# Random Walk on Plane

Seyyed Aliasghar Hosseini  Luis Goddyn

Department of Mathematics, Simon Fraser University
Burnaby, BC V5A 1S6, Canada

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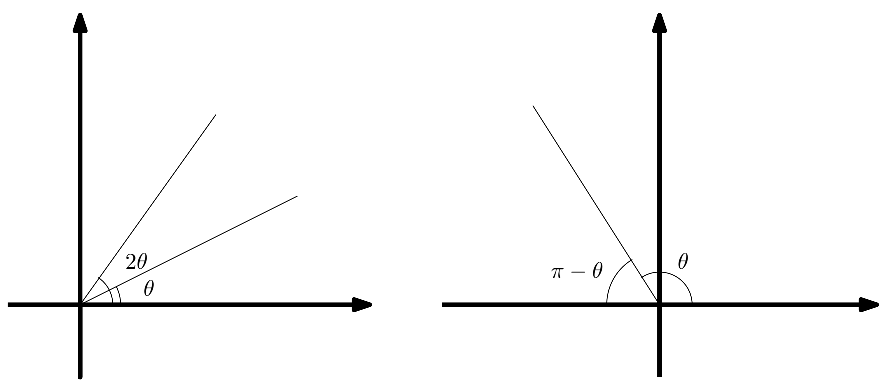
**Abstract**

In this short note we will solve the following question:
What is the probability that a random motion particle which is currently at $(x, y)$, hits the $x$-axis at $x > 0$ before hitting the negative side?

## 1 Answer

Let us first solve the problem for $(x, y) = (1, 1)$. In this case the probability that the particle hits the $x$-axis before hitting the $y$-axis is $\frac{1}{2}$ by symmetry and if the particle is on the $y$-axis it will hit the positive half of the $x$-axis (before hitting the negative half of it) with probability $\frac{1}{2}$. Therefore a particle at point (1,1) hits the $x$-axis at $x > 0$ with probability $\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

We can generalize this idea to solve the original question. Without loss of generality we can assume that $y > 0$.




It is easy to see that all points on a straight line that passes through origin have the same answer. Let $\theta = arctan\left(\frac{y}{x}\right)$ (the slope of the line) and let $Pr(\theta)$ be the answer for the points on this line. Then by generalizing the above strategy we have formula 1. See the above figure.

$$Pr(\theta) = \begin{cases} \frac{1}{2} + \frac{1}{2}Pr(2\theta) & 0 < \theta \leq \frac{\pi}{2} \\ 1 - Pr(\pi - \theta) & \frac{\pi}{2} < \theta < \pi \end{cases} \tag{1}$$

This is a recursive formula and we claim that

$$Pr(\theta) = \frac{\pi - \theta}{\pi} \tag{2}$$

which holds in 1. Therefore if 2 is true for $\theta_0$ it will be true for $\frac{\theta_0}{2^n}$ and $\pi - \frac{\theta_0}{2^n}$ for every $n$.

We know that $Pr\left(\frac{\pi}{2}\right) = \frac{1}{2}$ and this holds in 1 and 2. To show that equation 2 is true for $\theta = \frac{k\pi}{2^n}$ for $0 \leq k \leq 2^n$, assume that 2 is true for $\theta_0 = \frac{k\pi}{2^{n-1}}$ for $0 \leq k \leq 2^{n-1}$, therefore 2 is true for $\theta_1 = \theta_0/2 = \frac{k\pi}{2^n}$ and also for $\theta_2 = \pi - \theta_1 = \pi - \frac{k\pi}{2^n}$ for $0 \leq k \leq 2^{n-1}$. Hence it is true for $\theta = \frac{k\pi}{2^n}$ for $0 \leq k \leq 2^n$.

The set $\{\frac{k\pi}{2^n} : \forall n, 0 \leq k \leq 2^n\}$ is dense in $[0, \pi]$ and $Pr(\theta)$ is continuous by definiton, therefore 2 is true for all values of $\theta \in [0, \pi]$.