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Dissipativity, analyticity and viscous shocks in the (de)stabilized Kuramoto–Sivashinsky equation

Ralf W. Wittenberg

Department of Mathematics, University of Michigan, East Hall, 525 East University Avenue, Ann Arbor, MI 48109-1109, USA

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Abstract

We investigate an extension of the Kuramoto–Sivashinsky (KS) model for complex spatiotemporal dynamics, containing an additional linear stabilizing or destabilizing term. Generalizing previous results, we prove dissipativity and analyticity. In the destabilized case, a stable, attracting shock-like transition layer solution is observed numerically, and we obtain its asymptotic scaling. This “viscous shock” solution does not exhibit extensive scaling, which sheds light on the difficulties in obtaining optimal bounds and proving extensivity for the KS equation. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The Kuramoto–Sivashinsky (KS) equation in one space dimension,

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0, \quad x \in [-L/2, L/2] \tag{1}$$

has attracted a great deal of interest as a model for complex spatiotemporal dynamics in extended systems, and as a paradigm for finite-dimensional behavior in a partial differential equation (PDE). The reason for this interest is readily understood in Fourier (momentum) space, in which the dispersion relation, or linear growth rate of Fourier modes, is $\omega_0(k) = k^2 - k^4$, where we write $u(x,t) = \mathrm{i}\sum_k \hat{u}_k(t)\exp(\mathrm{i}kx)$, $k = nq$, $q = 2\pi/L$, $n \in \mathbb{Z}$. The number of linearly unstable modes with $|k| < 1$ increases proportionately to L; these modes are coupled to each other and to damped modes at $|k| > 1$ through the nonlinear term. As L increases, therefore, the stationary cellular (roll) states born at the primary instability at $L = 2\pi$ undergo secondary instabilities, yielding a complex bifurcation sequence [1] in which increasingly many unstable modes interact to give oscillating and (modulated) traveling structures, heteroclinic connections, and eventually a spatiotemporally chaotic state characterized by localized events such as local motion, creation and annihilation of peaks [2,3].

The KS equation generically describes the dynamics near long-wavelength primary instabilities in the presence of appropriate symmetries [4], and has been independently derived in a wide variety of contexts, including plasma ion mode instabilities [5], flame front instabilities [6], phase dynamics in reaction-diffusion systems [7] and delay-diffusion population models [8], and fluctuations in liquid films on inclines [9].

In its role as a paradigmatic amplitude equation describing dynamics and growth in dissipative systems subject to long-wave instabilities, the KS equation is stripped down to the simplest necessary terms—

E-mail address: ralf@umich.edu (R.W. Wittenberg).

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large-scale driving, small-scale damping, and a nonlinear coupling term providing energy transfer between modes. As such, many details pertinent to particular physical systems have been neglected which, when retained, may lead to additional terms in KS-type equations. Thus there has been increased recent interest in modified KS equations which have been derived and studied in various applications, in which (1) is augmented by, for instance, regularizing, dispersive, nonlocal or noisy forcing terms.

Arguably the simplest modification to the KS equation involves the inclusion of an additive linear term in (1). To describe this formulation, which is the subject of this Letter, it is convenient to rewrite the KS equation (1) in a rescaled form which explicitly isolates the negative definite part of the operator

$$u_t = -\left(\partial_x^2 + 1\right)^2 u + \alpha u - u u_x, \quad \alpha = 1. \tag{1$'$}$$

In this form, we see immediately that the αu term on the right-hand side is the source of the linear instability, at those modes for which it dominates the first, stable term. This provides an easy way to strengthen or weaken the instability, by modifying α. For $\alpha < 0$, one readily sees from the Fourier space formulation or energy estimates that the trivial zero solution is attractive, and $\limsup_{t\to\infty} \|u\| = 0$ (where $\|\cdot\|$ is defined to be the norm on $\mathrm{L}^2([-L/2, L/2])$: $\|u\|^2 = \int_{-L/2}^{L/2} u^2(x', t)\, dx'$). We thus emphasize $\alpha \geqslant 0$ by setting $\alpha = \varepsilon^2$, so that our generalized KS equation is

$$u_t = -\left(\partial_x^2 + 1\right)^2 u + \varepsilon^2 u - u u_x, \tag{2}$$

or alternatively, for correspondence with (1)

$$u_t + u_{xxxx} + 2u_{xx} + \left(1 - \varepsilon^2\right) u + u u_x = 0. \tag{2$'$}$$

To simplify notation, we introduce the linear operators $\mathcal{L}_0 = -\partial_x^4 - 2\partial_x^2$, $\mathcal{L} = \mathcal{L}_0 - (1 - \varepsilon^2) = -\partial_x^4 - 2\partial_x^2 - (1 - \varepsilon^2)$, so that in this notation (2) becomes

$$\partial_t u = \mathcal{L} u - u \partial_x u; \tag{3}$$

and we impose periodic boundary conditions on the domain $\Omega = [-L/2, L/2]$ of length L.

In concordance with previous works [10,11], we shall refer to (2) or (3) as the *damped Kuramoto–Sivashinsky equation* (DKS equation), or alternatively, as the stabilized KS equation [4,12]. However, our notation and nomenclature may be misleading; for we impose no requirement on ε to be small, or even less than one. In fact, some of our most interesting results occur for $\varepsilon^2 > 1$, in which case we have added a *destabilizing* term, or "negative damping", to the KS equation. Nevertheless, for convenience we shall continue to refer to (3) as the DKS equation, for general ε. In both cases, the effect of the additional term implies a uniform vertical shift in the Fourier dispersion relation, $\omega(k) = \varepsilon^2 - (1 - k^2)^2 = \omega_0(k) + \varepsilon^2 - 1$.

As the simplest generalization of the KS equation, the DKS model (2) (with $\varepsilon^2 < 1$) has attracted attention, particularly in the study of the transition from regular cellular solutions to KS spatiotemporal chaos as ε increases towards 1 [4,10], as it was in this context that the mechanism of "spatiotemporal intermittency" was first proposed [13]. It is also important in applications, with the linear term arising in several early derivations [5,14,15]. For small ε^2, it was studied in the context of wave number selection as the "model b" variant of the Swift–Hohenberg model of convection [16–18]. More recently, the DKS equation has been shown to be relevant to the study of directional solidification [19,20], to the evolution of a terrace edge during step-flow growth [21], and (with an additive noise term) in the context of electrodeposition growth near equilibrium [12,22]. Like the "pure" KS equation, the stabilized KS equation is thus also a generic equation with a wide range of applicability [4]. Indeed, in its original form for the integral $h(x,t) = \int^x u(x', t)\, dx'$, with h interpreted as a front position, we see that the translational symmetry $h \to h + \delta h$ in the front direction, valid for the KS equation, no longer holds for the DKS equation. This is as expected if there is an external field in the growth direction, which imposes a preferred front location: in the context of directional solidification, a thermal gradient provides this external driving force, while it arises from step-step interaction in step-flow growth.

1.1. Review of analytical results for the KS equation

There has been considerable rigorous analysis for the KS equation (1) since the pioneering study of Nicolaenko et al. [23], concerning bounds on the magnitude, smoothness and asymptotic dynamics of the solutions. The equation has been shown to be dissipative for odd [23] and general [24–26] initial conditions; that is, initial data in L^2 will remain in L^2 for all time, and are attracted to an absorbing ball in L^2 with bounded L-dependent radius. Similar absorbing balls exist in higher Sobolev spaces [23], and the linear operator $\mathcal{L}_0$ has sufficiently strong smoothing properties that solutions are space-analytic, or bounded with respect to the Gevrey norm [27] (as well as time-analytic [28]). Furthermore, KS solutions approach a compact maximal global attractor with finite (L-dependent) fractal and Hausdorff dimension [23]. While the attractor can have very complex structure, the existence of a finite-dimensional inertial manifold has also been shown, positively invariant under the flow, exponentially absorbing and containing the attractor [29,30]. The restriction of the PDE to the inertial manifold yields a finite-dimensional dynamical system, demonstrating that the asymptotic dynamics are rigorously finite-dimensional.

An important aspect of many of the above studies is the explicit estimation of the scaling of bounds—for the radius of the absorbing ball in L^2 or various Sobolev spaces, for the dimension of the attractor or inertial manifold, and for the radius of the strip of analyticity—in terms of the bifurcation parameter L. This is especially interesting since theory has not yet caught up with (numerical) experiment in this case: numerical evidence makes a definite prediction for the scaling, while the best theoretical estimates give cruder bounds.

Since the attractor (and inertial manifold) must contain the unstable manifold of the zero solution, and the number of linearly unstable Fourier modes is proportional to L, the attractor and inertial manifold dimensions cannot be less than $\mathcal{O}(L)$. The conjecture that this linear scaling also holds as an upper bound is supported by the numerical computations of Manneville [31], who found the fractal dimension of the dynamics for large L to be proportional to L. More generally, for large L the dynamics appear to be *extensive*: the local properties are asymptotically independent of L [3]. The simplest such property is probably a bound on $|u(x,t)|$ for solutions on the attractor; all analytically approximated and numerically computed solutions seem to be uniformly bounded independent of L. However, a uniform L^∞ bound has not yet been proved in general, although some special cases have been shown, such as a uniform bound for all stationary solutions [32,33] and solutions near these on the attractor [34].

An L-independent bound $\|u\|_\infty$ for the amplitude of u implies that the L^2 norm is proportional to $L^{1/2}$, or equivalently, that the energy density is asymptotically finite. The radius $R_{L,\varepsilon}$ of the absorbing ball in L^2 turns out to be a fundamental bound, as the others may be derived from it; so the most attention has been devoted to improving this bound to its "optimal" value of $\mathcal{O}(L^{1/2})$. However, the best available bounds [25] give $R_{L,\varepsilon=1} = \mathcal{O}(L^{8/5})$ (an improvement on the original $\mathcal{O}(L^{5/2})$ estimate of [23]). A similar situation pertains to the radius of the strip of analyticity, or the rate of exponential decay of Fourier modes: it is conjectured [27], but not yet proved rigorously, that this decay is L-independent.

In this Letter, we report some analytical results concerning boundedness and analyticity for the DKS equation (3), generalizing those previously obtained for the KS equation (1) and summarized above. We are aware of two similar studies in which the effect of other perturbations of the KS equation is investigated: Duan and Ervin [35] have studied the effect of a well-behaved nonlocal operator, a Hilbert transform term, and Johnson [36] has looked at a KS equation with an x-dependent coefficient. In these other studies, as in ours, the techniques are essentially the same as those introduced for the pure KS equation; indicating that our success in adapting the previous methods for our perturbed KS equation is unsurprising. However, part of the reason for the general applicability of the bounds of Collet et al. [25] is that they are "too crude"; as indicated above, their $\mathcal{O}(L^{8/5})$ scaling for the absorbing ball in the L^2 norm is considerably larger than the optimal expected bound of $\mathcal{O}(L^{1/2})$. In Section 3 we partially clarify this issue of excessively crude estimates by providing an explicit example, in the destabilized case $\varepsilon^2 > 1$, of a solution which satisfies the $\mathcal{O}(L^{8/5})$ bound, and which does *not* have the extensive $\mathcal{O}(L^{1/2})$ scaling numerically observed and conjectured for the pure KS equation.

2. Dissipativity and analyticity

The results on dissipativity for the DKS equation (3) are completely analogous to those the KS equation (1), with the proofs of [25] going through almost line-by-line: the dynamics approach an absorbing ball in L^2, and we can find an L- and ε-dependent estimate on the bound of this ball. A caveat to this statement is that for $\varepsilon^2 \geqslant 1$, we must restrict ourselves to zero mean solutions: it is readily seen that the mean (or zeroth Fourier mode) $m(t) = \int_{-L/2}^{L/2} u(x', t)\, dx'$ evolves as

$$\frac{dm}{dt} = (\varepsilon^2 - 1)m,$$

so that for $\varepsilon^2 < 1$ the mean decays, for the KS equation $\varepsilon^2 = 1$ it is conserved, and for $\varepsilon^2 > 1$ the mean grows exponentially. A priori, therefore, we can only expect to get an absorbing ball in L^2 for $\varepsilon^2 < 1$ [37]. Since the DKS equation conserves the property of vanishing mean, however, requiring the initial data to have mean zero removes the difficulty.

2.1. Dissipativity

We summarize the results on dissipativity as follows (compare [25,27,30]).

Theorem 1. *If the initial condition $u_0(x)$ of the DKS equation* (3) *is L-periodic with zero mean on the domain $\Omega = [-L/2, L/2]$, then so is the solution $u(x,t)$ at time t; if $u_0(x)$ is antisymmetric* (*odd*), *satisfying $u_0(-x) = -u_0(x)$, then the same is true for $u(x,t)$.*

Periodic, mean zero solutions of (3) *are attracted to an absorbing ball of radius $R_{L,\varepsilon}$ in $L^2(\Omega)$*

$$\limsup_{t\to\infty} \| u(\cdot,t) \|^2 \leqslant R^2_{L,\varepsilon}, \tag{4}$$

and furthermore, if $\|u(\cdot,t)\| \leqslant R_{L,\varepsilon}$ for some t, then the same is true for all $t' > t$. Here there is a constant K, independent of L, ε and u_0, in terms of which we may estimate $R_{L,\varepsilon}$ as follows:

(a) $\varepsilon^2 \leqslant 1$, *odd initial data*: $R^2_{L,\varepsilon} = K(\varepsilon^{16/5}L^{16/5} + (1-\varepsilon^2)\varepsilon^2 L^3)$;
(b) $\varepsilon^2 \leqslant 1$, *general periodic initial data*: $R^2_{L,\varepsilon} = K\varepsilon^2 L^{16/5}$;
(c) $\varepsilon^2 \geqslant 1$, *general periodic initial data*: $R^2_{L,\varepsilon} = K\varepsilon^{32/5}L^{16/5}$.

Note that in the stabilized case $\varepsilon^2 \leqslant 1$, the ε-dependence on the bounds is better for odd solutions; and that for the pure KS equation ($\varepsilon^2 = 1$) the bounds reduce to those of Collet et al. [25, Theorem 2.2]: $\limsup_{t\to\infty} \|u\| \leqslant K \cdot L^{8/5}$.

The demonstration of dissipativity in the proof of Theorem 1 is based on the method of Nicolaenko et al. [23], who introduced the idea of bounding the distance in L^2 between u and a suitably chosen comparison (or gauge) function ϕ; the effect is that the nonlinear interaction between v and ϕ balances the linear term. Specifically, for odd initial data one writes $u(x,t) = v(x,t) + \phi(x)$, and rewrites the DKS equation as

$$v_t = (\mathcal{L} - \phi')v - vv' + \mathcal{L}\phi - \phi v' - \phi\phi',$$

where $v' = \partial_x v$; the goal is then to choose $\phi(x)$ so that the operator $\mathcal{L} - \phi'$ is negative definite. Such an indirect approach via ϕ is needed since the nonlinear term uu_x in the DKS equation is energy-preserving, and thus does not provide the desired damping effect directly; see [25,26,30] for discussions.

Multiplying by v and integrating by parts, one writes the evolution of the norm of v as

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = -(v,v)_{\phi/2} - (v,\phi)_\phi, \tag{5}$$

where $(\cdot,\cdot)_{\gamma\phi}$ is a bilinear form defined for sufficiently smooth v_1, v_2 and for $\gamma \in \mathbb{R}$ as

$$\begin{aligned}(v_1,v_2)_{\gamma\phi} &= \int v_1'' v_2'' - 2\int v_1' v_2' + (1-\varepsilon^2)\int v_1 v_2 \\ &\quad + \gamma \int v_1 v_2 \phi' \\ &= -\int v_1(\mathcal{L} - \gamma\phi')v_2,\end{aligned}$$

and one defines the associated quadratic form $R_{\gamma\phi}(v) = (v,v)_{\gamma\phi}$.

Applying the Cauchy–Schwarz and Young's inequalities in (5), one obtains [25]

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leqslant -\frac{2}{3}R_{\phi/4}(v) + \frac{1}{6}R_\phi(\phi). \tag{6}$$

The crucial step in the argument now is the careful choice of $\phi(x)$ to guarantee the coercivity of the ϕ-dependent bilinear form $(\cdot,\cdot)_{\gamma\phi} = R_{\gamma\phi}(\cdot)$ as follows:

$$R_{\gamma\phi}(v) \geqslant \frac{\eta}{4}\left(\|v''\|^2 + \|v\|^2\right) \tag{7}$$

for all $v \in L^2_{\mathrm{odd}}$ and all $\gamma \in [1/4, 1]$; in our construction we choose $\eta = \varepsilon^2$ for the stabilized case $\varepsilon^2 \leqslant 1$, and $\eta = 1$ for $\varepsilon^2 \geqslant 1$. Simultaneously, one attempts to minimize $R_\phi(\phi) = R_0(\phi) = -\int \phi \mathcal{L} \phi$ as a function of L and ε. The coercivity estimate (7) both guarantees the positivity of $R_{\gamma\phi}(v)$, permitting the use of the Cauchy–Schwarz inequality above in (5), and allows us to bound the first term on the right-hand side of (6) by $-\frac{\eta}{6}\|v\|^2$. An argument via the Gronwall inequality then allows us to deduce the boundedness of $\|v\|$ and hence (via the finiteness, by construction, of $\|\phi\|$) the boundedness of $\|u(\cdot,t)\|$ in Theorem 1.

Once the antisymmetric case has been dealt with, and a comparison function $\phi(x)$ defined, the essential idea of the generalization to arbitrary periodic data [25,26] is to use the given ϕ to construct a *family* of comparison functions $\phi(x+b(t))$ by time translation. An appropriate evolution equation for $b(t)$ then ensures that at each time, the translation is chosen in such a way that the distance between u and the set is minimized. Again, our computation generalizes that of Collet et al. [25], although some care must be taken for the stabilized case $\varepsilon^2 < 1$ [38, Section 2.2.2].

The main technical aspect of the proof is our construction of $\phi(x)$, which closely follows the Fourier space construction of Collet et al. [25] (which improved the bound of [23]; see also [30]). The large-scale Fourier modes $\hat{\psi}_k$ of the even function $\psi = -\phi'$ are chosen to be constant in k, to balance the unstable part of the linear dispersion relation $\omega(k)$ and guarantee the coercivity (7) of $R_{\gamma\phi}(v)$, while the decay of the $\hat{\psi}_k$ for large k is dictated by the need to make $R_0(\phi)$ small and obtain a better bound in Theorem 1. Since the extra term in (2′) (compared with (1)) merely vertically shifts the linear dispersion relation, $\omega(k) = \omega_0(k) + \varepsilon^2 - 1$, the essential modification to the computation of [25] in the proof for the DKS equation is an ε-dependent rescaling of the large-scale Fourier modes of ψ to balance the modified linear operator $\mathcal{L}$. Note that the cases $\varepsilon^2 \leqslant 1$ and $\varepsilon^2 \geqslant 1$ must be treated separately, which accounts for the different bounds in Theorem 1.

Further details of our calculations are given in [38, Chapter 2]; we have not presented them here, as they involve adaptations of the calculations given in detail in [25] and also discussed in depth, with some refinements, in [30]. Detailed constructions of comparison functions for the KS equation (1) are also given in [23,26,39,40], and for related equations in [35,36,40]. We observe that the dissipativity for the DKS equation for $\varepsilon^2 \in (0, 1/4)$ and odd $u(x,t)$ has previously been shown by Ziegra [41], and our results concerning the antisymmetric case for $\varepsilon^2 \leqslant 1$ are based on his, with minor changes, mainly to ensure that the bounds are uniformly valid over the entire range $\varepsilon^2 \in (0, 1]$. The extensions to $\varepsilon^2 \geqslant 1$ and to general periodic (asymmetric) solutions have not previously been reported for the DKS equation.

2.2. Analyticity

Using techniques developed in [42], Collet et al. [27] established the analyticity of solutions to the KS equation (see also [43] for a similar proof of analyticity in the two-dimensional KS equation). They did this by obtaining the boundedness of a suitably defined norm, the Gevrey norm, which allows one to deduce analyticity in a strip of finite width about the real axis from the exponential decay of Fourier modes. As we did for dissipativity above, we can immediately generalize these previous results to derive the real analyticity of solutions u of the DKS equation (3). In fact, since the only modification to the KS equation we consider is in the linear operator $\mathcal{L}$, which does not affect the fundamental bounds of [27] on the nonlinear term, their proofs carry over essentially line by line, and we shall thus just state the theorems on Gevrey regularity and analyticity, referring to [27] for the details:

Theorem 2 (cf. [27, Theorem 3.1]). *If the initial condition $u_0(x)$ of the DKS equation* (3) *with L-periodic boundary conditions satisfies $\|u_0\|^2 \leqslant \rho^2$, then the solution $u(x,t)$ satisfies the bound*

$$\left\| e^{\bar{\alpha}\min(t,\bar{t})A} u(\cdot,t) \right\| \leqslant 2\rho, \tag{8}$$

where

$$A = \sqrt{-\partial_x^2}, \qquad \bar{\alpha} = \alpha_0 \rho^{6/5}, \qquad \bar{t} = t_0 \rho^{-8/5},$$

and α_0 and t_0 are L- and ε-independent constants.

Since we know from Theorem 1 that initial data in L^2 is attracted to an absorbing ball, so that within finite time t' it enters a region of L^2 of radius $R_{L,\varepsilon}$ given in Theorem 1, we can shift the origin of time to t', and use the radius $R_{L,\varepsilon}$ of the absorbing ball for ρ.

Corollary 3 (cf. [27, Theorem 1.2]). *For large t, the function $u(x,t)$ satisfying* (3) *is analytic in x in a strip of width*

$$\beta_{L,\varepsilon} \geqslant \text{const}\, R_{L,\varepsilon}^{-2/5} \tag{9}$$

about the real axis.

Note that according to this estimate, the width of the strip of analyticity shrinks as $L \to \infty$. However, consistent with other numerical observations of extensivity [3], Collet et al. [27] present strong numerical evidence that the width of the strip of analyticity is asymptotically independent of L for the KS equation $\varepsilon = 1$, and we expect this to hold also for the stabilized case $\varepsilon^2 \leqslant 1$.

2.3. Scaling for small ε and the Ginzburg–Landau formalism

While the L-dependence of the rigorous bounds is weaker than that predicted by the numerics, the ε-dependence (for small ε) is consistent with our expectations. For instance, the growth of the domain of analyticity as $\varepsilon \to 0$ indicated by Corollary 3 agrees with the intuition that increased damping should lead to increased smoothness.

For general periodic solutions and $\varepsilon^2 \leqslant 1$, the radius of the absorbing ball in L^2 is $R_{L,\varepsilon} \sim \varepsilon L^{8/5}$. For odd u, our construction of the comparison function ϕ implies that the scaling as $\varepsilon \to 0$ may in fact be found with a slightly improved L-dependence [38] (provided L does not increase too rapidly; we need $\varepsilon L \leqslant \mathcal{O}(1)$): $R_{L,\varepsilon} \sim \varepsilon L^{3/2}$ for $\varepsilon \to 0$, u odd. In the case of both odd and general periodic solutions, a rigorous estimate for the bounds for $\varepsilon \to 0$ is thus $\|u\| = \mathcal{O}(\varepsilon)$. This is reassuring, as it coincides with the scaling predicted via the Ginzburg–Landau formalism: for $\varepsilon \ll 1$, the only linearly unstable modes are located in a narrow band near $k = 1$, and one expects, and observes numerically, that the solution u to (3) in this limit is a perturbation of a purely sinusoidal solution. Formal multiple scale analysis indicates that in this case u may be written as

$$u = \varepsilon\left[A(T,X)e^{ix} + \bar{A}(T,X)e^{-ix}\right] + \text{h.o.t.},$$

where we have introduced the slow time scale $T = \varepsilon^2 t$ and large spatial scale $X = \varepsilon x$, h.o.t. denotes higher-order terms in ε, and the evolution of $A = \mathcal{O}(1)$ is governed by the Ginzburg–Landau (GL) equation. From this expression, it is clear that we expect $\|u\| \sim \varepsilon$.

The rigorous correspondence between u and A is discussed in [37], where it is shown that for sufficiently small ε and for sufficiently "nice" initial data the function u is uniformly bounded by a constant times ε (specifically, we need $u_0 \in H^1_{l,u}(\mathbb{R})$, where the "locally uniform" translationally invariant space $H^1_{l,u}$ contains functions with a finite Sobolev H^1_ρ norm with respect to a weight function $\rho(x)$ and all its translates; and we require $\|u_0\|_{H^1_{l,u}} \leqslant K\varepsilon^{1/2}$). In fact, in this case we obtain the desired scaling $\|u\| \leqslant K\varepsilon L^{1/2}$, establishing extensivity in this limit. However, we cannot extend this result to larger values of ε, let alone to the pure KS equation: firstly, rather strong conditions have been placed on the initial data, and secondly, the GL formalism in this form presupposes a rather narrow band of unstable modes; it breaks down at or before $\varepsilon = 3/5$, at which value there is a wave number k such that both k and $2k$ lie in the band of instability.

3. A viscous shock solution

For small ε, we were able to compare our bounds with rigorous results obtained through the Ginzburg–Landau formalism; where under more restricted conditions, one can also show extensivity. For large ε, on the other hand, it turns out that there is a stable stationary solution, whose existence and asymptotic scaling clarifies the *failure* of existing approaches to establishing dissipativity for the KS equation.

Numerical simulations of the DKS equation (3) for sufficiently large ε ($\varepsilon^2 \gtrsim 1.4$ seems large enough) indicate that arbitrary initial conditions rapidly converge, as in Fig. 1, to a shock-like solution, a stationary solution with a sharp interior transition layer, reminiscent of viscous shocks; a typical profile of this solution is shown in Fig. 2. Similar oscillatory shock-like solutions have been observed in numerical simulations of the KS equation on the real line with homogeneous Neumann boundary conditions [32,44], and in the simplified model for KS dynamics proposed and investigated in detail by Goren et al. [45], in which the usual KS dispersion relation was replaced, in the un-

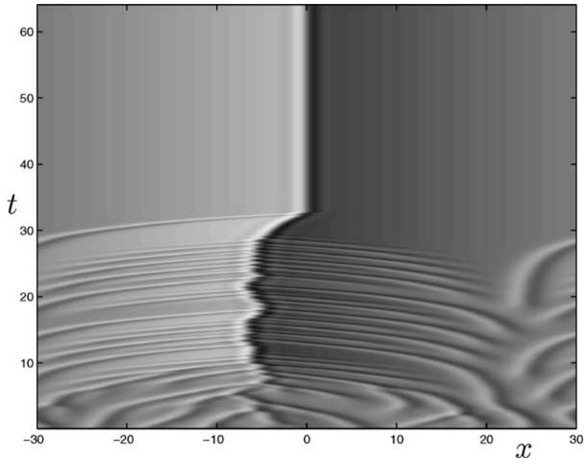

Fig. 1. Evolution of destabilized KS equation (3), with $L = 60$ and $\varepsilon^2 = 1.4$, showing the rapid convergence to a stable shock-like solution; in this gray-scale representation of the space–time evolution, lighter shading indicates local maxima, darker shows minima, and the shading interpolates between extremes of $u = \pm 25$.

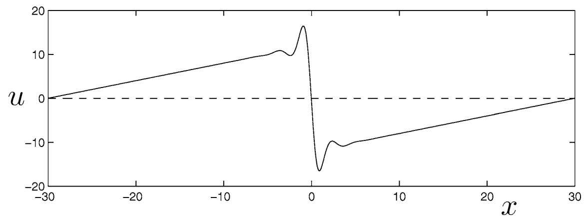

Fig. 2. Cross-section of the antisymmetric, long-time stationary shock-like transition layer profile of Fig. 1 ($L = 60, \varepsilon^2 = 1.4$), showing the constant slope in the outer region and the narrow internal layer.

stable range, by a nonlocal operator with two unstable modes. Numerical experiments in which large-scale (wavelet) modes are subject to large-amplitude driving (relative to their normal dynamics on the attractor), also display such sharp internal layers, consistent with the fact that the main effect of the destabilizing term for $\varepsilon^2 > 1$ is to provide additional energy at long wavelengths [38,46].

3.1. Asymptotic scaling of the transition layer

A more detailed analysis of the structure and stability of this shock-like solution will be reported elsewhere; Goodman [26] has given an analysis for the second-order "Burgers–Sivashinsky" equation, which displays similar shock transition layers. Of particular interest for further investigation is the transition from this steady solution to the spatiotemporally chaotic KS state as ε^2 decreases towards 1. In the remainder of this Letter, we show how numerical evidence and a quick asymptotic analysis for large $\varepsilon^2 L$ yield the scaling of the energy, or L^2 norm, with interesting implications for the search for optimal bounds for the KS equation.

Setting $\nu = \varepsilon^2 - 1$, we seek an appropriate stationary solution ($u_t = 0$) to (3), satisfying

$$u_{xxxx} + 2u_{xx} - \nu u + u u_x = 0. \tag{10}$$

It is apparent from the profile of Fig. 2 that there are two distinguished limits and an interior layer of width $\sim \delta$. By antisymmetry of the profile about its midpoint, without loss of generality we may choose the center of the transition layer in the domain $x \in [-L/2, L/2]$ to lie at $x = 0$. A systematic asymptotic analysis would begin by rescaling the spatial variable by $y = x/L$, to give a fixed-length domain $[-1/2, 1/2]$, and the solution amplitude by $v = u/\nu L$. In the rescaled equation, it then becomes apparent that the appropriate parameters are νL and L, the relevant asymptotic limit is $(\varepsilon^2 - 1)L = \nu L \gg 1$, and in the rescaled variables the layer thickness is small, $\delta \ll 1$. An analysis of (10), however, is sufficient to obtain the lowest-order scaling of solutions.

We have performed numerical simulations of the DKS equation (3) using a Fourier pseudo-spectral method, in which we integrated the linear terms exactly, used an Adams–Bashforth time-stepping scheme for the nonlinear terms, and ensured that the small scales were well-resolved by retaining Fourier modes well into the strongly damped regime; integration continued until the solution had clearly converged to its stationary equilibrium. Fig. 3 shows representative viscous shock solutions for fixed $\varepsilon^2 = 1.5$ ($\nu = 0.5$) and varying L. In the figure we see that the constant outer slope depends only on ν, the maximal amplitude increases linearly with L, and the width δ of the internal layer decreases with L.

Fig. 4 shows the scaling of numerical estimates of the height and width of the transition layer of the stationary solution, as a function of νL, for 128 computations ranging over ε^2 values from 1.4 to 16, and lengths L from 30 to 130. The linear growth of the maximum solution amplitude with νL seen in Fig. 4(a), and the decrease in the width observed in

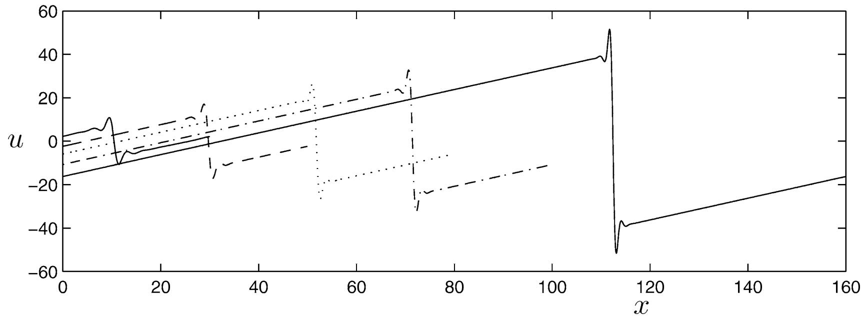

Fig. 3. Stationary viscous shock solutions for $\varepsilon^2 = 1.5$, and (from left to right, in increasing order of height) for $L = 30$, $L = 50$, $L = 80$, $L = 100$ and $L = 160$ (here the periodic domain is taken as $\Omega = [0, L]$).

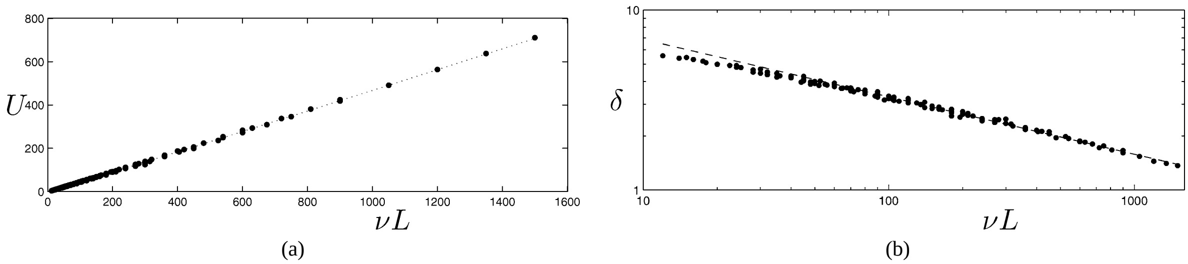

Fig. 4. (a) The maximal height U of the outer solution, estimated as the amplitude of the local minimum immediately preceding the transition layer, computed for 128 simulations of the DKS equation; the least-squares best fit straight line through the last 70 points shows the linear dependence on $\nu L = (\varepsilon^2 - 1)L$. (b) Log–log representation of the width δ of the interior layer as a function of νL, showing a power-law decrease. The width is numerically estimated as the separation between the last local minimum for $x < 0$ and the first local maximum for $x > 0$ (see Fig. 2), and the dashed line represents a least-squares best fit straight line (to the log–log graph) for the 70 data points with $\nu L \geqslant 100$, corresponding to the scaling $\delta \approx 14.3(\nu L)^{-0.319}$.

Fig. 4(b), are consistent with the two distinguished asymptotic limits as follows.

Since in the outer region, $|x| \gtrsim \delta/2$, we have $\partial_x \sim 1/L$, the approximately linear outer solution satisfies the asymptotic balance

$$uu_x \sim \nu u \quad \text{or} \quad u_x \sim \nu,$$

which implies

$$u_o(x) \approx \begin{cases} \nu(x - L/2), & x \in (\delta/2, L/2], \\ \nu(x + L/2), & x \in [-L/2, -\delta/2). \end{cases}$$

The maximum height of the outer solution at $x \approx \mp\delta/2$ is of the order $\pm\nu(L - \delta)/2 \approx \pm\nu L/2$, and the change Δu across the transition layer is $\sim \nu L$, as seen in Fig. 4(a). Since $\nu = \varepsilon^2 - 1$, we see already that u is uniformly bounded in neither L nor ε for this viscous shock solution. Observe that $u_x = \nu$ is an exact outer solution, and would solve the DKS equation in the absence of boundary conditions; the jump arises because u is constrained to be periodic.

The appropriate dominant balance for the inner solution is $u_{xxxx} \sim -uu_x$. Within the transition layer, $\partial_x \sim 1/\delta$, so that this balance gives $u/\delta^4 \sim u^2/\delta$, or

$$\delta \sim u^{-1/3} \sim (\nu L)^{-1/3},$$

for $\nu L \gg 1$; in this limit, the width of the internal layer is $\delta \sim (\nu L)^{-1/3} \ll 1$. This scaling is consistent with the numerical data of Fig. 4(b), with improved agreement for larger νL. We remark that the analytic form of the oscillatory shock front appears to be approximately that of the sine integral function, $u_i(x) \approx C\,\mathrm{Si}(x/\delta) = C\int_0^{x/\delta}(\sin t/t)\,dt$, as suggested by Goren et al. for their simplified model.

3.2. Bounds for the viscous shock solution

From the above scaling, we may read off bounds on the norms of solutions. We have already seen that the amplitude, for a fixed ε^2, grows linearly in

L: $\|u\|_\infty = \mathcal{O}(\nu L)$. This indicates that the dynamics of the destabilized KS equation are not extensive; local amplitudes are not asymptotically independent of system size. A similar conclusion is deduced from consideration of the L^2 bound: the contribution of the outer layer to the energy is

$$\|u\|^2 = 2\int_{\delta/2}^{L/2} u^2\,dx \sim 2\int_0^{L/2}\left[\nu\left(x-\frac{L}{2}\right)\right]^2 dx$$
$$= \frac{1}{12}\nu^2 L^3 = \frac{1}{12}\left(\varepsilon^2-1\right)^2 L^3 \quad (\nu L \gg 1).$$

The corresponding contribution of the internal layer is clearly of lower order. Thus we find for the transition layer shock-like solution

$$\|u\| \sim \left(\varepsilon^2-1\right)L^{3/2}. \tag{11}$$

This bound falls within the range given by Theorem 1, and is consistent with our previous results.

The scaling (11) is particularly interesting, since it shows that for sufficiently large, but fixed $\varepsilon^2 > 1$, there exists a solution to the DKS equation with $\|u\| \sim L^{3/2}$. For the KS equation with $\varepsilon^2 = 1$, the best bound conjectured is $\|u\| \sim L^{1/2}$ (corresponding to a global L-independent bound for the amplitude of u); this (counter)example shows that such extensive scaling no longer holds for the linearly unstable KS equation. It implies that while the $\mathcal{O}(\varepsilon^{16/5}L^{8/5})$ bound may be improved, it cannot be better than $\mathcal{O}(\varepsilon^2 L^{3/2})$ for $\varepsilon^2 > 1$.

4. Discussion

The dissipativity and analyticity of solutions of the DKS equation (3) are unsurprising, though it is useful to establish these results in the light of the importance of this equation in applications. However, our viscous shock example is particularly instructive, as it sheds light on the failure to date of methods establishing dissipativity for the KS equation, to prove the conjectured extensivity, $\|u\| \sim L^{1/2}$. The methods in [23,25,26] all involve the construction of comparison functions whose effect is an essentially uniform vertical displacement in the linear dispersion relation for the unstable modes. As we have shown in this Letter, any such techniques also work for the DKS equation, even in the case of a linear destabilizing term, simply by modifying the extent of the shift in the dispersion relation.

For any future proof to succeed in establishing extensivity for the KS equation (1), and a uniform L^∞ bound on the solutions, it must thus necessarily be *inapplicable* to the DKS equation with $\varepsilon^2 > 1$. We note that in the unstable DKS equation, we cannot achieve extensivity because as L increases, we have Fourier modes with positive linear growth rates arbitrarily close to $k = 0$. A successful demonstration of extensivity for the KS equation, $\varepsilon^2 = 1$, or the stabilized DKS equation with $\varepsilon^2 < 1$, should probably make explicit use of the fact that the dispersion relation $\omega(k)$ satisfies $\lim_{k\to 0}\omega(k) \leqslant 0$. We conjecture that the proof of extensivity for the KS equation will require one to study and estimate the nonlinear interactions between modes, especially for small k, in detail.

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