum [50] are L-independent, while "extensive" quantities such as the Lyapunov dimension of the spatiotemporally chaotic attractor [28] and the energy (square of the L^2 norm) $\|u\|^2 \equiv \|u(\cdot,t)\|_2^2 \equiv \int_{-L/2}^{L/2} u^2(x',t)\,dx'$ scale proportionally to the system size. That is, one expects that (after decay of transients) local quantities such as $\|u(\cdot,t)\|_\infty = \sup_x |u(x,t)|$ are bounded independent of L — a notable partial result in this direction is Michelson's proof [30] of the uniform boundedness of all stationary periodic solutions of (1) — while the L^2 norm scales as $\|u\| \leq cL^{1/2}$ (note that this second estimate would follow from the first, via $\|u\|^2 \leq L\|u\|_\infty^2$). An ongoing challenge has been to derive such extensive behavior analytically by obtaining bounds on norms of u directly from (1), the difficulty arising because the nonlinear uu_x coupling term does not appear in the global energy balance $\frac{1}{2}\frac{d}{dt}\|u\|^2 = \|u_x\|^2 - \|u_{xx}\|^2$; this program of deriving *a priori* bounds, briefly reviewed below, and its extensions to (2)–(5) form the basis for the present work.

Damped (stabilized) Kuramoto-Sivashinsky equation. The *stabilized* (damped) KS equation, given by (2) with $\varepsilon^2 < 1$, which was studied as an early model for wavelength selection [37, 38], has received attention in the context of directional solidification and step-flow growth [1, 34]. The dispersion relation of (2) is $\omega(k) \equiv \omega_{\mathrm{sKS}}(k) = \varepsilon^2 - (1-k^2)^2$, so that while Fourier modes with $|k^2 - 1| < \varepsilon$ are linearly unstable about zero, all other modes are damped, in particular the long-wave modes for $k \to 0$; for $\varepsilon^2 < 1$ the Galilean invariance is broken and the spatial mean decays. For sufficiently small ε, a multiple-scale analysis of the form $u(x,t) \sim \varepsilon A(X,T)\mathrm{e}^{\mathrm{i}x} + \text{c.c.} + \text{h.o.t.}$ (where $X = \varepsilon x$, $T = \varepsilon^2 t$) reveals that the slowly-varying amplitude $A = \mathcal{O}(1)$ satisfies a Ginzburg-Landau equation, and that stable stationary "roll" (or "cellular") solutions of wave number k exist for each $k = 1 + \mathcal{O}(\varepsilon)$ in the narrow band of unstable modes. As ε increases, these roll solutions become unstable, and a rich variety of secondary and tertiary bifurcations is observed [7, 31] in the transition to spatiotemporal chaos in the KS limit $\varepsilon^2 \to 1^-$ [8, 14]. In the present work we shall establish the rigorous ε-dependence of the long-time scaling of solutions of the sKS equation (2) ($\varepsilon^2 < 1$) in the form $\|u\| = \mathcal{O}(\varepsilon)$, consistent with the expectation from the asymptotics.

Destabilized Kuramoto-Sivashinsky equation. For the *destabilized* KS equation (3) with $\alpha = \varepsilon^2 - 1 > 0$, the αu term shifts the entire dispersion relation $\omega_{\mathrm{dKS}}(k) = \alpha + 2k^2 - k^4 = \omega_{\mathrm{KS}}(k) + \alpha$ upward, with the main effect of driving the long-wave modes and again breaking the continuous symmetry. Indeed, since the spatial mean evolves as $d\hat{u}_{\mathbf{0}}/dt = \omega_{\mathrm{dKS}}(0)\hat{u}_{\mathbf{0}}$ with $\omega_{\mathrm{dKS}}(0) = \alpha > 0$, only mean zero solutions can be bounded, so we assume $\hat{u}_{\mathbf{0}}(0) = 0$. Numerical simulations of (3) [49] for sufficiently large α and/or L ($\alpha L \geq K^*$ for some constant K^* appears to be sufficient) then show distinctly non-extensive behavior (contrary to the KS limit $\alpha \to 0^+$), in that solutions approach a single attracting stationary "viscous shock" structure as in Fig. 2(c) (a similar viscous shock solution occurs in the so-called "Burgers-Sivashinsky" equation [18, 39]). Asymptotic and numerical investigations of these dKS viscous shock solutions reveal an outer linear region with $u_x \sim \alpha$, and an inner transition layer of width $\delta \sim (\alpha L)^{-1/3}$, indicating an amplitude $\|u\|_\infty$ proportional to αL, and a leading-order scaling of the L^2 norm as $\|u\|^2 \sim \alpha^2 L^3/12$ [39, 49]; in the present work we confirm this observed scaling rigorously, by proving a bound for long-time solutions of (3) of this form $\|u\| = \mathcal{O}(\alpha L^{3/2})$.

Nikolaevskiy equation. A sixth-order analogue of the KS equation, the Nikolaevskiy equation (5) was originally proposed to model longitudinal seismic waves [2], but has more recently been studied as a canonical model for *short-wave* pattern formation with Galilean invariance [11, 29, 45], with potential applications to phase dynamics in reaction-diffusion systems [42] and transverse instabilities of moving fronts [11]. By contrast, for fourth-order dissipative PDEs in the KS family (2), the continuous symmetry is associated with *long-wave* instability in the KS equation (4); in the damped KS case $\varepsilon^2 < 1$, in which only a finite band of Fourier modes bounded away from $k = 0$ is unstable, the symmetry is lost ($\omega(0) \neq 0$). The additional symmetry combined with finite-wavelength instability in the sixth-order case can be traced to the linear operator $\mathcal{L}_{\mathrm{Nik}} = -\partial_x^2 \mathcal{L}_{\mathrm{sKS}}$, whose dispersion relation satisfies $\omega_{\mathrm{Nik}}(k) = k^2 \omega_{\mathrm{sKS}}(k) = k^2 \left[\varepsilon^2 - (1-k^2)^2\right]$ (see Fig. 1(b)): for $0 < \varepsilon^2 < 1$, the unstable band is restricted to a narrow finite-wavelength interval $|k^2 - 1| < \varepsilon$, but in addition $\omega_{\mathrm{Nik}}(0) = 0$.

The Nikolaevskiy model has been receiving considerable attention recently due to its remarkable dynamical and scaling properties: Again, as for (2) for any $0 < \varepsilon^2 \ll 1$ rolls of amplitude $\mathcal{O}(\varepsilon)$ can be shown to exist by weakly nonlinear analysis, but unlike in the sKS equation, all roll solutions of (5) are unstable [29, 47], so that as ε increases from 0, there is a direct transition from a uniform equilibrium to a spatiotemporally chaotic state [46]. Furthermore, a multiple-scale analysis for $0 < \varepsilon^2 \ll 1$ of the form $u(x,t) \sim \varepsilon^{\gamma_1} A(X,T) \mathrm{e}^{\mathrm{i}x} + \mathrm{c.c.} + \varepsilon^{\gamma_2} f(X,T) + \mathrm{h.o.t.}$, incorporating the long-wave ($k = 0$) as well as the patterned ($k = 1$) mode, reveals that the only asymptotically consistent scaling is $\gamma_1 = 3/2$, $\gamma_2 = 2$ [29] (though numerical investigations of the scaling behavior [43] reveal further potential anomalies for the long-wave mode [51], indicating that the situation is not yet fully understood). On the chaotic attractor, the observed solutions of (5) for $0 < \varepsilon^2 \ll 1$ scale as $u \sim \varepsilon^{3/2}$, but the presence of the unstable $\mathcal{O}(\varepsilon)$ roll solutions means that the global L^2 absorbing ball for the Nikolaevskiy PDE can scale at best as $\|u\| = \mathcal{O}(\varepsilon)$; we shall establish a bound of this form below.

1.2. **Rigorous bounds on the absorbing ball.** The first demonstration of the L^2 boundedness of the dynamics of the KS equation (1) was provided by Nicolaenko *et al.* [33], who proved, within the invariant subspace of odd solutions, an estimate of the form $\limsup_{t\to\infty} \|u\| \leq K L^p$ with $p = 5/2$, and also showed how bounds on related quantities such as higher derivatives, the number of determining modes and attractor dimension could follow from the fundamental L^2 estimate (see also [10, 15, 44]). While subsequent work has succeeded in establishing the extension to general periodic initial data [9, 18, 23] and reducing the exponent p [6, 9, 17], the numerically predicted extensive scaling $\|u\| = \mathcal{O}(L^{1/2})$ has long remained analytically elusive (although considerable progress towards the optimal scaling has recently been achieved by Otto [35]).

Most bounding approaches have been based on the idea, introduced in [33], of showing that the L^2 distance between u and a suitably chosen "comparison" (or "gauge") function ϕ acts as a Lyapunov function [6, 18]; improvements in the bounds could then be achieved by improved constructions of ϕ. An indication of the limitations of such analytic methods was provided by the discovery [49] that while Lyapunov function-based bounds (following [9]) may be proved for the destabilized KS equation (3), this equation supports viscous shock solutions for which $\|u\| \sim \alpha L^{3/2}$; this example suggests that bounding approaches for (1) which also apply to the dKS equation for $\alpha > 0$ cannot attain L-dependent scaling better than $L^{3/2}$. A breakthrough in the understanding of the Lyapunov function methods was achieved by Bronski and Gambill [6], who not only explicitly constructed ϕ to prove the bound $\limsup_{t\to\infty} \|u\| \le KL^{3/2}$ for the KS equation (1), but also presented scaling arguments to show that for a large class of comparison functions ϕ, the $\mathcal{O}(L^{3/2})$ bound is indeed the best achievable by such methods.

(We note that the $\mathcal{O}(L^{3/2})$ scaling for the KS equation (1) has in fact been improved using non-Lyapunov methods: Giacomelli and Otto [17] derived a bound of $\limsup_{t\to\infty} \|u\| = o(L^{3/2})$ by relating the KS equation at large scales to entropy solutions of the inviscid Burgers equation. In a remarkable recent advance, Otto [35] obtained an estimate of the form $\langle \|u\| \rangle \le \mathcal{O}(L^{5/6+})$ (where $\langle\cdot\rangle$ represents the time average), and in fact achieved extensive scaling up to logarithms for the time averages of some higher fractional Sobolev norms, again by relating the KS equation to Burgers' equation, and utilizing careful Littlewood-Paley-type estimates for Sobolev and Besov norms. It does not appear as straightforward to generalize or extract parameter dependence from these approaches as for Lyapunov function methods, though, and they are necessarily inapplicable to the dKS equation for $\alpha > 0$.)

Lyapunov function bounding approaches have proved to be quite amenable to generalization to various PDEs related to the KS equation, for instance those with different or additional linear terms [3, 13, 16, 19, 20, 41, 48, 49] or in higher space dimensions [12, 32, 36]. The Bronski-Gambill (BG) construction, which yields the best L-dependence achievable through such methods, gives an alternative bounding approach and in many cases may improve previously derived bounds; the method has been applied in [4, 5, 41]. However, in problems containing additional parameters, the BG approach in its original form does not attempt to optimize the parameter dependence of the *a priori* bounds. A main goal of the present work is to show, using the examples of the dKS, sKS and Nikolaevskiy PDEs, how to adapt Lyapunov function methods for KS-like equations to incorporate and optimize the dependence on parameters (in addition to the system size L). In the particular case of the destabilized KS equation (3) for sufficiently large αL, our approach establishes (to our knowledge for the first time) a demonstrably optimal parameter- and L-dependent scaling of *a priori* bounds for a KS-like PDE. (For comparison, a direct application of the BG approach, while permitting bounds to be derived for the dKS equation, does not yield the optimal scaling with α, as discussed in Section 3.1.)

1.3. **Outline.** Since we are treating several related PDEs with slightly different linear operators $\mathcal{L}$, in Section 2 we obtain some general results on *a priori* bounds on solutions of $u_t = \mathcal{L}u - uu_x$, assuming certain estimates on $\mathcal{L}$ and properties of the comparison function $\phi(x)$ which we check in each individual instance. Our approach to dealing with parameter dependence is developed, for the case of a large parameter α, in Section 3 in the context of the destabilized KS equation (3). Specifically, we are able to use the *same* potential function $\widetilde{q}(y)$ [6] in our estimates, with all dependence on L and the parameter being incorporated in a scaling Ansatz for the comparison function, for which we derive the optimal exponents (Section 3.2). To keep our derivations sufficiently self-contained, and since the detailed properties of the potential are essential to all our bounds, following [6] we explicitly construct $\widetilde{q}(y)$ and verify its properties in Appendix A; to obtain the scaling exponents we also need the norms of the comparison function $\phi(x)$, which are derived in Appendix B.

The resulting bounds for the dKS equation (Section 3.3) are optimal, in that they capture the scaling of the viscous shock solutions which exist for large αL.

One needs to choose the scaling properties of $\phi(x)$ differently when the parameter is small; this situation is explored for the stabilized KS equation (2) for small ε, for which the bounds recover the $\mathcal{O}(\varepsilon)$ scaling of the roll solutions (Section 4.1). While in general the potential $\widetilde{q}(y)$ used in the construction of $\phi(x)$ might be expected to depend on the order of the PDE, a Poincaré estimate for odd solutions permits a reduction of order that allows us to treat the sixth-order Nikolaevskiy PDE (5) using the same potential as before, as shown in Section 4.2, for which our bounds are again sharp in the small parameter ε.

We remark that our bounds are restricted to odd solutions of (2)–(5); the results may be extended to general periodic initial data following the approach of Collet *et al.* [9] or Goodman [18].

A further application of how the methods developed here may be used to improve the parameter dependence of bounds is given in Appendix C. In [20], Hilhorst, Peletier, Rotariu and Sivashinsky analyzed a KS equation with additional local stabilizing and nonlocal destabilizing terms; among their results was a proof of nonlinear stability of the zero solution under certain conditions on the parameters α and κ, and a bound on the L^2 absorbing ball under some restrictions on the parameters. Applying the methods of [6], Bronski, Fetecau and Gambill [5] reconsidered the estimates on the absorbing ball, removing the constraints on the parameters, improving the L-dependence and explicitly tracking the parameter dependence of their estimates. In Appendix C we revisit this problem, sharpening both the estimates of the nonlinear stability boundary and the scaling of the bounds with κ and α, by using a similar approach and the same potential function $\widetilde{q}(y)$ as in Sections 2–4.

1.4. **Notation.** The time evolution of PDEs of the form (2)–(5) preserves L-periodicity, a vanishing spatial mean and the subspace of odd (antisymmetric) solutions. Throughout this work, solutions $u(x,t)$ of the governing PDEs are assumed to be L-periodic (typically considered on the domain $[-L/2, L/2]$) and odd (which automatically implies the mean zero condition $\hat{u}_{\mathbf{0}}(t) = 0$); the extension to general periodic initial data can be obtained along similar lines to the arguments in [9, 18]. Following [9], we define the space of odd L-periodic functions

$$\mathcal{A}_L = \{u : u(x - L/2) = u(x + L/2),\ u(-x) = -u(x)\}. \tag{7}$$

It will be convenient for our calculations to assume sufficient smoothness: specifically, that the highest derivatives in the PDEs are continuous. This smoothness assumption may be relaxed, but it is not restrictive in our case, as one can show (see [10, 49]) that solutions of the general KS-like PDEs are analytic, while (see Theorem A.2) the comparison function $\phi(x) \in C^\infty$ by construction. Thus we define

$$\begin{aligned}\mathcal{A}_{L,2m} &= \mathcal{A}_L \ \cap\ C^{2m}[-L/2, L/2] \\ &= \{u : u(x - L/2) = u(x + L/2),\ u(-x) = -u(x),\ u(\cdot) \in C^{2m}[-L/2, L/2]\}.\end{aligned} \tag{8}$$

An integral with no indicated limits is assumed to be over the full spatial domain: $\int \cdot \equiv \int_{-L/2}^{L/2} \cdot \, dx$.

For each of the PDEs (2)–(5), for sufficiently small L all solutions decay to 0 since no Fourier modes lie in the unstable band; for simplicity we shall assume throughout that $L \geq 2\pi$.

2. **Derivation of general bounding principle.** We first outline a formulation of the Lyapunov function approach which, by generalizing previous works on the KS equation [6, 9, 18, 33], permits the analogous treatment of a wider class of related KS-like PDEs, largely following the approach of Collet *et al.* [9].

Let $\mathcal{L}$ be a $2m$-th order real self-adjoint linear operator (containing only even-order x-derivatives); the sKS, dKS and Nikolaevskiy equations can be written in the form (with $\mathcal{L} = \mathcal{L}_{\rm sKS}$ or $\mathcal{L}_{\rm dKS}$ for $m = 2$, or $\mathcal{L}_{\rm Nik}$ with $m = 3$):

$$u_t = \mathcal{L}u - uu_x \ . \tag{9}$$

Introduce an odd comparison function $\phi(x)$, and write

$$u(x,t) = v(x,t) + \phi(x); \tag{10}$$

then v satisfies the PDE

$$v_t \ = \ u_t \ = \ \mathcal{L}v + \mathcal{L}\phi - (v+\phi)(v+\phi)_x. \tag{11}$$

Multiplying by v and integrating over the domain, integrating by parts and using the periodic boundary conditions, we find

$$\frac{1}{2}\frac{d}{dt}\int v^2 = \int v\left(\mathcal{L} - \tfrac{1}{2}\phi'\right)v + \int v\left(\mathcal{L} - \phi'\right)\phi \tag{12}$$

$$= -\left(v,v\right)_{\phi/2} - \left(v,\phi\right)_{\phi} . \tag{13}$$

In the above, following [9], we have defined the bilinear form

$$\left(v_1,v_2\right)_{\gamma\phi} = -\int v_1\left(\mathcal{L} - \gamma\phi'\right)v_2 \ ; \tag{14}$$

for $\gamma \in [0,1]$, with $\phi' = \phi_x$; note that since $\mathcal{L}$ is self-adjoint, $\left(v_1,v_2\right)_{\gamma\phi} = \left(v_2,v_1\right)_{\gamma\phi}$.

The crucial step in the Lyapunov function approach to the derivation of *a priori* bounds for solutions $u(x,t)$ of (9) is the choice of a comparison function $\phi(x)$ and a scalar $\lambda > 0$ so that the hypotheses (15)–(16) of Theorem 2.1 below, which generalizes a main result of Collet *et al.* [9], are satisfied. The construction of a suitable $\phi(x)$ for different linear operators $\mathcal{L}$ is discussed in the following Sections; for now we show how such a comparison function leads to bounds on the radius of the L^2 absorbing ball:

Theorem 2.1. *Assume that $\phi \in \mathcal{A}_{L,2m}$ is an odd L-periodic function and $\lambda > 0$ is a scalar such that for all $v \in \mathcal{A}_{L,2m}$, we have that*

$$\left(v,v\right)_{\phi/4} \equiv -\int v\left(\mathcal{L} - \frac{1}{4}\phi'\right)v \geq \lambda\int v^2 \equiv \lambda\|v\|^2, \tag{15}$$

and also that

$$\left(v,v\right)_{\phi} \equiv -\int v\left(\mathcal{L} - \phi'\right)v \geq 0. \tag{16}$$

Then there is a universal constant $K \leq 17/4$ such that if $u(x,t) \in \mathcal{A}_{L,2m}$ satisfies the PDE (9), then

$$\limsup_{t\to\infty}\|u(\cdot,t)\|^2 \leq \frac{K}{\lambda}\left(\phi,\phi\right)_0 . \tag{17}$$

Remark 2.2. Following the above approach, the problem of obtaining the *optimal* scaling for the radius of an absorbing ball reduces to choosing $\phi(x)$ and λ to minimize the bound in (17), subject to satisfying the constraints (15)–(16). Note that the existence of *some* function $\phi(x)$ and constant λ leading to bounds was previously established for the KS equation in, for instance, [6, 9] and for the dKS equation in

[49], and follows also for fairly general $\mathcal{L}$ from the results of [16]. In the present work we have attempted to optimize neither the prefactor K nor the constants in $(\phi,\phi)_0$ in the bound (17), but rather concentrated on the scaling dependence on L and any parameters.

Proof. Following the discussion above, if $u(x,t)$ satisfies (9), then $v(x,t) = u(x,t) - \phi(x)$ satisfies the energy equality (13). The positivity, by hypothesis (16), of $(v,v)_\phi$ (see [24]) and the symmetry of the bilinear form $(v_1,v_2)_\phi$ allow the use of the Cauchy-Schwarz inequality followed by Young's inequality: for any $\epsilon > 0$,

$$\left|(v,\phi)_\phi\right| \le (v,v)_\phi^{1/2}(\phi,\phi)_\phi^{1/2} \le \frac{\epsilon}{2}(v,v)_\phi + \frac{1}{2\epsilon}(\phi,\phi)_\phi \,. \tag{18}$$

Following [9], we choose $\epsilon = 2/3$: substituting into (13), we find

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le -(v,v)_{\phi/2} + \frac{1}{3}(v,v)_\phi + \frac{3}{4}(\phi,\phi)_\phi \tag{19}$$

$$= \frac{2}{3}\int v\left(\mathcal{L} - \frac{1}{4}\phi'\right)v - \frac{3}{4}\int \phi\mathcal{L}\phi \quad = -\frac{2}{3}(v,v)_{\phi/4} + \frac{3}{4}(\phi,\phi)_0 \,, \tag{20}$$

since ϕ is periodic. The fundamental estimate (15) on the quadratic form $(v,v)_{\phi/4}$ now allows us to derive a differential inequality for $\|v\|$,

$$\frac{d}{dt}\|v\|^2 \le -\frac{4}{3}\lambda\|v\|^2 + \frac{3}{2}(\phi,\phi)_0 \,; \tag{21}$$

application of Gronwall's inequality to (21) then yields

$$\|v(\cdot,t)\|^2 \le \|v(\cdot,0)\|^2 e^{-\frac{4}{3}\lambda t} + \frac{9}{8\lambda}(\phi,\phi)_0\left(1 - e^{-\frac{4}{3}\lambda t}\right). \tag{22}$$

Consequently we have the long-time bound

$$\limsup_{t\to\infty}\|v(\cdot,t)\|^2 \le \frac{9}{8\lambda}(\phi,\phi)_0 \,, \tag{23}$$

which implies a bound on $\|u\|^2 = \|v+\phi\|^2 \le 2\|v\|^2 + 2\|\phi\|^2$. Since by assumption $\phi \in \mathcal{A}_{L,2m}$, the comparison function $\phi(x)$ itself satisfies the inequality (15), giving

$$\|\phi\|^2 \le \frac{1}{\lambda}(\phi,\phi)_{\phi/4} = \frac{1}{\lambda}(\phi,\phi)_0 \,. \tag{24}$$

Hence the bound (17) follows, since we have

$$\limsup_{t\to\infty}\|u(\cdot,t)\|^2 \le 2\left(\frac{9}{8}+1\right)\frac{1}{\lambda}(\phi,\phi)_0 = \frac{17}{4}\frac{1}{\lambda}(\phi,\phi)_0 \,. \tag{25}$$

□

The retention of the complete linear operator $\mathcal{L}$ (as in [9]), in addition to giving the somewhat general Theorem 2.1, will permit us to utilize the structure of $\mathcal{L}$ to achieve cancellations and hence better overall parameter-dependent control of the bounds (particularly for the sKS and Nikolaevskiy PDEs discussed in Section 4), compared with estimating each term of $\int v\mathcal{L}\phi$ separately (as in [6]). Observe that the discussion so far has not used any particular properties of $\mathcal{L}$ except for self-adjointness and the estimates (15)–(16).

2.1. **Estimates on the linear operator and general form of the comparison function.** Successful applications of the Lyapunov function method to the Kuramoto-Sivashinsky and related equations [6, 9, 16, 18, 33] have depended on estimates replacing the full linear operator $\mathcal{L}$ with a related operator $\mathcal{L}'$ which dominates $\mathcal{L}$ in the sense that $\int v\mathcal{L}v \leq \int v\mathcal{L}'v$ for all $v \in \mathcal{A}_{L,2m}$, and which is easier to work with.

For example (using the destabilized KS equation (3) for definiteness), by the Cauchy-Schwarz and Young inequalities, for $\sigma > 0$ we have $2\int v_x^2 = -2\int v_{xx}v \leq \sigma\int v_{xx}^2 + \sigma^{-1}\int v^2$, so that

$$-\int v\mathcal{L}_{\mathrm{dKS}}v = \int \left[v_{xx}^2 - 2v_x^2 - \alpha v^2\right] \geq \int \left[(1-\sigma)v_{xx}^2 - \left(\frac{1}{\sigma} + \alpha\right)v^2\right]. \tag{26}$$

This may be written as $\int v\mathcal{L}_{\mathrm{dKS}}v \leq \int v\mathcal{L}'_{\mathrm{dKS}}v$, for

$$\mathcal{L}'_{\mathrm{dKS}} = -(1-\sigma)\,\partial_x^4 + \sigma^{-1} + \alpha, \tag{27}$$

or equivalently, in Fourier space, $\omega_{\mathrm{dKS}}(k) = \alpha + 2k^2 - k^4 \leq \alpha + \sigma^{-1} - (1-\sigma)k^4 \equiv \omega'_{\mathrm{dKS}}(k)$.[1]

Continuing to use the dKS equation to motivate this approach, using (26) we observe that (15) is satisfied for some $\lambda > 0$ and $0 < \sigma < 1$ if

$$-\int v\left(\mathcal{L}'_{\mathrm{dKS}} - \frac{1}{4}\phi'\right)v - \lambda\int v^2 = \int \left[(1-\sigma)v_{xx}^2 + \left(\frac{1}{4}\phi' - \frac{1}{\sigma} - \alpha - \lambda\right)v^2\right] \geq 0. \tag{28}$$

Now this condition is trivially satisfied by choosing a constant $\phi'(x) = \mu \geq 4(\sigma^{-1} + \alpha + \lambda) > 0$. Without further modification this is not a permissible choice of comparison function, however, since the resulting $\phi(x)$, being linear, is not L-periodic; the periodicity constraint requires that $\phi'(x)$ has mean zero. However, as shown in [6, 18], this idea may be adapted to construct a suitable $\phi(x)$ in real space: one lets $\phi'(x)$ be a positive constant throughout most of the domain, and adds a spatially localized correction, having the form of a smoothed-out delta function, which introduces a "jump" in $\phi(x)$ to maintain periodicity. (The Fourier space constructions of $\phi(x)$ of [9, 33] incorporate a similar idea; see [6] for further discussion.) Specifically, the comparison function $\phi(x)$ is constructed on $[-L/2, L/2]$ via its derivative (uniquely, using $\phi(0) = 0$) as

$$\phi'(x) = \mu - q(x), \tag{29}$$

for some suitably chosen constant $\mu > 0$ and spatially localized function $q(x)$, where $q \in C_0^\infty$ has compact support completely contained in $(-L/2, L/2)$, and $\phi'(x)$ is extended to be an L-periodic function. To preserve periodicity, we require that $\int \phi' = 0$, so that μ must be chosen as the spatial average of $q(x)$:

$$\mu = L^{-1}\int q(x). \tag{30}$$

[1]For reference, in [6] bounds on the KS equation (1) are obtained with $\sigma = 1$, using the real space estimate $\int\left[v_{xx}^2 - v_x^2\right] \geq \int\left[\frac{1}{2}v_{xx}^2 - \frac{1}{2}v^2\right]$, while the approach of [9] is equivalent to the same inequality in Fourier space, $k^2 \leq \frac{1}{2}k^4 + \frac{1}{2}$. In [18, 33] the fourth-order operator $\partial_x^2 - \partial_x^4$ is bounded by a second-order operator $\mathcal{L}'$, for instance in [18] via the $\sigma = 1/2$ estimate in the form $\int\left[v_x^2 - \frac{1}{2}v_{xx}^2\right] \leq \int\left[-v_x^2 + 2v^2\right]$; as pointed out in [6], this reduction of order of the operator accounts for the increase in the scaling of the bound on $\limsup_{t\to\infty}\|u\|$ to $\mathcal{O}(L^{5/2})$.

Substituting the Ansatz (29) for ϕ' into (28), we find that $-\int v\left(\mathcal{L}_{\mathrm{dKS}} - \frac{1}{4}\phi'\right)v \geq \lambda\|v\|^2$ (15) is satisfied for a given $\lambda > 0$ provided that for all $v \in \mathcal{A}_{L,4}$,

$$\int \left[(1-\sigma)v_{xx}^2 - \frac{1}{4}q(x)v^2\right] + \left(\frac{1}{4}\,\mu - \frac{1}{\sigma} - \alpha - \lambda\right)\int v^2 \geq 0. \tag{31}$$

An obvious sufficient condition for (31) (which can also be shown to be necessary [6]) is that the two terms are separately positive. Thus to construct a comparison function $\phi(x)$ of the form (29) satisfying (31), we shall choose $\mu \geq 4(\sigma^{-1} + \alpha + \lambda)$; the positivity of the first integral in (31) is then ensured for all (odd) $v \in \mathcal{A}_{L,4}$ by localizing the potential $q(x)$ near $x = 0$, where via $v(0) = 0$ the $\int q(x)v^2$ term is controlled by the higher derivative term $\int v_{xx}^2$. A similar argument pertains to the other requirement (16).

We generalize the above ideas into sufficient conditions for the hypotheses of Theorem 2.1 to be satisfied in the following:

Theorem 2.3. *Given the $2m$-th order real self-adjoint differential operator $\mathcal{L}$, assume that there exist constants ν_1, $\nu_2 > 0$ such that for all $v \in \mathcal{A}_{L,2m}$*

$$\int v\mathcal{L}v \leq \int v\mathcal{L}'v = -\int \nu_1\,(\partial_x^m v)^2 + \int \nu_2\,v^2, \tag{32}$$

where the operator $\mathcal{L}'$ is defined by $\mathcal{L}' = (-1)^{m+1}\nu_1\partial_x^{2m} + \nu_2$. Given a scalar $\lambda > 0$, assume that the even spatially localized function $q(x) \in C_0^\infty$ has the properties that for all $v \in \mathcal{A}_{L,2m}$,

$$\int \left[\nu_1(\partial_x^m v)^2 - q(x)v^2\right] \geq 0 \tag{33}$$

and furthermore, for $\mu = L^{-1}\int q(x)$, that

$$\mu \geq 4\,(\nu_2 + \lambda)\,. \tag{34}$$

Then there exists a universal constant $K \leq 17/4$ such that if $u(x,t) \in \mathcal{A}_{L,2m}$ is a solution of the PDE $u_t = \mathcal{L}u - uu_x$ (9), then

$$\limsup_{t\to\infty} \|u(\cdot,t)\|^2 \leq -\frac{K}{\lambda}\int \phi\,\mathcal{L}\phi, \tag{35}$$

where the L-periodic function $\phi(x)$ is defined via $\phi'(x) = \mu - q(x)$, $\phi(0) = 0$.

Proof. We need only to verify that the hypotheses (15)–(16) of Theorem 2.1 are satisfied; that theorem then immediately implies the conclusion (35). Thus, for a given $q(x)$ with mean μ (30) satisfying (33)–(34), we define the comparison function $\phi(x)$ as above (29). Then from $\int v\mathcal{L}v \leq \int v\mathcal{L}'v$ we have

$$(v,v)_{\gamma\phi} = -\int v\,(\mathcal{L} - \gamma\phi')\,v \geq -\int v\,(\mathcal{L}' - \gamma\phi')\,v \equiv (v,v)'_{\gamma\phi}\,, \tag{36}$$

so that to satisfy (15)–(16) it is sufficient to show that

$$(v,v)'_{\phi/4} \geq \lambda\|v\|^2 \qquad \text{and} \qquad (v,v)'_{\phi} \geq 0. \tag{37}$$

Substituting $\phi' = \mu - q(x)$, conditions (37) are equivalent to

$$(v,v)'_{\phi/4} - \lambda\|v\|^2 = \int \left[\nu_1(\partial_x^m v)^2 - \frac{1}{4}q(x)v^2\right] + \left[\frac{1}{4}\,\mu - \nu_2 - \lambda\right]\int v^2 \geq 0 \tag{38}$$

and

$$(v,v)'_{\phi} = \int \left[\nu_1(\partial_x^m v)^2 - q(x)v^2\right] + \left[\mu - \nu_2\right]\int v^2 \geq 0. \tag{39}$$

Since ν_1, ν_2 and $\lambda > 0$, these inequalities are immediately satisfied whenever (33)–(34) hold, since then we also have $\int \left[\nu_1(\partial_x^m v)^2 - \frac{1}{4}q(x)v^2\right] = \frac{3}{4}\int \nu_1(\partial_x^m v)^2 + \frac{1}{4}\int \left[\nu_1(\partial_x^m v)^2 - q(x)v^2\right] \geq 0$ and $\mu - \nu_2 \geq 4\,(\nu_2 + \lambda) - \nu_2 > 0$. □

In the following, we approach the bounding problem in the reformulated form of Theorem 2.3. The derivation of L^2 bounds for KS-type PDEs of the form (9) thus rests on, first, estimating $\mathcal{L}$ in terms of an operator $\mathcal{L}'$ as in (32); and second, constructing a potential $q(x)$ satisfying the fundamental constraints (33)–(34) (for some $\lambda > 0$), since once such a $q(x)$ is constructed, one may immediately obtain a comparison function $\phi(x)$ and hence derive a bound via (35).

Remark 2.4. The *a priori* bounds on the absorbing ball for KS-like equations derived via Lyapunov function methods are thus, in fact, also bounds on odd solutions of the PDE $u_t = \mathcal{L}'\,u - uu_x$, for which the optimal L-dependence is indeed $\|u\| = \mathcal{O}(L^{3/2})$. Tighter bounds on solutions of the KS equation [17, 35] have exploited the more detailed structure of the KS linear operator $\mathcal{L}_{\mathrm{KS}}$, which is lost as soon as one replaces $\mathcal{L}$ by $\mathcal{L}'$ via Cauchy-Schwarz/Young estimates of the form (26) (or more generally (32)).

3. Optimal parameter-dependent bounds for the destabilized KS equation.

3.1. Comments on the comparison function.

A key insight due to Bronski and Gambill [6], which paved the way to their improvement of the Lyapunov function bound on the L^2 absorbing ball for the KS equation (1) (with $\alpha = 0$) to $\mathcal{O}(L^{3/2})$, was their recognition of the *scaling behavior* of comparison functions $\phi(x)$ of the form (29) with the system size L. Briefly (in the present notation), for the condition (34) $\mu \geq 4(\sigma^{-1} + \lambda)$ to be satisfied as $L \to \infty$, in the light of (30) one needs $\int q(x)$ to grow at least as fast as L; however, for some $\sigma \in (0, 1)$ the condition $\int (1-\sigma)v_{xx}^2 \geq \int q(x)v^2$ should also hold for all (sufficiently large) L. Introducing an argument outlined and generalized below, they demonstrated in [6] that both these conditions could be satisfied if $q(x)$ has the form $q(x) = L^{4/3}\bar{q}(xL^{1/3})$, where $\bar{q}(y)$ is some smooth L-independent "potential" function with compact support in $[-\tilde{\eta}, \tilde{\eta}]$ for some $\tilde{\eta} = \mathcal{O}(1)$; and explicitly constructed a $\bar{q}(y)$ with the desired properties.

The Bronski-Gambill construction of the potential function is outlined in Appendix A. A central feature of this construction is that $\bar{q}(y)$ may be chosen with $\mu = L^{-1}\int q(x) = \int_{-\tilde{\eta}}^{\tilde{\eta}} \bar{q}(y)\,dy$ arbitrarily large; this is achieved through a small parameter δ so that $\int_{-\tilde{\eta}}^{\tilde{\eta}} \bar{q}(y)\,dy \gtrsim \mathcal{O}(\delta^{-1})$ (see Lemma A.4). As pointed out in [6] (see also [5]), this in fact permits one to prove dissipativity and an $\mathcal{O}(L^{3/2})$ bound on the L^2 absorbing ball also for the dKS equation (3), with $\alpha > 0$. However, direct implementation of this approach, letting $q(x) = L^{4/3}\bar{q}(xL^{1/3})$ for a *parameter-dependent* potential $\bar{q}(y)$, yields non-optimal scaling in terms of the parameter α:

Specifically, for the dKS equation we have $\limsup_{t\to\infty} \|u\|^2 \leq -K\lambda^{-1}\int \phi\mathcal{L}_{\mathrm{dKS}}\phi \leq K\lambda^{-1}\|\phi_{xx}\|^2 = K\lambda^{-1}\int q'(x)^2 = KL^3\lambda^{-1}\int_{-\tilde{\eta}}^{\tilde{\eta}} \bar{q}'(y)^2\,dy$ (using (26), (29), and (35)), and by (103) the dependence on δ scales as $\int_{-\tilde{\eta}}^{\tilde{\eta}} \bar{q}'(y)^2\,dy \lesssim \mathcal{O}(\delta^{-5})$. Now in this case condition (34) becomes $\mu \geq 4(\sigma^{-1} + \alpha + \lambda)$, which may be satisfied according to the approach of [5, 6] by choosing $\delta = \mathcal{O}(\alpha^{-1})$ for large α. If in addition we let $\lambda = \mathcal{O}(\alpha)$, the resulting bounds for the dKS equation (3) become $\limsup_{t\to\infty} \|u\|^2 \leq \mathcal{O}(\alpha^4 L^3)$.

Asymptotic and numerical evidence, however, indicates that the viscous shock solutions of the dKS equation (see Fig. 2(c)) scale for large αL as $\|u\| = \mathcal{O}(\alpha L^{3/2})$ [39, 49], as pointed out in the Introduction; so that while the L-dependence of the above bound is optimal, the α-dependence is not.

A main result of the present work is to show how an improved (and in some cases, including that of (3), demonstrably optimal) parameter dependence of the L^2 bounds may be achieved, developing our methods in the context of the dKS equation in the next section: Instead of letting $\bar{q}(y)$ depend on α via δ, we shall introduce a *fixed* function $\widetilde{q}(y)$, and incorporate the full parameter dependence into the scaling of the argument of $q(x)$, and hence $\phi'(x)$, in the same manner in which the L-dependence was treated in [6].

3.2. **Parameter-dependent scaling form of the comparison function.** In the light of the above discussion and motivated by the scaling of viscous shock solutions of the dKS equation (3), to construct a comparison function $\phi(x)$ of the form (29) satisfying the conditions of Theorem 2.3 we generalize the Ansatz of [6], and explicitly isolate the parameter dependence as well as the L-dependence of $q(x)$:

Let β represent some parameter appearing in $\mathcal{L}$, such as a function of α in $\mathcal{L}_{\mathrm{dKS}}$; assuming a power-law dependence, we write

$$q(x) = \beta^{d_2} L^{c_2} \widetilde{q}\left(x\beta^{d_1} L^{c_1}\right), \tag{40}$$

where $\widetilde{q}(y)$ is a fixed C^∞ function, assumed to be even, with positive mean and compact support in $[-\tilde{\eta}, \tilde{\eta}]$ for some $\tilde{\eta} = \mathcal{O}(1)$. Of particular importance is that $\widetilde{q}(y)$ is *independent* of both L and the parameters, so that the entire parameter dependence of $q(x)$ is contained in the scaling Ansatz. We shall verify later that for all L and β of interest, the support of $\widetilde{q}(x\beta^{d_1}L^{c_1})$ lies completely in $(-L/2, L/2)$. (The Ansatz (40) is readily modified for more than one parameter, possibly with different scalings for different parameters.)

Defining also the (L- and parameter-independent) constant

$$\widetilde{\mu} = \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}(y)\, dy > 0, \tag{41}$$

using (30) and (40) we can then find the spatial average μ of $q(x)$ in terms of $\widetilde{\mu}$, L and β. Substituting into (29), the assumed form of the comparison function $\phi(x)$ is

$$\phi'(x) = \beta^{d_2-d_1} L^{c_2-c_1-1}\widetilde{\mu} - \beta^{d_2} L^{c_2} \widetilde{q}\left(x\beta^{d_1} L^{c_1}\right). \tag{42}$$

Scaling exponents for a large parameter: The destabilized KS equation. For the particular case of the dKS equation (3), using the scaling forms of $q(x)$ and μ and substituting $y = x\beta^{d_1}L^{c_1}$ in the integral, we find that (33)–(34) are satisfied provided

$$\beta^{3d_1} L^{3c_1} \int_{-\bar{L}/2}^{\bar{L}/2} \left[(1-\sigma)v_{yy}^2 - \beta^{d_2-4d_1} L^{c_2-4c_1} \widetilde{q}(y) v^2\right] dy \geq 0 \tag{43}$$

for all $v \in \mathcal{A}_{L,4}$ (where $\bar{L} = \beta^{d_1} L^{1+c_1}$), and in addition

$$\beta^{d_2-d_1} L^{c_2-c_1-1}\widetilde{\mu} \geq 4\left(\sigma^{-1} + \alpha + \lambda\right); \tag{44}$$

observe that here all the L-dependence is shown explicitly. Now as discussed in [6], necessary conditions for (43) and (44) to be satisfied as $L \to \infty$ are that $c_2 - 4c_1 \leq 0$ (consider in (43) a function v with support only where $\widetilde{q}(y) \geq 0$) and $c_2 - c_1 - 1 \geq 0$, respectively.

Regarding the parameter dependence, for the dKS equation we wish to obtain bounds uniform in $\alpha \geq 0$ (noting that $\alpha = 0$ corresponds to the KS equation (4)), and optimize their scaling as $\alpha \to \infty$ (for fixed L). Now (44) can only be satisfied for if $\beta^{d_2-d_1}$ is at least $\mathcal{O}(\alpha)$ as $\alpha \to \infty$; assuming that β is a non-decreasing function of α which coincides with α for sufficiently large α, this implies $d_2 - d_1 \geq 1$. Noting from (25) that a larger λ gives a tighter bound, the scaling of the terms in (44) now also allows us to choose $\lambda = \mathcal{O}(\alpha)$ for large α. On the other hand, since $\sigma^{-1} > 1$, for (44) to hold in the KS limit $\alpha \to 0^+$, β should not vanish, as we need $\beta^{d_2-d_1}$ to remain at least $\mathcal{O}(1)$ in this limit.

A formulation satisfying these requirements, leading to appropriate α-dependent scaling as $\alpha \to \infty$ while giving uniform bounds as $\alpha \to 0^+$, is as follows: For some fixed L- and parameter-independent $\tilde{\lambda} = \mathcal{O}(1)$, let

$$\beta = \max\{1, \alpha\}, \qquad \lambda = \beta\,\tilde{\lambda}. \tag{45}$$

Then (44) implies $\beta^{d_2-d_1} L^{c_2-c_1-1} \widetilde{\mu} > 4(1 + \alpha + \beta\,\tilde{\lambda})$, which may be satisfied for all $\beta = \alpha \geq 1$ only if $d_2 - d_1 \geq 1$; while similarly inequality (43) can only be satisfied as $\alpha \to \infty$ (and hence $\beta \to \infty$) if $d_2 - 4d_1 \leq 0$.

In summary, satisfaction of (43)–(44) as $L \to \infty$ and/or $\beta \to \infty$ thus leads to the constraints on the scaling exponents (see [6]):

$$c_2 \geq c_1 + 1, \quad c_2 \leq 4c_1; \qquad d_2 \geq d_1 + 1, \quad d_2 \leq 4d_1. \tag{46}$$

Subject to ensuring (43)–(44), so that the assumptions (33)–(34) of Theorem 2.3 are satisfied, we wish to optimize the L- and β-dependence of the scaling of the *a priori* bound (35), which scales as $\lambda^{-1} (\phi, \phi)_0 = -\lambda^{-1} \int \phi \mathcal{L} \phi$. Now for a comparison function $\phi(x)$ given in the form (42), the scaling properties of L^2 norms of ϕ and its derivatives are derived in Appendix B, from which we can deduce the leading scaling behavior of $(\phi, \phi)_0$.

For the particular case of the destabilized KS equation (3) with $\alpha \geq 0$, using (26), (45) and (96), we find that the bound on $\limsup_{t\to\infty} \|u\|^2$ for a comparison function $\phi(x)$ of the form (42) scales as

$$\begin{aligned} \frac{1}{\lambda}(\phi, \phi)_0 &= \frac{1}{\lambda} \int \left[\phi_{xx}^2 - 2\phi_x^2 - \alpha\phi^2\right] \leq \frac{1}{\lambda}\|\phi_{xx}\|^2 \\ &= \frac{1}{\beta\tilde{\lambda}} \cdot \beta^{2d_2+d_1} L^{2c_2+c_1} \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}'(y)^2\, dy \;=\; \mathcal{O}\left(\beta^{2d_2+d_1-1} L^{2c_2+c_1}\right), \end{aligned} \tag{47}$$

provided $\int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}'(y)^2\, dy < \infty$. The best scaling of the bound as $L \to \infty$ is then obtained at the minimum of $2c_2 + c_1$ [6], while the β-dependent scaling (for $\beta \to \infty$) is optimized upon minimizing $2d_2 + d_1 - 1$, where the scaling exponents satisfy the constraints (46). The solutions of these two (equivalent) constrained minimization problems are

$$c_1 = 1/3, \quad c_2 = 4/3; \qquad d_1 = 1/3, \quad d_2 = 4/3. \tag{48}$$

We thus find that the optimal scaling form for a comparison function $\phi(x)$ of the form (42) for large L and parameter β is given by

$$\phi'(x) = \beta\widetilde{\mu} - (\beta L)^{4/3} \widetilde{q}\left(x(\beta L)^{1/3}\right). \tag{49}$$

For such a $\phi(x)$ to be an admissible comparison function, it remains to verify, given that supp $\widetilde{q}(y) \subset [-\tilde{\eta}, \tilde{\eta}]$, that the support of $\widetilde{q}\left(x(\beta L)^{1/3}\right)$ lies completely in

$(-L/2, L/2)$, or equivalently,

$$\bar{L}/2 = (\beta L)^{1/3} L/2 > \tilde{\eta}. \tag{50}$$

For $L \geq 2\pi$ and $\beta \geq 1$ this condition is certainly satisfied for $\tilde{\eta} \leq \pi$.

Conditions on the potential function $\widetilde{q}(y)$. We may now substitute the scaling exponents appropriate to the dKS equation as derived above into (43)–(44), which become

$$\int_{-\bar{L}/2}^{\bar{L}/2} \left[(1-\sigma)v_{yy}^2 - \widetilde{q}(y)v^2\right] \, dy \geq 0 \tag{51}$$

(where now $\bar{L} = \beta^{1/3} L^{4/3}$) and

$$\beta\widetilde{\mu} \geq 4\left(\sigma^{-1} + \alpha + \beta\,\tilde{\lambda}\right); \tag{52}$$

and we are still free to choose the $\mathcal{O}(1)$ constants σ and $\tilde{\lambda}$. As we are concentrating on the *scaling* of the bounds and are not trying to optimize their $\mathcal{O}(1)$ prefactor, for simplicity we choose $\tilde{\lambda} = 1$ and $\sigma = 1/2$. It is further convenient to use (45) to simplify (52) at no cost to scaling via $\sigma^{-1} + \alpha + \beta\,\tilde{\lambda} = 2 + \alpha + \beta \leq 4\beta$.

With these σ and $\tilde{\lambda}$, we can now finally state sufficient conditions on $\widetilde{q}(y)$ to satisfy the assumptions (33)–(34) of Theorem 2.3: we require that a smooth function $\widetilde{q}$ supported on $[-\tilde{\eta}, \tilde{\eta}]$ is chosen so that for all sufficiently smooth, odd L-periodic v,

$$\int_{-\bar{L}/2}^{\bar{L}/2} \left[\frac{1}{2}v_{yy}^2 - \widetilde{q}(y)v^2\right] \, dy \geq 0 \tag{53}$$

provided also that its mean $\widetilde{\mu}$ satisfies (using (41))

$$\widetilde{\mu} = \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}(y)\, dy \geq 16. \tag{54}$$

It remains to construct a (universal) potential function $\widetilde{q}(y)$ and its integral $\widetilde{\mu} = \int \widetilde{q}(y)\, dy$ satisfying (53) and (54); the construction of such a function $\widetilde{q}(y)$ and verification of its properties is discussed in Appendix A, closely following Bronski and Gambill [6]. We have derived the scaling form for $\phi(x)$ and necessary conditions on $\widetilde{q}(y)$ rather carefully for this problem, as a similar approach is used for the other related PDEs considered. A distinct advantage of the present formulation is that all of the explicit dependence on the system size L and parameter $\alpha \in [0, \infty)$ are taken care of in the scaling Ansatz.

3.3. **Estimates and bounds on the destabilized KS equation.**

In the aforegoing, the scaling exponents for the comparison function, and sufficient conditions on $\widetilde{q}(y)$ to permit the derivation of bounds for the dKS equation (3), were derived. We now summarize the bounding calculation by using the parameter values and scaling forms motivated above, verifying the assumptions of Theorem 2.3 and computing the resulting bounds directly:

From (26)–(27) (with $\sigma = 1/2$) we have $\int v\mathcal{L}_{\text{dKS}}v = -\int \left[v_{xx}^2 - 2v_x^2 - \alpha v^2\right] \leq -\int \frac{1}{2}v_{xx}^2 + \int (2+\alpha)\, v^2$, giving (32) with $\nu_1 = 1/2$, $\nu_2 = 2 + \alpha$.

Let $\widetilde{q}(y)$ be the function defined in Section A.3 following the arguments of Section A.2 [6], having the properties (93)–(94). Now choose any $v \in \mathcal{A}_{L,4}$; we wish to verify (33)–(34): Since v is odd, we can use the Hardy-Rellich inequality (84) of

Section A.1 to bound $\int v_{yy}^2\,dy$ from below by $\int w_y^2\,dy$, where $v = yw$. Since $\widetilde{q}(y)$ is supported on $[-\pi/2, \pi/2]$, using (84) we thus have

$$\int_{-\bar{L}/2}^{\bar{L}/2} \left[\frac{1}{2}v_{yy}^2 - \widetilde{q}(y)v^2\right] dy \geq \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}v_{yy}^2 - \widetilde{q}(y)v^2\right] dy$$
$$\geq \int_{-\pi/2}^{\pi/2} \left[w_y^2 - y^2\widetilde{q}(y)w^2\right] dy \geq 0, \tag{55}$$

where the last inequality is (94), proved in Theorem A.2. Now define $q(x) = (\beta L)^{4/3}\widetilde{q}\left(x(\beta L)^{1/3}\right)$; then using (55)

$$\int_{-L/2}^{L/2} \left[\frac{1}{2}v_{xx}^2 - q(x)v^2\right] dx = \int_{-L/2}^{L/2} \left[\frac{1}{2}v_{xx}^2 - (\beta L)^{4/3}\widetilde{q}\left((\beta L)^{1/3}x\right)v^2\right] dx$$
$$= \beta L \int_{-\bar{L}/2}^{\bar{L}/2} \left[\frac{1}{2}v_{yy}^2 - \widetilde{q}(y)v^2\right] dy \geq 0, \tag{56}$$

showing (33). Furthermore, we have $\mu = L^{-1}\int q(x) = \beta \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}(y)\,dy = \beta\widetilde{\mu} \geq 16\beta$ by (93). Choosing $\lambda = \beta\,\tilde{\lambda} = \beta$, using (45) we thus verify (34) via

$$\nu_2 + \lambda = 2 + \alpha + \beta \leq 4\beta \leq \frac{\mu}{4}. \tag{57}$$

[For completeness we should confirm that functions $v(x,t) = u(x,t) - \phi(x)$, where u solves the dKS equation (3), satisfy the regularity assumptions of the lemmas and estimates used. In fact, it is known [10] that solutions $u(x,t)$ of the KS equation are Gevrey regular and hence analytic for all $t > 0$; this result readily carries over the dKS equation (see [49]). Since the potential $\widetilde{q}(y)$ is by construction a C_0^∞ function (see Lemma A.4), we know that $\phi(x) \in C^\infty$, and hence that $v(x,t)$ is a C^∞ function of x for each $t > 0$.]

We can now apply Theorem 2.3 to obtain an upper bound $K\lambda^{-1}(\phi,\phi)_0$ on the long-time behavior of $\|u\|^2$: specifically, using (25), (35) and (47)–(48), we find (for $\beta = \max\{1,\alpha\}$)

$$\limsup_{t\to\infty} \|u(\cdot,t)\|^2 \leq \frac{17}{4}\left[\int_{-\pi/2}^{\pi/2} \widetilde{q}'(y)^2\,dy\right]\beta^2 L^3, \tag{58}$$

or

$$\limsup_{t\to\infty} \|u(\cdot,t)\| \leq \mathcal{O}(\beta L^{3/2}). \tag{59}$$

In particular, for $\alpha \geq 1$ the bound is $\mathcal{O}(\alpha L^{3/2})$.

Discussion. The results above improve on previous bounds on the L^2 absorbing ball for the dKS equation, namely the estimate $\|u\| \leq \mathcal{O}(\alpha^{8/5}L^{8/5})$ proved in [49] (following the approach of [9]), as well as the scaling, implied in [6], obtained by letting $\delta = \mathcal{O}(\alpha^{-1})$ in the construction of $\bar{q}(y)$, which gives bounds of at best $\mathcal{O}(\alpha^2 L^{3/2})$ (see Section 3.1).

As mentioned in the Introduction, for sufficiently large αL solutions of the dKS equation (3) beginning from arbitrary initial conditions approach a stationary viscous shock solution (see Fig. 2(c)) for which the L^2 norm scales as $\|u\| = \mathcal{O}(\alpha L^{3/2})$. Hence the bound obtained in (59) using the full L- and α-scaling form for $\phi'(x)$ in fact has the *optimal scaling* in both the domain length L and the parameter α as α and/or $L \to \infty$ (although the numerical prefactors are not close to being sharp).

Indeed, the agreement goes beyond the overall L^2 bound: as suggested by asymptotic arguments and confirmed numerically [39, 49], in an outer region the observed stationary solution $u(x)$ of (3) is linear, with $u_x \sim \alpha$, while the width of the transition layer scales as $(\alpha L)^{-1/3}$. That is, the scaling of the comparison function $\phi(x)$ constructed above (see (49)) coincides exactly with that of the attracting viscous shock profile of (3). This raises the intriguing possibility that the viscous shock might itself act as a valid comparison function or even be shown to be attracting [6]; but establishing such results appears to be beyond the reach of present analytical methods.

4. **Bounds on related equations.** We now generalize the bounding approach discussed above to two related KS-like equations, important in applications, which display slightly different features.

4.1. **Stabilized (damped) Kuramoto-Sivashinsky equation.** The stabilized, or damped, KS equation (2) has the same form as the destabilized KS equation (3), with $\varepsilon^2 = \alpha + 1$, but over a different range of parameter values; specifically, we are interested in $\varepsilon^2 \in (0, 1]$ (that is, $\alpha \in (-1, 0]$). As mentioned in Section 1.1, for small ε solutions $u(x,t)$ of (2) are expected to have $\mathcal{O}(\varepsilon)$ amplitude; indeed, for $\varepsilon \ll 1$ one may construct $\mathcal{O}(\varepsilon)$ roll solutions by a weakly nonlinear asymptotic analysis. We shall prove a rigorous bound of the form $\|u\| \leq \mathcal{O}(\varepsilon)$.

The essential difference between this situation and the destabilized KS case treated earlier is that now we have a *small* parameter, and focus on the scaling behavior for $\varepsilon \to 0$; we shall see that the choice of scaling of the optimal comparison function differs for small and large parameters.

Estimates on the differential operator. We begin by choosing a suitable dominating operator $\mathcal{L}'$: Applying (26) with $\alpha = \varepsilon^2 - 1$, we find

$$\int v\mathcal{L}_{\mathrm{sKS}}v = -\int \left[v_{xx}^2 - 2v_x^2 + (1-\varepsilon^2)v^2\right] \leq -\int (1-\sigma)v_{xx}^2 + \int \left(\sigma^{-1} - 1 + \varepsilon^2\right) v^2, \tag{60}$$

giving (32) with $\nu_1 = 1 - \sigma$, $\nu_2 = \sigma^{-1} - 1 + \varepsilon^2$; where $\nu_2 = \max_k[\omega'_{\mathrm{sKS}}(k)] = \omega'_{\mathrm{sKS}}(0) > 0$ is a measure of the instability of the modified operator $\mathcal{L}'_{\mathrm{sKS}} = -\nu_1\partial_x^4 + \nu_2$ used to estimate $\mathcal{L}_{\mathrm{sKS}}$. Now we note that for small ε the dKS operator is weakly unstable, with maximum Fourier growth rate $\max_k[\omega_{\mathrm{sKS}}(k)] = \omega_{\mathrm{sKS}}(1) = \varepsilon^2$. In order to capture the correct ε-dependent scaling of this instability as $\varepsilon \to 0$, we should choose σ so that $\nu_2 = \mathcal{O}(\varepsilon^2)$; this requires that $\sigma^{-1} - 1 = \mathcal{O}(\varepsilon^2)$, or $1 - \sigma = \mathcal{O}(\varepsilon^2)$. We thus let $\sigma = 1 - \chi\varepsilon^2$ for some $\chi = \mathcal{O}(1)$, so $\sigma^{-1} - 1 = \chi\varepsilon^2/(1-\chi\varepsilon^2)$. In order that $\sigma > 0$ uniformly for $0 < \varepsilon^2 \leq 1$, we need $\chi < 1$; for simplicity (and consistency with Section 3.3) we choose $\chi = 1/2$. Thus we have $\sigma = 1 - \frac{1}{2}\varepsilon^2 \geq 1/2$ and hence $\nu_2 = \frac{1}{2}\varepsilon^2/(1-\frac{1}{2}\varepsilon^2) + \varepsilon^2 \leq 2\varepsilon^2$ for $\varepsilon^2 \in (0, 1]$.

In summary, using $\sigma = 1 - \frac{1}{2}\varepsilon^2$ in (60), we can estimate the differential operator in the form (32) (with a slightly weaker ν_2) by

$$\int v\mathcal{L}_{\mathrm{sKS}}v = -\int \left[v_{xx}^2 - 2v_x^2 + (1-\varepsilon^2)v^2\right] \leq -\int \tfrac{1}{2}\varepsilon^2 v_{xx}^2 + \int 2\varepsilon^2 v^2 \tag{61}$$

for $0 < \varepsilon^2 \leq 1$. The conditions (33)–(34) for Theorem 2.3 then become

$$\int \left[\tfrac{1}{2}\varepsilon^2 v_{xx}^2 - q(x)v^2\right] \geq 0 \tag{62}$$

and, for $\mu = L^{-1} \int q(x)$,

$$\mu \geq 4(2\varepsilon^2 + \lambda). \tag{63}$$

Scaling exponents for a small parameter: The stabilized KS equation. As in Section 3.2 we now postulate a scaling Ansatz for $q(x)$ of the form (40); since the L-dependence is exactly as before, we may immediately set $c_1 = 1/3$, $c_2 = 4/3$ as in (48). We thus concentrate on the dependence on β, which again represents a parameter in the PDE, in this case setting $\beta = \varepsilon^2 \leq 1$. The essential difference from the previous case is that we are now interested in the scaling as $\beta \to 0$ rather than $\beta \to \infty$, which modifies the constraints on the scaling exponents:

Substituting $q(x) = \beta^{d_2} L^{4/3} \widetilde{q}\left(x \beta^{d_1} L^{1/3}\right)$ as in (43)–(44), and furthermore allowing λ to scale with β via $\lambda = \beta^{d_3} \tilde{\lambda}$ for some $\tilde{\lambda} = \mathcal{O}(1)$, conditions (62)–(63) become

$$\beta^{3d_1} L \int_{-\bar{L}/2}^{\bar{L}/2} \left[\tfrac{1}{2} \beta v_{yy}^2 - \beta^{d_2 - 4d_1} \widetilde{q}(y) v^2\right] dy \geq 0 \tag{64}$$

and

$$\beta^{d_2 - d_1} \widetilde{\mu} \geq 4\left(2\beta + \beta^{d_3} \tilde{\lambda}\right), \tag{65}$$

where now the entire parameter dependence is explicit. For such $q(x)$, μ and λ, using (96) and (100) the bound (35) on $\limsup_{t\to\infty} \|u\|^2$ for the sKS equation scales as (compare (47))

$$\begin{aligned} \frac{1}{\lambda} (\phi, \phi)_0 &= \frac{1}{\lambda} \int \left[\phi_{xx}^2 - 2\phi_x^2 + (1 - \varepsilon^2)\phi^2\right] \leq \frac{1}{\lambda} \left[\|\phi_{xx}\|^2 + \|\phi\|^2\right] \\ &\leq \frac{1}{\tilde{\lambda}} \beta^{-d_3} \left[\beta^{2d_2 + d_1} L^3 \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}'(y)^2 \, dy + \beta^{2d_2 - 2d_1} L^3 \|\widetilde{q}\|_1^2\right]. \end{aligned} \tag{66}$$

Now for fixed $d_3 \leq 1$, the constraints on d_1 and d_2 for (64)–(65) to be satisfied as $\beta \to 0$ are $d_2 - 4d_1 \geq 1$ and $d_2 - d_1 \leq d_3$ (note the opposite direction of the inequalities compared with (46), as we are dealing with a small rather than large parameter). Since these imply that $4d_1 \leq d_2 - 1 \leq d_1 + d_3 - 1 \leq d_1$, so $d_1 \leq 0$, the $\|\phi_{xx}\|^2$ term dominates the $\|\phi\|^2$ term in (66) for $\beta \leq 1$; and the best scaling of the bound as $\beta \to 0$ is obtained by *maximizing* $2d_2 + d_1 - d_3$ subject to the constraints. For each fixed $d_3 \leq 1$, this maximum is $2d_3 - 1$, at $d_1 = \frac{1}{3}(d_3 - 1)$, $d_2 = \frac{1}{3}(4d_3 - 1)$; now maximizing this over $d_3 \leq 1$, we find that the optimal scaling is $\lambda^{-1} (\phi, \phi)_0 = \mathcal{O}(\beta^1)$ when $d_3 = 1$ (so $d_1 = 0$, $d_2 = 1$). Performing a similar analysis for $d_3 \geq 1$, the constraint from (65) is now $d_2 - d_1 \leq 1$, and the maximum of $2d_2 + d_1 - d_3$ over the allowable set is $2 - d_3$ (at $d_1 = 0$, $d_2 = 1$), maximized at $d_3 = 1$, as before.

For a small parameter β, the best bound is thus obtained for $d_1 = 0$, $d_2 = d_3 = 1$, when all terms in (64)–(65) are $\mathcal{O}(\beta)$; so the optimal scaling form for $\phi(x)$ of the form (40) for large L and small $\beta = \varepsilon^2$ is given by

$$\phi'(x) = \beta\widetilde{\mu} - \beta L^{4/3} \widetilde{q}\left(x L^{1/3}\right) = \varepsilon^2 \left[\widetilde{\mu} - L^{4/3} \widetilde{q}\left(x L^{1/3}\right)\right] = \varepsilon^2 \phi_{\mathrm{KS}}'(x), \tag{67}$$

where $\phi_{\mathrm{KS}}(x)$ is the comparison function for the KS equation as in [6], obtained by setting $\varepsilon^2 = 1$ (or $\alpha = 0$ in Section 3).

Substituting the scaling exponents and choosing again $\tilde{\lambda} = 1$ for simplicity, (64)–(65) reduce to $\int_{-\bar{L}/2}^{\bar{L}/2} \left[\frac{1}{2} v_{yy}^2 - \widetilde{q}(y) v^2\right] dy \geq 0$ and $\widetilde{\mu} \geq 4(2 + \tilde{\lambda}) = 12$, where now $\bar{L} = L^{4/3}$. Comparing with the analogous conditions (53) and (the stronger) (54), we now observe that we can use exactly the same function $\widetilde{q}$ to bound the sKS equation as we used for the dKS equation.

Bounds on the stabilized KS equation. The direct demonstration of bounds for the sKS equation (2) thus proceeds, in brief, as follows: Begin by estimating the linear operator via (61), with $\nu_1 = \frac{1}{2}\varepsilon^2$, $\nu_2 = 2\varepsilon^2$. Choose again $\widetilde{q}(y)$ to be the function defined in Section A.3, and $\widetilde{\mu} = \int_{-\pi/2}^{\pi/2} \widetilde{q}(y)\, dy \geq 16$. Let $q(x) = \varepsilon^2 L^{4/3}\widetilde{q}\left(xL^{1/3}\right)$; then since for any $v \in \mathcal{A}_{L,4}$, (55) holds, we have

$$\int \left[\frac{1}{2}\varepsilon^2 v_{xx}^2 - q(x)v^2\right] dx = \varepsilon^2 L \int_{-\bar{L}/2}^{\bar{L}/2} \left[\frac{1}{2}v_{yy}^2 - \widetilde{q}(y)v^2\right] dy \geq 0, \tag{68}$$

showing (62) and hence (33). With $\mu = L^{-1}\int q(x) = \varepsilon^2 \widetilde{\mu} \geq 16\,\varepsilon^2$, we have similarly

$$\nu_2 + \lambda = 2\varepsilon^2 + \varepsilon^2 \leq \frac{\mu}{4}, \tag{69}$$

giving (63) and consequently (34). Thus we can apply Theorem 2.3, proving a long-time upper bound in L^2 on solutions $u(x,t)$ of the sKS equation (2): Using the scaling exponents derived above with (25), (35) and (66), we have

$$\limsup_{t\to\infty} \|u(\cdot,t)\|^2 \leq \frac{17}{4}\left[\int_{-\pi/2}^{\pi/2} \widetilde{q}'(y)^2\, dy + \|\widetilde{q}\|_1^2\right]\varepsilon^2 L^3, \tag{70}$$

or

$$\limsup_{t\to\infty} \|u(\cdot,t)\| \leq \mathcal{O}(\varepsilon L^{3/2}). \tag{71}$$

Discussion. In the $\varepsilon \to 0$ asymptotic limit, by weakly nonlinear analysis one may readily construct stationary roll solutions $u(x) \sim \varepsilon a \sin(x-\theta) + \text{h.o.t.}$ of the damped KS equation (2); indeed, for small ε these rolls are stable and easily computed (see Fig. 2(b)), and the $\|u\| = \mathcal{O}(\varepsilon)$ scaling may be verified numerically. Our estimate (71) is thus sharp in the parameter ε. (Note however that the L-dependence is non-optimal; solutions of the sKS equation for $\varepsilon^2 \leq 1$ appear to be uniformly bounded independent of L, so that one expects the extensive scaling $\|u\| \sim \mathcal{O}(L^{1/2})$, which has so far eluded proof. The $\mathcal{O}(\varepsilon)$ scaling of the bounds was previously shown in [49], though with a worse L-dependence.)

We remark that an essential ingredient in being able to attain the optimal $\mathcal{O}(\varepsilon)$ scaling was to treat the linear operator $\mathcal{L}_{\mathrm{sKS}}$ as a whole in Theorem 2.1 and to bound the entire operator by $\mathcal{L}'_{\mathrm{sKS}} = \varepsilon^2(-\frac{1}{2}\partial_x^4 + 2)$ (see (61)) for Theorem 2.3, so that ν_1, $\nu_2 = \mathcal{O}(\varepsilon^2)$; if we had instead applied the Cauchy-Schwarz inequality to the terms in $\int v\mathcal{L}_{\mathrm{sKS}}\phi$ separately, the resulting bound on $\|u\|$ would have been at best $\mathcal{O}(1)$.

4.2. **Nikolaevskiy equation.** For the Nikolaevskiy equation (5), as for the stabilized KS equation (2) treated above, we are again interested in small ε solutions; however, that we now have a higher-order PDE affects the scaling behavior. Though one might expect to need a higher-order estimate analogous to that derived in Theorem A.2, it turns out that by exploiting the assumed oddness of the solutions $u(x,t)$ of (5), we are able to obtain bounds using the *same* potential $\widetilde{q}(y)$ as before.

Estimates on the differential operator. As before, we begin by choosing a suitable operator $\mathcal{L}'_{\rm Nik} = \nu_1 \partial_x^6 + \nu_2$ to dominate the Nikolaevskiy differential operator $\mathcal{L}_{\rm Nik} = \partial_x^6 + 2\partial_x^4 + (1-\varepsilon^2)\partial_x^2$, giving a sixth-order analogue of (32) or (61). Since $\mathcal{L}_{\rm Nik}$, like $\mathcal{L}_{\rm sKS}$, is weakly unstable, with a maximum Fourier growth rate of $\max_k[\omega_{\rm Nik}(k)] = \max_k \left[k^2\left(\varepsilon^2 - (1-k^2)^2\right)\right] = \varepsilon^2 + \mathcal{O}(\varepsilon^4)$, we wish to choose $\nu_1, \nu_2 = \mathcal{O}(\varepsilon^2)$. A convenient choice of constants in our estimate is $\nu_1 = \frac{1}{3}\varepsilon^2$, $\nu_2 = \frac{8}{3}\varepsilon^2$, as shown in the following:

Lemma 4.1. *For all $v \in \mathcal{A}_{L,6}$ we have*

$$\int v\mathcal{L}_{Nik} v = -\int \left[v_{xxx}^2 - 2v_{xx}^2 + (1-\varepsilon^2)v_x^2\right] \le -\int \tfrac{1}{3}\varepsilon^2 v_{xxx}^2 + \int \tfrac{8}{3}\varepsilon^2 v^2 = \int v\mathcal{L}'_{Nik} v. \tag{72}$$

Proof. We begin by estimating the destabilizing $\int v_{xx}^2$ term in terms of the others, via $2\int v_{xx}^2 \le \sigma \int v_{xxx}^2 + \sigma^{-1}\int v_x^2$; choosing $\sigma = 1 - \frac{1}{2}\varepsilon^2$ as in Section 4.1, and using $1/(1-\frac{1}{2}\varepsilon^2) \le 2$ for $\varepsilon^2 \le 1$, we get

$$-\int v\mathcal{L}_{\rm Nik} v = \int \left[-v_{xxx}^2 + 2v_{xx}^2 - (1-\varepsilon^2)v_x^2\right] \le -\int \tfrac{1}{2}\varepsilon^2 v_{xxx}^2 + \int 2\varepsilon^2 v_x^2. \tag{73}$$

Now we substitute the interpolation estimate $\int v_x^2 \le \frac{1}{12}\int v_{xxx}^2 + \frac{4}{3}\int v^2$, obtained by combining the inequalities $2\int v_x^2 \le \frac{1}{2}\int v_{xx}^2 + 2\int v^2$ and $2\int v_{xx}^2 \le \frac{1}{2}\int v_{xxx}^2 + 2\int v_x^2$, to give (72). □

Theorem 2.3 with $m = 3$, $\nu_1 = \frac{1}{3}\varepsilon^2$ and $\nu_2 = \frac{8}{3}\varepsilon^2$ now immediately implies a bound on solutions u of the Nikolaevskiy equation (5), provided a comparison function $\phi(x)$ given by $\phi' = \mu - q(x)$ can be found for which $q(x) \in C_0^\infty$ and $\mu = L^{-1}\int q(x)$ satisfy (33)–(34).

Scaling exponents for a 6th-order PDE: The Nikolaevskiy equation. We assume as usual that $q(x)$ has the scaling form (40) for suitable exponents $c_{1,2}$ and $d_{1,2}$. Again the parameter is $\beta = \varepsilon^2 \le 1$, and we are interested in the scaling in the limit $\varepsilon \to 0$; by an analysis similar to that in Section 4.1, we find that the optimal ε-dependence of the bound is obtained for $d_1 = 0$, $d_2 = 1$, and $\lambda = \beta^1\tilde{\lambda} = \varepsilon^2\tilde{\lambda}$. The bound (17) on $\limsup_{t\to\infty}\|u\|^2$ for the Nikolaevskiy equation then scales as (see (47), (66))

$$\begin{aligned}
-\frac{1}{\lambda}\int v\mathcal{L}_{\rm Nik} v &= \frac{1}{\lambda}\int \left[\phi_{xxx}^2 - 2\phi_{xx}^2 + (1-\varepsilon^2)\phi_x^2\right] \le \frac{1}{\lambda}\left[\|\phi_{xxx}\|^2 + \|\phi_x\|^2\right] \\
&\le \frac{1}{\varepsilon^2\tilde{\lambda}}\left[\varepsilon^4 L^{2c_2+3c_1}\int_{-\tilde{\eta}}^{\tilde{\eta}} \tilde{q}''(y)^2\,dy + \varepsilon^4 L^{2c_2-c_1}\int_{-\tilde{\eta}}^{\tilde{\eta}} \tilde{q}(y)^2\,dy\right] \\
&= \mathcal{O}\left(\varepsilon^2 L^{2c_2+3c_1}\right),
\end{aligned} \tag{74}$$

where we used (95) and (97).

Now the exponents c_1 and c_2 describing the L-dependence depend on the order of the differential operator, and thus differ from those found previously for the fourth-order KS case, but can be found straightforwardly via calculations analogous to those of Section 3.2: the best scaling of the bound as $L \to \infty$ is found upon minimizing $2c_2 + 3c_1$ subject to $c_2 - 6c_1 \le 0$ and $c_2 - c_1 - 1 \ge 0$, yielding a minimum of 3 for $c_1 = 1/5$, $c_2 = 6/5$. The optimal scaling form for $\phi(x)$ is thus

$$\phi'(x) = \mu - q(x) = \varepsilon^2\left[\tilde{\mu} - L^{6/5}\tilde{q}\left(xL^{1/5}\right)\right]; \tag{75}$$

substituting, conditions (33)–(34) reduce to $\int_{-\bar{L}/2}^{\bar{L}/2}\left[\frac{1}{3}v_{yyy}^2-\widetilde{q}(y)v^2\right]\,dy\geq 0$ with $\bar{L}=L^{6/5}$, and $\widetilde{\mu}\geq 4\left(\frac{8}{3}+\tilde{\lambda}\right)=16$, where we choose $\tilde{\lambda}=4/3$ (compare (54)). As shown below (exploiting the oddness of v), our previously constructed potential function $\widetilde{q}(y)$ satisfies these constraints, so it turns out that there is no need to construct a new potential function for this higher-order problem.

Bounds on the Nikolaevskiy equation. Using the scaling form derived above, we now directly derive long-time L^2 bounds on solutions $u(x,t)$ of the sixth-order Nikolaevskiy PDE (5):

Begin by bounding the linear operator $\mathcal{L}_{\mathrm{Nik}}$ using (72); we then need to satisfy (33)–(34), so that the conclusion (35) of Theorem 2.3 implies a bound scaling as (74). Choose $\widetilde{q}(y)$ to be the smooth, compactly supported function defined in Section A.3, and let $q(x)=\varepsilon^2L^{6/5}\widetilde{q}\left(xL^{1/5}\right)$. Rescaling and using the higher-order Hardy-Rellich type inequality (81), for each $v\in\mathcal{A}_{L,6}$ we have

$$\begin{aligned}\int\left[\tfrac{1}{3}\varepsilon^2v_{xxx}^2-q(x)v^2\right]&\geq\varepsilon^2L\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{3}v_{yyy}^2-\widetilde{q}(y)v^2\right]\,dy\\&\geq\varepsilon^2L\int_{-\pi/2}^{\pi/2}\left[2w_{yy}^2-y^2\widetilde{q}(y)w^2\right]\,dy.\end{aligned}\tag{76}$$

Since here $w(y)=v(y)/y$ is even, so that $w_y(0)=0$, we now bound $\int_{-\pi/2}^{\pi/2}w_{yy}^2\,dy$ from below via a Poincaré inequality: for $y\in[0,\pi/2]$ we have $w_y(y)=\int_0^y w_{y'y'}\,dy'\leq y^{1/2}\left(\int_0^{\pi/2}w_{y'y'}^2\,dy'\right)^{1/2}$, and hence $\int_0^{\pi/2}w_y^2\,dy\leq(\pi^2/8)\int_0^{\pi/2}w_{y'y'}^2\,dy'$; and similarly on $[-\pi/2,0]$. Thus

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\left[2w_{yy}^2-y^2\widetilde{q}(y)w^2\right]\,dy&\geq\int_{-\pi/2}^{\pi/2}\left[\frac{16}{\pi^2}w_y^2-y^2\widetilde{q}(y)w^2\right]\,dy\\&\geq\int_{-\pi/2}^{\pi/2}\left[w_y^2-y^2\widetilde{q}(y)w^2\right]\,dy\geq 0,\end{aligned}\tag{77}$$

where the last inequality is (94); combining (76) and (77), we have shown (33) in this case. Since using (93) we have $\mu=L^{-1}\int q(x)=\varepsilon^2\widetilde{\mu}\geq 16\varepsilon^2$, with $\lambda=\frac{4}{3}\varepsilon^2$ we have $\frac{8}{3}\varepsilon^2+\lambda=4\varepsilon^2\leq\mu/4$, showing (34). Hence the conditions of Theorem 2.3 are satisfied, and we have proved a long-time L^2 upper bound on solutions of the Nikolaevskiy equation, of the form (using (35) and (74) with $c_1=1/5$, $c_2=6/5$)

$$\limsup_{t\to\infty}\|u(\cdot,t)\|^2\leq\frac{51}{16}\left[\int_{-\tilde{\eta}}^{\tilde{\eta}}\widetilde{q}''(y)^2\,dy+L^{-4/5}\int_{-\tilde{\eta}}^{\tilde{\eta}}\widetilde{q}(y)^2\,dy\right]\varepsilon^2L^3,\tag{78}$$

or

$$\limsup_{t\to\infty}\|u(\cdot,t)\|\leq\mathcal{O}\left(\varepsilon L^{3/2}\right).\tag{79}$$

Discussion. The existence of an absorbing ball for the Nikolaevskiy equation (5) (order $2m=6$) has previously been proved in [16], as a special case of a more general result for a class of pseudo-differential equations of arbitrary even order, with a bound scaling as $\|u\|\leq\mathcal{O}(L^{15/2})$; the above result (79) improves on this estimate and incorporates parameter dependence.

As in the case of the sKS equation (2), a weakly nonlinear analysis demonstrates that the Nikolaevskiy equation with $\varepsilon^2 \ll 1$ supports stationary periodic roll solutions of amplitude $\mathcal{O}(\varepsilon)$ (although they are unstable as solutions of (5) for all $\varepsilon > 0$ [11, 29, 47]). The $\|u\| = \mathcal{O}(\varepsilon L^{1/2})$ scaling of these rolls indicates that the ε-dependence proved in (79) is again optimal (although the L-dependence is unlikely to be so).

Remark 4.2. For the stabilized KS and Nikolaevskiy equations (2), (5) we may also consider the case $\varepsilon^2 = 0$, so that by the dispersion relation the trivial solution $u \equiv 0$ is linearly *marginally stable* (Fourier modes with $|k| = 1$ have zero growth rate). A corollary of our results (71), (79) is that for $\varepsilon^2 = 0$ the trivial solution is in fact *nonlinearly stable*, $u(x,t) \to 0$ as $t \to \infty$; since our proofs can readily be adapted to show that $\limsup_{t\to\infty} \|u\| \leq K\varepsilon' L^{3/2}$ for *all* $\varepsilon' > 0$, and hence necessarily $\limsup_{t\to\infty} \|u\| = 0$.

5. Conclusions.

While it gives the optimal scaling for the destabilized Kuramoto-Sivashinsky equation, a system dominated by large-scale driving, the Lyapunov function approach is often insufficiently sensitive to the details of the linear operator to give sharp bounds with domain size L; for instance, it fails to capture the extensive scaling of the Kuramoto-Sivashinsky equation, and that presumed to hold for some related PDEs such as the Nikolaevskiy equation. We have shown, however, that this Lyapunov function bounding approach may be adapted to yield estimates whose dependence on system parameters (other than L) is in many cases demonstrably sharp; and anticipate that these methods may prove fruitful elsewhere.

Acknowledgments. I would like to thank Razvan Fetecau for helpful discussions and comments. This research was partially supported by the National Science and Engineering Research Council of Canada (NSERC).

Appendix A. The Bronski-Gambill construction.

Fundamental to the derivation by Bronski and Gambill [6] of an $\mathcal{O}(L^3)$ bound for the KS equation, the best possible by Lyapunov function methods, is the construction of a comparison function $\phi(x)$ of the form (29). As shown above for slightly more general situations, after appropriate rescalings the problem reduces to finding a potential function $\widetilde{q}(y)$ with sufficiently large mean (54) for which an estimate such as (53) holds. In this Appendix we review the essential features of their construction [6] of such a $\widetilde{q}(y)$, with slight modifications and extensions as needed for our results.

A.1. Hardy-Rellich inequality for reduction of order.

A crucial step in [6] in satisfying an estimate like (53) for a Schrödinger-type operator was the realization that, via a Hardy-Rellich type inequality, the order of the differential operator may be reduced by one at the cost of sacrificing a Dirichlet boundary condition. We need this result [6, Lemma 3] and a straightforward higher-order generalization:

Lemma A.1. *For each $b > 0$, suppose that $v \in C^4[0,b]$ with $v(0) = 0$. Then if we define $w(y) = v(y)/y$, we have the inequalities*

$$\int_0^b \frac{1}{2} v_{yy}^2 \, dy \geq \int_0^b \left(\frac{v(y)}{y}\right)_y^2 dy = \int_0^b w_y^2 \, dy \tag{80}$$

and

$$\int_0^b \frac{1}{2} v_{yyy}^2 \, dy \geq 3 \int_0^b \left(\frac{v(y)}{y}\right)_{yy}^2 dy = 3 \int_0^b w_{yy}^2 \, dy. \tag{81}$$

Proof. The conditions on the function $v(y)$ ensure that $w(y) \in C^3[0,b]$. Then with $v(y) = yw(y)$, we have $v_{yy} = yw_{yy} + 2w_y$ and $v_{yyy} = yw_{yyy} + 3w_{yy}$, so that (integrating by parts)

$$\int_0^b v_{yy}^2\,dy = \int_0^b \left(yw_{yy} + 2w_y\right)^2\,dy = \int_0^b y^2 w_{yy}^2\,dy + \int_0^b 2w_y^2\,dy + 2bw_y^2(b) \\ \geq 2\int_0^b w_y^2\,dy, \tag{82}$$

and similarly

$$\int_0^b v_{yyy}^2\,dy = \int_0^b y^2 w_{yyy}^2\,dy + \int_0^b 6w_{yy}^2\,dy + 3bw_{yy}^2(b) \geq 6\int_0^b w_{yy}^2\,dy, \tag{83}$$

which imply (80) and (81), respectively. □

In particular, for any *odd* $v \in C^4[-b,b]$, the conditions at $y = 0$ are automatically satisfied, the above lemma holds also on $[-b,0]$, and we have

$$\int_{-b}^b \frac{1}{2}v_{yy}^2\,dy \geq \int_{-b}^b w_y^2\,dy, \qquad \int_{-b}^b \frac{1}{2}v_{yyy}^2\,dy \geq 3\int_{-b}^b w_{yy}^2\,dy, \tag{84}$$

where $w(y) = v(y)/y$ is *even*. We comment that while the smoothness conditions on v may be weakened, the hypotheses of the above lemma are sufficient for our purposes, due to the analyticity for $t > 0$ of solutions of KS-like equations [10, 49] and the smoothness of the potentials constructed in Theorem A.2 below.

A.2. **The potential function.** Closely following Bronski and Gambill [6] (with some minor modifications) we next show that a general class of functions $\bar{q}(y)$ satisfying estimates of the form (53)–(54) may be constructed in real space. For definiteness, in Appendix A.3 we then specify the arbitrary parameters to obtain a *fixed* $\widetilde{q}(y)$ which is used to obtain the bounds for all the PDEs (2)–(5).

Theorem A.2. *For each constant $b > 0$ and constant $\zeta > 0$, there exists a C^∞ function $\bar{q}(y)$, supported on $[-b,b]$, with the properties that*

(1) $\int_{-b}^b \bar{q}(y)\,dy > \zeta$ *and*
(2) *For each $w(y) \in C^2[-b,b]$, $\int_{-b}^b w_y^2 - y^2\bar{q}(y)w^2\,dy \geq 0$.*

First we observe that we can choose $\bar{q}(y)$ to be even, so that the above result follows immediately if we ensure the inequality in (2) on $[0,b]$, with $\int_0^b \bar{q}(y)\,dy > \zeta/2$.

The function $\bar{q}(y)$ is then obtained by constructing first the function $\bar{Q}(y) = y^2\bar{q}(y)$; note that we need $\int \bar{q}(y)\,dy = \int \bar{Q}(y)/y^2\,dy$ to be positive (and, in fact, arbitrarily large), while $\bar{q}(y)$ should be smooth, so that $\bar{Q}(y)$ should vanish (at least) quadratically at $y = 0$.

$\bar{Q}(y)$ is constructed in two parts:

- Define a piecewise constant function $Q_c(y)$ satisfying (2), for which $Q_c(y) > 0$ for small $|y|$.
- Construct $\bar{Q}(y)$ as a mollified approximation of $Q_c(y)$ to satisfy both conditions (1) and (2) in Theorem A.2.

Specifically, we have:

Lemma A.3. *For each $a > 0$, define the piecewise constant function $Q_c(y)$ for $y \geq 0$ as follows:*

$$Q_c(y) = \begin{cases} q_0 & \text{for } 0 \leq y \leq \frac{a}{2}, \\ -q_1 & \text{for } \frac{a}{2} < y \leq a, \\ 0 & \text{for } y > a, \end{cases} \tag{85}$$

where q_0 and q_1 are positive constants satisfying the inequalities

$$\frac{1}{2} q_0 a^2 < 1, \qquad q_1 > \frac{q_0}{1 - \frac{1}{2} q_0 a^2} \ . \tag{86}$$

Then we have that for all $b \geq a$ and all $w \in C^1[0, b]$,

$$\int_0^b w_y^2 - Q_c(y) w^2 \, dy \geq 0. \tag{87}$$

Proof. See [6, Lemma 4], with trivial modifications to account for an additional factor of 2. □

Lemma A.4. *For each $\zeta > 0$ and $a > 0$, and for $\delta < a/4$ sufficiently small, there exists a function $\bar{Q}(y)$ such that for all $b \geq a + \delta$,*

(1) $\bar{q}(y) = \bar{Q}(y)/y^2 \in C_0^\infty$,
(2) $\int_0^b w_y^2 - \bar{Q}(y) w^2 \, dy \geq 0$ *for all $w \in C^1[0, b]$,*
(3) $\int_0^b \bar{q}(y) \geq \zeta/2$.

Proof. Following [6, Lemma 5], such a function $\bar{Q}(y)$ may be constructed as a mollification of the piecewise constant $Q_c(y)$ obtained in Lemma A.3, in such a way as to ensure that $\bar{q}(y) = \bar{Q}(y)/y^2$ is smooth and satisfies estimate (3) above.

Choose $f(y)$ to be a nondecreasing C^∞ function such that

$$f(y) \equiv 0, \quad y \leq 0, \qquad f(y) \equiv 1, \quad y \geq 1. \tag{88}$$

For the given $a > 0$, let $q_0 > 0$ and $q_1 > 0$ satisfy (86) as in Lemma A.3; then using this mollifier, for $\delta < a/4$ define $\bar{Q}(y)$ for $y \geq 0$ by

$$\bar{Q}(y) = \begin{cases} q_0 f\left(\frac{y}{\delta}\right) & \text{for } 0 \leq y \leq \delta, \\ q_0 & \text{for } \delta \leq y \leq \frac{a}{2} - \delta, \\ q_0 - (q_0 + q_1) f\left(\frac{y - (a/2 - \delta)}{\delta}\right) & \text{for } \frac{a}{2} - \delta \leq y \leq \frac{a}{2}, \\ -q_1 & \text{for } \frac{a}{2} \leq y \leq a, \\ -q_1 f\left(\frac{(a+\delta) - y}{\delta}\right) & \text{for } a \leq y \leq a + \delta. \end{cases} \tag{89}$$

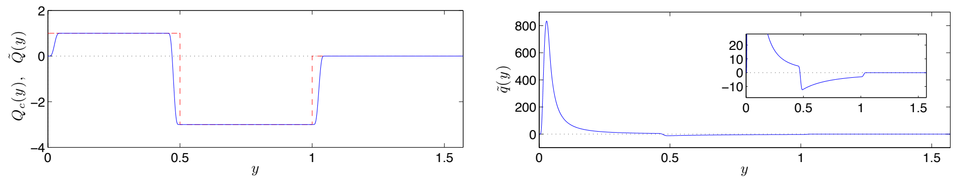
Since $\bar{Q}(y)$ is supported on $[0, a+\delta]$, and by construction $\bar{Q}(y) \leq Q_c(y)$ on $[0, a+\delta]$, (87) from Lemma A.3 implies that for $b \geq a + \delta$,

$$\int_0^b w_y^2 - \bar{Q}(y) w^2 \, dy \geq \int_0^{a+\delta} w_y^2 - Q_c(y) w^2 \, dy \geq 0. \tag{90}$$

Also, from the properties of f we clearly have that $\bar{q}(y) = \bar{Q}(y)/y^2 \in C_0^\infty$, and $\bar{q}(0) = 0$.

Lastly, using $\bar{Q}(y) \geq 0$ on $[0, \delta]$, $\bar{Q}(y) \geq -q_1$ on $[a/2 - \delta, a + \delta]$, we compute

$$\int_0^b \bar{q}(y) \, dy = \int_0^{a+\delta} \frac{\bar{Q}(y)}{y^2} \, dy \geq \frac{q_0}{\delta} \frac{a - 4\delta}{a - 2\delta} - q_1 \frac{a + 4\delta}{(a+\delta)(a - 2\delta)}. \tag{91}$$



(a) (b)

FIGURE 3. (a) Piecewise constant function $\widetilde{Q}_c(y)$ defined in Lemma A.3 (see (85)) with $a = 1$, $q_0 = 1$, $q_1 = 3$ (dashed line); and its mollification $\widetilde{Q}(y)$ from Lemma A.4 (solid line), using $\delta = 1/25$ and the smoothing function $f(y)$ defined in the text. (b) The even potential function $\widetilde{q}(y) = \widetilde{Q}(y)/y^2$, shown for $y \geq 0$; the values near $\widetilde{q} = 0$ are magnified in the inset.

In order to show that $\int_0^b \bar{q}(y)\,dy$ may be made arbitrarily large, it is convenient to simplify this expression; for instance, one may choose δ small enough that by straightforward estimates (91) implies, say,

$$\int_0^b \bar{q}(y)\,dy \geq \frac{1}{2}\frac{q_0}{\delta} - \frac{3}{2a}q_1 \tag{92}$$

(here $\delta \leq a/16$ is sufficient). For fixed $a, q_0, q_1 > 0$, by choosing $\delta < a/4$ sufficiently small we may thus ensure $\int_0^b \bar{q}(y)\,dy \geq \zeta/2$ for any $\zeta > 0$. □

To complete the proof of Theorem A.2, for any fixed $b > 0$ and $\zeta > 0$ we choose $a > 0$ so that $a \leq 4b/5$. Choosing sufficiently small $\delta < a/4$ and defining $\bar{q}(y)$ on $[0, a+\delta]$ as in Lemma A.4, we now extend the definition of $\bar{q}(y)$ from Lemma A.4 to be an *even* function supported on $[-(a+\delta), a+\delta] \subset [-b, b]$. The above arguments and lemmas then hold also on $[-b, 0]$, so that the function $\bar{q}(y)$ has the properties stated in the theorem. □

A.3. **Specification of a fixed potential $\widetilde{q}(y)$.** In the above we have constructed a family of functions $\bar{q}(y)$ (and shown it to be nonempty) whose members satisfy the conditions needed to define comparison functions $\phi(x)$ (42) and prove *a priori* bounds of the form (17). For definiteness, we now specify a *particular* member $\widetilde{q}(y)$ from this family, which we use for all our bounds. We have not attempted to optimize $\widetilde{q}$-dependent constants, which depend on our (somewhat arbitrary) choices of a, q_0, q_1, δ, $\tilde{\lambda}$ and the smoothing function $f(y)$.

We begin by (arbitrarily) choosing $a = 1$; then the conditions (86) are satisfied with the choices $q_0 = 1$, $q_1 = 3$, thereby fixing the piecewise constant function $\widetilde{Q}_c(y)$ defined by (85); see Fig. 3(a). To obtain the mollification $\widetilde{Q}(y)$ of $\widetilde{Q}_c(y)$, we need to select a particular smoothing function $f \in C^\infty$ satisfying (88); the one we shall use is $f(y) = G^{-1}\int_0^y g(s)\,ds$, where $g \in C_0^\infty$ is defined as $g(y) = \mathrm{e}^{-1/y(1-y)}$ for $0 < y < 1$, $g = 0$ elsewhere, and $G = \int_0^1 g(s)\,ds$. The choice $\delta = 1/25 < a/4$, motivated below, completes via (89) the definition of $\widetilde{Q}(y)$, also shown in Fig. 3(a).

The fixed potential function $\widetilde{q}(y)$ is then an *even* function defined for $y \geq 0$ by $\widetilde{q}(y) = y^{-2}\widetilde{Q}(y) \in C^\infty$. It follows that $\widetilde{q}(y)$ is supported on $[-\tilde{\eta}, \tilde{\eta}]$ for any $\tilde{\eta} \geq a+\delta$; for convenience we let $\tilde{\eta} = \pi/2$. The value of δ is chosen for $\widetilde{q}$ to have sufficiently large integral; it suffices to have (54)

$$\widetilde{\mu} = \int_{-b}^{b} \widetilde{q}(y)\,dy = \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}(y)\,dy \geq 16 \tag{93}$$

for any $b \geq \tilde{\eta} = \pi/2$. Now by (92) we are guaranteed (for $\delta \leq a/16$) that $\widetilde{\mu} = 2\int_0^{\tilde{\eta}} \widetilde{q}(y)\,dy \geq \delta^{-1}q_0 - 3q_1/a = \delta^{-1} - 9$, which is at least as large as an arbitrary $\zeta > 0$ for $\delta \leq (\zeta+9)^{-1}$; thus to ensure (93), $\delta = 1/25$ is sufficient. We recall that by construction (Lemmas A.1–A.4), for any odd function $v \in C^4[-b,b]$ we have (with $w(y) = v(y)/y$)

$$\int_{-b}^{b} \frac{1}{2}v_{yy}^2 - \widetilde{q}(y)v^2\,dy \geq \int_{-b}^{b} w_y^2 - y^2\widetilde{q}(y)w^2\,dy = 2\int_0^b w_y^2 - \widetilde{Q}(y)w^2\,dy \geq 0. \tag{94}$$

The potential function $\widetilde{q}(y)$ is shown (for $y \geq 0$) in Fig. 3(b).

Appendix B. Scaling of norms of the comparison function. For a comparison function $\phi(x)$ taking the scaling form (42), where the even function $\widetilde{q}(y) \in C_0^\infty[-\tilde{\eta},\tilde{\eta}]$ with integral $\widetilde{\mu}$ is parameter- and L-independent, we may readily obtain elementary estimates on the scaling of the norms of ϕ:

Using (29), (41), (42) and the substitution $y = x\beta^{d_1}L^{c_1}$, and assuming $\bar{L} = \beta^{d_1}L^{c_1+1} > 2\tilde{\eta}$, we compute

$$\begin{aligned}\|\phi_x\|^2 &= \int (\mu - q(x))^2\,dx = \int q(x)^2\,dx - \mu^2 L \\ &= \beta^{2d_2-d_1}L^{2c_2-c_1}\left[\int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}(y)^2\,dy - \beta^{-d_1}L^{-c_1-1}\widetilde{\mu}^2\right],\end{aligned} \tag{95}$$

while for higher derivatives,

$$\|\phi_{xx}\|^2 = \int \left(-\frac{d}{dx}q(x)\right)^2 dx = \beta^{2d_2+d_1}L^{2c_2+c_1}\int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}'(y)^2\,dy \tag{96}$$

and

$$\|\phi_{xxx}\|^2 = \int \left(-\frac{d^2}{dx^2}q(x)\right)^2 dx = \beta^{2d_2+3d_1}L^{2c_2+3c_1}\int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}''(y)^2\,dy. \tag{97}$$

Furthermore, since $\phi(0) = 0$, we have

$$\begin{aligned}\phi(x) &= \int_0^x \phi'(s)\,ds = \mu x - \int_0^x q(s)\,ds \\ &= \beta^{d_2-d_1}L^{c_2-c_1-1}\widetilde{\mu}x - \beta^{d_2-d_1}L^{c_2-c_1}\int_0^{xL^{c_1}\beta^{d_1}} \widetilde{q}(y)\,dy \\ &\leq \beta^{d_2-d_1}L^{c_2-c_1}\left[L^{-1}\widetilde{\mu}|x| + \int_0^{|x|L^{c_1}\beta^{d_1}} |\widetilde{q}(y)|\,dy\right],\end{aligned} \tag{98}$$

which leads to the simple estimate for $x \in [-L/2, L/2]$

$$|\phi(x)| \leq \beta^{d_2-d_1}L^{c_2-c_1}\left[L^{-1}\widetilde{\mu}\frac{L}{2} + \int_0^{\tilde{\eta}} |\widetilde{q}(y)|\,dy\right] = \beta^{d_2-d_1}L^{c_2-c_1}\left(\frac{1}{2}\widetilde{\mu} + \frac{1}{2}\|\widetilde{q}\|_1\right), \tag{99}$$

since $\widetilde{q}$ is supported on $[-\tilde{\eta}, \tilde{\eta}]$. Hence, since by (41) we have $\widetilde{\mu} \leq \int_{-\tilde{\eta}}^{\tilde{\eta}} |\widetilde{q}(y)|\, dy = \|\widetilde{q}\|_1$, we obtain

$$\|\phi\|^2 \leq \int \|\phi\|_\infty^2 \, dx \leq \beta^{2d_2-2d_1} L^{2c_2-2c_1+1} \|\widetilde{q}\|_1^2. \tag{100}$$

The above bounds on the L^2 norms of ϕ and its derivatives may for each linear operator $\mathcal{L}$ now be used to estimate the scaling of the quadratic form (14) $(\phi, \phi)_0 = -\int \phi \mathcal{L} \phi$, which provides the overall bound (17) on the absorbing ball for $\|u\|$.

The bounds for the nonlocal KS equation discussed in Appendix C also depend on the scaling of $\|x\phi_x\|^2$, for which we can estimate (following [5], and working on the domain $[-L, L]$)

$$\begin{aligned} \|x\phi_x\|^2 &= \int_{-L}^{L} x^2 (\mu - q(x))^2 \, dx \leq 2 \int_{-L}^{L} x^2 \mu^2 \, dx + 2 \int_{-L}^{L} x^2 q(x)^2 \, dx \\ &= \frac{4}{3} \beta^{2(d_2-d_1)} L^{2(c_2-c_1-1)+3} \widetilde{\mu}^2 + 2\beta^{2d_2-3d_1} L^{2c_2-3c_1} \int_{-\tilde{\eta}}^{\tilde{\eta}} y^2 \widetilde{q}(y)^2 \, dy. \end{aligned} \tag{101}$$

It is also instructive to consider the dependence of the norms of the potential and its derivatives on the parameters q_0, q_1 and δ introduced in the construction of $\bar{q}(y) = \bar{Q}(y)/y^2$ in Theorem A.2 (assuming a fixed mollifier $f \in C^\infty$ as introduced in (88)). Straightforward scaling calculations based on (89) show that to leading order, we may estimate

$$\|\widetilde{q}\|_1 = 2 \int_0^{\tilde{\eta}} |\widetilde{q}(y)| \, dy \leq 2 \frac{q_0}{\delta} \left[\int_0^1 \frac{f(u)}{u^2} \, du + 1 \right] + \frac{4}{a}(q_1 - q_0) \quad \Longrightarrow \quad \|\widetilde{q}\|_1^2 \leq \mathcal{O}\left(\frac{q_0^2}{\delta^2} \right), \tag{102}$$

and similarly

$$\int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}(y)^2 \, dy \leq \mathcal{O}\left(\frac{q_0^2}{\delta^3} \right), \quad \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}'(y)^2 \, dy \leq \mathcal{O}\left(\frac{q_0^2}{\delta^5} \right), \quad \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}''(y)^2 \, dy \leq \mathcal{O}\left(\frac{q_0^2}{\delta^7} \right). \tag{103}$$

Appendix C. A note on a nonlocal Kuramoto-Sivashinsky equation.

Introduction. In [20], Hilhorst, Peletier, Rotariu and Sivashinsky studied a nonlocal fourth-order PDE of the form

$$u_t = -u_{xxxx} - u_{xx} - u u_x + 2\pi\kappa I[u] - \alpha(x u_x + 2u), \tag{104}$$

where u satisfies the homogeneous Dirichlet boundary conditions $u = u_{xx} = 0$ at $x = \pm L$,[2] and the nonlocal operator I is defined by

$$\pi I[u] = \sum_n \left[\frac{n\pi}{L} a_n(t) \sin\left(\frac{n\pi x}{L} \right) + \frac{(2n+1)\pi}{2L} b_n(t) \cos\left(\frac{(2n+1)\pi x}{2L} \right) \right], \tag{105}$$

where

$$a_n(t) = \frac{1}{L} \int_{-L}^{L} u(y,t) \sin\left(\frac{n\pi y}{L} \right) dy \,, \quad b_n(t) = \frac{1}{L} \int_{-L}^{L} u(y,t) \cos\left(\frac{(2n+1)\pi y}{2L} \right) dy \,; \tag{106}$$

note the identity $\|\pi I[u]\|^2 = \|u_x\|^2 \equiv \int_{-L}^{L} u_x^2 \, dx$. This equation models the evolution of a diffusively destabilized flame front subject also to hydrodynamic instabilities

[2] In this Appendix we work on a domain of length $2L$ for consistency with [5, 20].

due to the effects of thermal expansion, modelled by $2\pi\kappa I[u]$, and which is stabilized by fluid stretching in a stagnation point flow, represented by the $\alpha(xu_x + 2u)$ term.

Hilhorst *et al.* [20] showed that if the stabilization coefficient α is sufficiently large relatively to the destabilization κ, $\alpha > \alpha_0(\kappa)$, then the trivial solution $u \equiv 0$ is the unique global attractor; they estimated $\alpha_0 = \frac{2}{3}(1+\kappa^2)$. Furthermore, for *odd* solutions, under the assumption $\kappa < 3^{-7/4}(1+3\alpha)^{3/4}$ they were able to prove the existence of an L^2 absorbing ball for solutions of (104)–(105), of the form $\|u\| \leq CL^{11/5}$ (note that in this case $b_n(t) = 0$ in (106)), largely using the Fourier space methods of Collet *et al.* [9] to construct their comparison function.

Using the real space techniques of [6], Bronski, Fetecau and Gambill [5] were able to improve the results of [20]. Specifically, the constraint on κ and the restriction to odd solutions were removed, and the dependence on the parameters α and κ was considered explicitly: the resulting bounds of [5] took the form $\|u\| \leq C(\kappa,\alpha)L^{3/2}$ for odd solutions, and $\|u\| \leq C(\kappa,\alpha)L^{13/6}$ for general initial data.

In this Appendix, we outline how the parameter dependence of the results of [5, 20] may be further improved, using some of the approaches developed in Sections 2–4.

C.1. **General bounding principle.** We introduce a comparison function $\phi(x)$ and write $u(x,t) = v(x,t) + \phi(x)$ as usual, but here follow [5] in obtaining estimates in terms of u rather than v. Multiplying (104) by $v = u - \phi$, integrating, and integrating by parts using the boundary conditions gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 = \int_{-L}^{L}\Big[&-u_{xx}^2 + u_x^2 + 2\pi\kappa u I[u] - \frac{3}{2}\alpha u^2 - \frac{1}{2}\phi_x u^2 \\ &+ u_{xx}\phi_{xx} - u_x\phi_x - 2\pi\kappa\phi I[u] + \alpha u\phi - \alpha x u \phi_x\Big]\,dx. \qquad (107)\end{aligned}$$

Nonlinear stability of the zero solution. Conditions for nonlinear stability of the trivial solution $u \equiv 0$ may be found by setting $\phi = 0$, which gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 &= -\|u_{xx}\|^2 + \|u_x\|^2 - \frac{3}{2}\alpha\|u\|^2 + 2\pi\kappa\int_{-L}^{L} uI[u]\,dx \\ &\leq -\frac{1}{2}\|u_{xx}\|^2 + \frac{1}{2}(1-3\alpha)\|u\|^2 + 2\kappa\|u\|\|u_x\| \qquad (108)\end{aligned}$$

upon estimating the nonlocal $\int_{-L}^{L} uI[u]\,dx$ term using the Cauchy-Schwarz inequality and $\|\pi I[u]\| = \|u_x\|$, and using $\|u_x\|^2 \leq \|u_{xx}\|\|u\| \leq \frac{1}{2}(\|u_{xx}\|^2 + \|u\|^2)$. The optimal κ-dependence (for large κ) in estimates of the last term in (108) is obtained using $2\kappa\|u_x\|\|u\| \leq 2\|u_{xx}\|^{1/2}\cdot\kappa\|u\|^{3/2} \leq 2\left(\frac{1}{4}\|u_{xx}\|^2 + \frac{3}{4}\kappa^{4/3}\|u\|^2\right)$, where in the last step we used Young's inequality with $p = 4$, $q = 4/3$. Substituting into (108), we find

$$\frac{d}{dt}\|u\|^2 \leq \left(1 + 3\kappa^{4/3} - 3\alpha\right)\|u\|^2 \equiv 3(\alpha_0 - \alpha)\|u\|^2. \qquad (109)$$

Hence $u \equiv 0$ is the unique global attractor if $\alpha > \alpha_0$, where the value $\alpha_0 = \frac{1}{3} + \kappa^{4/3}$ obtained here improves on the result of [20, Thm. 1.2].

Lyapunov function approach: Preliminary estimates. When $\alpha \leq \alpha_0$, (109) is no longer helpful in bounding $\|u\|^2$, but we derive *a priori* bounds on the attractor using the Lyapunov function method.[3] We begin by estimating the terms in (107) using the Cauchy-Schwarz and Young's inequalities in the form $|\int fg| \leq s_i\|f\|^p/p + \|g\|^q/(qs_i^{q-1})$ for constants $s_i > 0$, where $p^{-1}+q^{-1}=1$ (variously using $p = 2$, 3 or 4); we have not attempted to optimize $\mathcal{O}(1)$ prefactors.

Instead of estimating $\int_{-L}^{L} u_{xx}\phi_{xx}\,dx$ and $\int_{-L}^{L} u_x\phi_x\,dx$ separately, we exploit the structure of the linear operator $-\partial_x^4 - \partial_x^2 = -\left(\partial_x^2 + \frac{1}{2}\right)^2 + \frac{1}{4}$: completing the square gives $\int_{-L}^{L}[u_{xx}\phi_{xx} - u_x\phi_x]\,dx = \int_{-L}^{L}\left[(u_{xx}+\frac{1}{2}u)(\phi_{xx}+\frac{1}{2}\phi) - \frac{1}{4}u\phi\right]\,dx \leq \frac{1}{2}\|u_{xx}+\frac{1}{2}u\|^2 + \frac{1}{2}\|\phi_{xx}+\frac{1}{2}\phi\|^2 + \frac{1}{8}\|u\|^2 + \frac{1}{8}\|\phi\|^2$. The α-dependent terms are bounded via $\alpha\int_{-L}^{L}\left[-\frac{3}{2}u^2 + u\phi - xu\phi_x\right]\,dx \leq \alpha\int_{-L}^{L}\left[\frac{1}{2}\phi^2 + \frac{1}{4}x^2\phi_x^2\right]\,dx$, while the parameter dependence in the estimate

$$\int_{-L}^{L} 2\pi\kappa\phi I[u]\,dx \leq 2\|u_{xx}\|^{1/2}\cdot\kappa\|u\|^{1/2}\|\phi\| \leq 2\left(\frac{1}{32}\|u_{xx}\|^2 + \frac{3}{2}\kappa^{4/3}\|u\|^{2/3}\|\phi\|^{4/3}\right)$$
$$\leq \frac{1}{16}\|u_{xx}\|^2 + \kappa^{4/3}\|u\|^2 + 2\kappa^{4/3}\|\phi\|^2 \tag{110}$$

is chosen to balance and minimize the κ-dependence of the prefactors of $\|u\|^2$ and $\|\phi\|^2$, given that the coefficient of $\|u_{xx}\|^2$ should be $\mathcal{O}(1)$; similarly $\int_{-L}^{L} 2\pi\kappa u I[u]\,dx \leq \frac{1}{16}\|u_{xx}\|^2 + 3\kappa^{4/3}\|u\|^2$. Substituting these estimates into (107) and in addition using $\|u_x\|^2 \leq \frac{1}{4}\|u_{xx}\|^2 + \|u\|^2$, we find

$$\frac{d}{dt}\|u-\phi\|^2 \leq \int_{-L}^{L}\left[-\frac{1}{2}u_{xx}^2 + \left(\frac{3}{2} + 8\kappa^{4/3}\right)u^2 - \phi_x u^2\right]dx + R(\phi,\phi), \tag{111}$$

where the bilinear functional $R(\phi,\phi)$ is given by

$$R(\phi,\phi) = \int_{-L}^{L}\left[\phi_{xx}^2 - \phi_x^2 + \left(\frac{1}{2} + 4\kappa^{4/3} + \alpha\right)\phi^2 + \frac{\alpha}{2}x^2\phi_x^2\right]dx; \tag{112}$$

compare [5, Lemma 1].

Following the Lyapunov function bounding approach, the comparison function $\phi(x)$ is now chosen so that a fundamental coercivity estimate holds, namely that for some $\lambda > 0$ and for all sufficiently smooth u satisfying the boundary conditions, we have

$$\int_{-L}^{L}\left[\frac{1}{2}u_{xx}^2 - \left(\frac{3}{2} + 8\kappa^{4/3}\right)u^2 + \phi' u^2\right]dx \geq \lambda\|u\|^2; \tag{113}$$

as usual, $\phi(x)$ is written in the form $\phi'(x) = \mu - q(x)$, where $\mu = (2L)^{-1}\int_{-L}^{L} q(x)\,dx$, so that condition (113) becomes

$$\int_{-L}^{L}\left[\frac{1}{2}u_{xx}^2 - q(x)u^2\right]dx + \int_{-L}^{L}\left[\mu - \frac{3}{2} - 8\kappa^{4/3} - \lambda\right]u^2\,dx \geq 0. \tag{114}$$

Once estimate (113) is established, the desired bound on the radius of the L^2 absorbing ball follows immediately: using (113) and $-2\|u\|^2 \leq 2\|\phi\|^2 - \|u-\phi\|^2$ in (111) gives

$$\frac{d}{dt}\|u-\phi\|^2 \leq -\lambda\|u\|^2 + R(\phi,\phi) \leq -\frac{\lambda}{2}\|u-\phi\|^2 + \lambda\|\phi\|^2 + R(\phi,\phi), \tag{115}$$

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[3]The general bounding results from Section 2 do not directly carry over to this problem, as the linear operator is slightly different and our estimates here are in terms of u rather than $v = u - \phi$; but the arguments are completely analogous.

so that via Gronwall's inequality, solutions $u(x,t)$ of (104) satisfy

$$\limsup_{t\to\infty} \|u\|^2 \le 2\|\phi\|^2 + 2\limsup_{t\to\infty}\|u-\phi\|^2 \le 6\|\phi\|^2 + \frac{4}{\lambda}R(\phi,\phi). \tag{116}$$

C.2. **Parameter-dependent bounds for the nonlocal KS equation.** In order to satisfy (114) for a range of domain sizes and parameter values, following [6] and our previous discussions we choose $q(x)$ to have a suitable scaling form. We seek bounds uniform in $\kappa \ge 0$, but desire their scaling as $\kappa \to \infty$, so the situation is analogous to that for the destabilized KS equation discussed in Section 3. Defining the parameter $\beta = \max\{1, 2\kappa^{4/3}\}$, to satisfy $\mu \ge 3/2 + 8\kappa^{4/3} + \lambda$ uniformly in $\kappa \in [0,\infty)$ it is sufficient that the mean μ should satisfy $\mu \ge \frac{11}{2}\beta + \lambda$, so that we want $\mu = \mathcal{O}(\beta)$, and can then also choose $\lambda = \beta\tilde{\lambda}$ for $\tilde{\lambda} = \mathcal{O}(1)$.

Obtaining the scaling exponents as in Section 3.2 (see (48)) the comparison function thus takes the form (see (49), adapted for a domain of length $2L$)

$$\phi'(x) = \frac{1}{2}\beta\widetilde{\mu} - \beta^{4/3}L^{4/3}\widetilde{q}(x\beta^{1/3}L^{1/3}) \tag{117}$$

(where $\widetilde{q}(y)$ is supported in $[-\tilde{\eta},\tilde{\eta}]$ and $\widetilde{\mu} = \int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}(y)\,dy$) and a sufficient condition to satisfy (114) for all $L \ge 2\pi$ and $\kappa \ge 0$ is

$$\beta L \int_{-\bar{L}}^{\bar{L}} \left[\frac{1}{2}u_{yy}^2 - \widetilde{q}(y)u^2\right] dy + \beta\int_{-L}^{L}\left[\frac{1}{2}\widetilde{\mu} - \frac{11}{2} - \tilde{\lambda}\right]u^2\,dx \ge 0, \tag{118}$$

where $\bar{L} = \beta^{1/3}L^{4/3}$; compare conditions (53)–(54) on $\widetilde{q}(y)$ and $\widetilde{\mu}$.

Bounds for odd solutions. Consider for now the case of odd initial data $u(x,0)$ for (104), as considered in [5, 20]: We now observe that it suffices again to choose the same specific (parameter-independent) potential $\widetilde{q}(y)$ constructed in Appendix A (following [6]; see also Section 3.3), since its properties (93)–(94) imply that (118) is satisfied for all sufficiently smooth odd $u(x,t)$ provided $\tilde{\lambda} \le 5/2$ (for simplicity below choose $\tilde{\lambda} = 2$). Consequently, the corresponding comparison function $\phi(x)$ (117) satisfies (113), so that (116) gives a bound on $\|u\|^2$.

To evaluate this bound, note from (112) (using $\alpha \le \alpha_0 < \beta$) that

$$\begin{aligned} 6\|\phi\|^2 + \frac{4}{\lambda}R(\phi,\phi) &\le 6\|\phi\|^2 + \frac{4}{2\beta}\left[\|\phi_{xx}\|^2 + \frac{7}{2}\beta\|\phi\|^2 + \frac{\alpha}{2}\|x\phi_x\|^2\right] \\ &\le \frac{2}{\beta}\|\phi_{xx}\|^2 + 13\|\phi\|^2 + \|x\phi_x\|^2; \end{aligned} \tag{119}$$

using the estimates for $\|\phi\|^2$, $\|\phi_{xx}\|^2$ and $\|x\phi_x\|^2$ from Appendix B with the scaling exponents (48), we thus find from (116) that *odd* solutions of (104) satisfy

$$\limsup_{t\to\infty}\|u\|^2 \le \beta^2 L^3\left[2\int_{-\tilde{\eta}}^{\tilde{\eta}} \widetilde{q}'(y)^2\,dy + 13\|\widetilde{q}\|_1^2 + \frac{4}{3}\widetilde{\mu}^2 + 2(\beta L^4)^{-1/3}\int_{-\tilde{\eta}}^{\tilde{\eta}} y^2\widetilde{q}(y)^2\,dy\right] \tag{120}$$

for *fixed* ($\mathcal{O}(1)$) $\widetilde{q}(y)$ and $\widetilde{\mu}$, where $\beta = \max\{1, 2\kappa^{4/3}\}$. In particular, for $\kappa \ge 1$ we find: there exists a parameter- and L-independent constant C so that (sufficiently smooth) solutions of (104) with odd initial data approach an absorbing ball in L^2 given by

$$\limsup_{t\to\infty}\|u\| \le C\,\kappa^{4/3}L^{3/2}. \tag{121}$$

This result improves on the $\mathcal{O}(\kappa^5 L^{3/2})$ scaling found in part 1 of [5, Thm 1], with the tighter bounds in κ being due to three modifications: (i) the condition $\mu \geq \mathcal{O}(\beta)$ was satisfied (for a parameter $\beta \to \infty$) by choosing a fixed $\widetilde{q}(y)$ and letting the potential $q(x) = \beta^{4/3} L^{4/3} \widetilde{q}(x\beta^{1/3} L^{1/3})$ scale with β as well as L, rather than by using $\delta = \mathcal{O}(\beta^{-1})$ in the definition of $\widetilde{q}(y)$ to obtain a mean μ proportional to β, which permitted the estimate for $R(\phi, \phi)$ to be reduced from $\mathcal{O}(\beta^5)$ to $\mathcal{O}(\beta^3)$; (ii) the use of $\lambda = \mathcal{O}(\beta)$ in (113) further reduced the estimate of $\limsup_{t\to\infty} \|u\|^2$ by an additional factor of β, to $\mathcal{O}(\beta^2)$; and (iii) more careful estimates leading from (107) to (111)–(112) meant that β could be chosen to be $\mathcal{O}(\kappa^{4/3})$ rather than $\mathcal{O}(\kappa^2)$.

Bounds for general initial data. In the above calculation, the potential $q(x)$ is chosen to be localized at $x = 0$, and a crucial ingredient in the estimates is the Hardy-Rellich inequality of Appendix A.1 for reduction of order, which depends on u vanishing at the origin, and thus fails when the restriction to odd solutions $u(x, t)$ of (104) is lifted. However, as pointed out in [5], one can instead exploit the Dirichlet boundary conditions at $x = \pm L$, by modifying the potential function $q(x)$ to be localized at one or both endpoints rather than at the origin.

For definiteness, following [5], given the function $\phi(x)$ defined previously on $[-L, L]$ and used for odd initial data, in the general case we may simply use a comparison function $\widetilde{\phi}(x)$ defined to equal $\phi(x + L)$ for $-L \leq x < 0$, and $\phi(x - L)$ for $0 \leq x \leq L$. The bounding arguments then carry over directly, and by translation invariance we have $\|\widetilde{\phi}\|^2 = \|\phi\|^2$ and $\|\widetilde{\phi}_{xx}\|^2 = \|\phi_{xx}\|^2$; so it remains only to estimate the scaling of $\|x\widetilde{\phi}_x\|$, which worsens from the odd case (since we now have $|x| \approx L$ where the potential is localized) according to

$$\|x\widetilde{\phi}_x\|^2 \leq L^2 \|\widetilde{\phi}_x\|^2 = L^2 \|\phi_x\|^2 \leq \beta^{7/3} L^{13/3} \int_{-\tilde{\eta}}^{\tilde{\eta}} \tilde{q}(y)^2 \, dy, \tag{122}$$

where we used (95). Using this estimate and otherwise evaluating the bounds as before, it follows that there exist constants $\widetilde{C}_1$ and $\widetilde{C}_2$ so that general solutions of (104) satisfy

$$\limsup_{t\to\infty} \|u\|^2 \leq \widetilde{C}_1 \, \beta^2 L^3 + \widetilde{C}_2 \, \alpha \beta^{4/3} L^{13/3} \tag{123}$$

for $\alpha \leq \alpha_0 = \frac{1}{3} + \kappa^{4/3}$, $\beta = \max\{1, 2\kappa^{4/3}\}$; compare part 2 of [5, Thm.1]. In particular, if $\kappa \geq 1$ and $\alpha = \mathcal{O}(\beta) = \mathcal{O}(\kappa^{4/3})$, then the second term dominates *vis-à-vis* both the L- and κ-dependence, and we have

$$\limsup_{t\to\infty} \|u\| \leq \widetilde{C} \, \kappa^{14/9} L^{13/6} \tag{124}$$

for some parameter- and L-independent constant $\widetilde{C}$.

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Received August 2010; revised May 2013.

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