ON A CLASS OF POLYNOMIALS RELATED TO BARKER SEQUENCES

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ABSTRACT. For odd integer $n > 0$, we introduce the class $\mathcal{L}P_n$ of Laurent polynomials

$$P(z) = (n+1) + \sum_{\substack{k=1 \\ k \,-\, \mathrm{odd}}}^{n} c_k(z^k + z^{-k}),$$

with all coefficients c_k equal to -1 or 1. Such polynomials arise in the study of Barker sequences of even length – integer sequences having minimal possible autocorrelations. We prove that polynomials $P \in \mathcal{L}P_n$ have large Mahler measures, namely, $M(P) > (n+1)/2$. We conjecture that minimal Mahler measures in the class $\mathcal{L}P_n$ are attained by the polynomials $R_n(z)$ and $R_n(-z)$, where

$$R_n(z) = (n+1) + \sum_{\substack{k=-n \\ k \,-\, \mathrm{odd}}}^{n} z^k$$

is a polynomial with all the coefficients $c_k = 1$. We prove

$$M(R_n) > n - \frac{2}{\pi} \log n + O(1).$$

The results of experimental computations on polynomials in the class $\mathcal{L}P_n$ suggest two conjectures which could shed light on the long standing Barker problem.

1. INTRODUCTION

Let a_0, a_1, ..., a_n be the sequence of complex numbers, all of modulus $|a_j| = 1$. For this sequence, the *aperiodic autocorrelation* coefficients c_k, $-n \leqslant k \leqslant n$ are defined by

$$(1.1) \qquad c_k := \sum_{j=0}^{n-k} a_j \bar{a}_{k+j} \quad \text{for} \quad 0 \leqslant k \leqslant n \quad \text{and} \quad c_{-k} := c_k.$$

The sequences a_k having small autocorrelation values $|c_k|$ have a long history in signal processing, see [1], [6], [7]. In particular, the *Barker sequences* are those which have all numbers a_k equal to 1 or -1 and $|c_k| \leqslant 1$ for $k \neq 0$. It is widely believed that only finitely many Barker sequences exist. Turyn and Storer [18] proved that no Barker sequences of odd length exist for $n \geqslant 13$. Various restrictions on the possible values of n were derived in papers [4], [5], [8], [12], [16]. It is known that if a Barker sequence of even length $n > 13$ exists, then either $n = 189\,260\,468\,001\,034\,441\,522\,766\,781\,604$, or $n > 2 \cdot 10^{30}$ by the result of [9]. Though the non-existence problem still remains unsolved and has for more than 45 years. Kaltofen and Mossinghoff [2] introduced a polynomial version of the Barker conjecture in terms of irreducibility of polynomials. This is a theme we exploit further in this paper.

Recall that a polynomial $p(z) \in \mathbb{Z}[z]$ is called a *Littlewood polynomial*, if all coefficients of $p(z)$ are equal to 1 or -1. The set of Littlewood polynomials is denoted by

$$(1.2) \qquad \mathcal{L}_n := \left\{ p(z) = a_0 + \cdots + a_n z^n : a_j = 1 \text{ or } a_j = -1, 0 \leqslant j \leqslant n \right\}.$$

The polynomial $p \in \mathcal{L}_n$ is called a *Barker polynomial*, if the coefficients a_0, a_1, $\ldots$, a_n form a Barker sequence of length $n + 1$. Since the autocorrelations of the coefficients do not change in magnitude by replacing the polynomial $p(z)$ with $p(-z)$ or $-p(-z)$, one can normalize Barker polynomials using the conditions $a_n = a_{n-1} = 1$. By the results of [18], there exist precisely five normalized Barker polynomials of even degree n, namely: $z^2 + z - 1$, $z^4 + z^3 + z^2 - z + 1$, $z^6 + z^5 + z^4 - z^3 - z^2 + z - 1$, $z^{10} + z^9 + z^8 - z^7 - z^6 - z^5 + z^4 - z^3 - z^2 + z - 1$, and $z^{12} + z^{11} + z^{10} + z^9 + z^8 - z^7 - z^6 + z^5 + z^4 - z^3 + z^2 - z + 1$. The only known normalized Barker polynomials of odd degree n are three polynomials $z + 1$, $z^3 + z^2 - z + 1$ and $z^3 + z^2 + z - 1$. It is believed that there are no more Barker polynomials of odd degree. In [2], it was shown that the non-existence of Barker polynomials of even degree for $n > 13$ would follow from the irreducibility of certain integer polynomials in $\mathbb{Z}[x]$. By using the results of Saffari [13] on the property of *flatness*, Borwein and Mossinghoff [3] proved that if an infinite sequence of Barker polynomials exists, they would have extremely large L_s norms on the unit circle. This would give answers to the old conjectures of Littlewood [19], [20] and Newman [10], [11] for the class of Littlewood polynomials $p(z) \in \mathcal{L}_n$.

In the present paper we introduce and study a new class of polynomials associated to the Barker polynomials (with a focus on those of odd degree).

Definition 1.1. *Let n be an odd integer. Define the class $\mathcal{LP}_n$ as the class of polynomials of the form*

$$P(z) = (n + 1) + \sum_{k=-n, \ k \ odd}^{n} c_k z^k,$$

with coefficients $c_k = -1$ or 1 and $c_{-k} = c_k$.

Barker polynomials $p(z) \in \mathcal{L}_n$ are related to the polynomials in $\mathcal{LP}_n$ in the following way: if $p(z)$ is a Barker polynomial of odd degree, then the Laurent polynomial defined by $P(z) := p(z)p(1/z)$ belongs to the class $\mathcal{LP}_n$ as can be seen easily from the equations (1.1). It seems that the special form of the polynomials $P(z) \in \mathcal{LP}_n$ is natural in the investigations of Barker polynomials of odd degree. Our paper is just a starting point for these investigations. We shall concentrate the efforts in the two main directions. In the first place, we give the results on the L_s norms and especially the Mahler measure of the polynomials in the class $\mathcal{LP}_n$. Secondly, we state some conjectures on the measures and the irreducibility of $P \in \mathcal{LP}_n$.

2. Main Results

We start by making some definitions. Recall that a function $P(z)$ is called a *Laurent polynomial* (centered at origin), if it is a polynomial in z and $1/z$ with complex coefficients, $P(z) \in \mathbb{C}[z, 1/z]$. For any real number $s \geq 1$, the L_s norm of $P(z)$ on the unit circle $|z| = 1$ is defined by

$$||P||_s = \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^s dt \right)^{1/s}.$$

If $P \neq 0$ is a Laurent polynomial, then $P(z) = z^{-m}Q(z)$ for some polynomial $Q \in \mathbb{C}[z]$, $Q(0) \neq 0$ and some $m \in \mathbb{N}$:

$$Q(z) := c_0 + c_1 z^2 + \cdots + c_n z^n = c_n \prod_{j=1}^{n} (z - \alpha_j) \in \mathbb{C}[z].$$

The Mahler measure of $P(z)$ and $Q(z)$ is defined by

$$M(P) := M(Q) = |c_n| \prod_{j=1}^{n} \max\{1, |\alpha_j|\}.$$

In view of Jensen's formula, one has

$$(2.1) \qquad \log M(P) = \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{it})| dt.$$

In Proposition 2.1 we list some basic properties of the polynomials in the class $\mathcal{L}P_n$. Among all polynomials in $\mathcal{L}P_n$, polynomials with all coefficients c_k equal to 1 or, alternatively, all coefficients $c_k = -1$ are of special interest. We define

$$(2.2) \qquad R_n(z) := (n+1) + \sum_{\substack{k=-n \\ k - \text{odd}}}^{n} z^k.$$

We will show that $R_n(z)$ and $R_n(-z)$ has several interesting extremal properties.

Proposition 2.1. *Let $P \in \mathcal{L}P_n$. Then $P(e^{it})$ takes real non-negative values for $t \in [0, 2\pi)$. $P(z) = 0$ holds for some z of modulus 1 if and only if $P(z) = R_n(z)$ and $z = -1$ or $P(z) = R_n(-z)$ and $z = 1$. For each $P \in \mathcal{L}P_n$, one has*

$$||P||_1 = n + 1,$$

and

$$||P||_2 = \left((n+1)^2 + (n+1)\right)^{1/2}.$$

In addition, we also have

$$||P||_4 \leqslant ||R_n||_4 = \left((n+1)(3n^3 + 29n^2 + 49n + 24)/3\right)^{1/4}.$$

In Proposition 2.2 below we identify a more precise subclass of the polynomials in $\mathcal{L}P_n$, related to the products of Barker polynomials. We note that this is just a restatement of the theorem of Turyn and Storer [15], [18]. The proof is also given in [3].

Proposition 2.2. *Suppose that $p(z) \in \mathcal{L}_n$ is a Barker polynomial of odd degree n. Then $n = 4m^2 - 1$ for some $m \in \mathbb{Z}$ and the coefficients c_k of the polynomial $P(z) = p(z)p(1/z) \in \mathcal{L}P_n$ in the formula (1.1) satisfy $c_{n+1-k} = -c_k$ for $1 \leqslant k \leqslant n$, so that $P(z)$ can be written as*

$$P(z) = (n+1) + T(z) + T(1/z),$$

where the polynomial

$$T(z) := c_n z^n + \ldots c_k z^{k-1} + \cdots + c_1 z$$

is negative reciprocal: $z^{n+1}T(1/z) = -T(z)$.

The result of Theorem 4.1 in the paper [3] is that the hypothetical Barker polynomials $p(z)$ of degree n have Mahler measures $M(p) > \sqrt{n+1} - 1$ for n large enough. On the other hand, it is known that $M(p) < ||p||_1 < \sqrt{n+0.91}$ holds for all Littlewood polynomials of degree n, by Theorem 5.2 of [3] which was proven by refining the estimate of Newman. Therefore one possible approach for solving the non-existence problem of Barker would be to show that the Mahler measure of any Barker polynomial would be too large to satisfy an upper bound which we speculate is likely. However, completely new methods are needed to improve the earlier estimates - an observation made by Newman himself [10]. In search of a new approach, we give the next result on the uniform lower bound for the Mahler measures of polynomials in $\mathcal{L}P_n$.

Theorem 2.3. *Suppose that $P \in \mathcal{L}P_n$. Then we have*

$$M(P) > \frac{n+1}{2}.$$

The constant $1/2$ in the Theorem 2.3 does not seem to be optimal and we hope to refine it in subsequent attempts. We conjecture that:

Conjecture 2.4. *Among all the polynomials $P \in \mathcal{L}P_n$ the polynomials $R_n(z)$ and $R_n(-z)$ have minimal Mahler measures:*

$$\min_{P \in \mathcal{L}P_n} M(P) = M(R_n).$$

We have verified the Conjecture 2.4 computationally up to degree $n = 39$.

Conjecture 2.5. *The Mahler measures of polynomials $R_n(z)$ satisfy*

$$M(R_n) = n - 1 + \delta_n,$$

where the error term $\delta_n < 0.725$ for all n sufficiently large and the sequence δ_n is slowly decreasing. It seems that δ_n is bounded from below, probably $\delta_n > 0$. If this is true, then the monotone convergence theorem implies the existence of the limit $\lim_{n \to \infty} (M(R_n) - (n-1))$.

This second conjecture has been verified for polynomials R_n of degree $n < 250$. If proven, these conjectures would lead to the non-trivial improvement of Theorem 4.1 of [3]: if $p(z)$ is a Barker polynomial, then $M(p)^2 = M(p(z)(p(1/z))$ for $p(z)p(1/z) \in \mathcal{L}P_n$ implies $M(p) > \sqrt{n-1}$ for n sufficiency large. Note that the trivial upper bound $M(P) < ||P||_1 = n+1$ for all $P \in \mathcal{L}P_n$ follows from the monotonicity of L norms and Proposition 2.1. We remark that the computation of exact values of Mahler measure of polynomials is quite difficult, see, for instance, a survey by C. Smyth [14]. As a first step towards Conjecture 2.4 we prove the inequality for the Mahler measures of polynomials R_n:

Theorem 2.6. *For the polynomials $R_n(z)$ defined above:*

$$M(R_n) > n - \frac{2}{\pi} \log n + O(1),$$

as $n \to \infty$.

We would like to conclude this section with a conjecture on the reducibility of polynomials in the class $\mathcal{L}P_n$ which, combined with the Proposition (2.2) implies the non-existence of Barker polynomials of odd degree $n \geqslant 5$. This conjecture has been verified computationally for $n < 75$. An analogous conjecture for Barker polynomials in the even degree case was introduced in [2].

Conjecture 2.7. *For each $n \geqslant 5$, the subset of polynomials $P \in \mathcal{L}P_n$ with coefficients $c_{n+1-k} = -c_k$ for $1 \leqslant k \leqslant n$ in (1.1) contains only two elements which are reducible in $\mathbb{Z}[z, 1/z]$ (up to the factor of the form $\pm z^m$, $m \in \mathbb{Z}$). These elements are $P(z) = R_n(z)$ and $P(z) = R_n(-z)$ defined by (2.2). The polynomial $R_n(z)$ factors into $(z-1)(1-1/z)Q(z)$, where $Q \in \mathbb{Z}[z, 1/z]$ is irreducible.*

3. Lemmas

For the proofs of Theorems 2.3 and 2.6, we need the following results.

Lemma 3.1. *Let $k \in \mathbb{Z}$. Then we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left(4 \cos^2(kt/2) \right) dt = 0,$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left(4 \sin^2(kt/2) \right) dt = 0.$$

Proof. Consider $f(z) = (z^k + 1)^2$. Then $|f(e^{ikt})| = |e^{ikt} + 1|^2 = |e^{ikt/2} + e^{-ikt/2}|^2 = 4\cos^2(kt/2)$. By (2.1), the first integral is just $\log M((z^k + 1)^2) = 0$. Similarly, the second integral is $\log M((z^k - 1)^2) = 0$. □

Lemma 3.2. *Let a, b, x and α be real numbers. Suppose $a + b \cos \alpha \geq 0$. Then we have*

$$a \sin x + b \sin (x + \alpha) = c \sin (x + \beta),$$

where

$$c = \sqrt{a^2 + b^2 + 2ab \cos \alpha}$$

and

$$\beta = \tan^{-1} \left(\frac{b \sin \alpha}{a + b \cos \alpha} \right).$$

The proof is straightforward and is omitted.

Lemma 3.3. *Let $n \in \mathbb{N}$ be odd and $I(a) := [a - 1/n, a + 1/n] \subseteq \mathbb{R}$ for $a \in \mathbb{R}$. Then the function*
$$h(t) := (n+1)\sin t + \sin (n+1)t$$

satisfies the following
$$\log |h(t)| = \begin{cases} \log n + \log |t| + O(1), & \text{if } t \in I(0), \\ \log n + O(1/n), & \text{if } t \in I(\pi/2) \cup I(3\pi/2), \\ 3 \log n + 3 \log |t - \pi| + O(1), & \text{if } t \in I(\pi). \end{cases}$$

Proof. Since $h(t)$ is $C^\infty(\mathbb{R})$, so $h(t)$ is well approximated by its Taylor expansion. Indeed, if we let $N := n + 1$, then the derivatives of $h(t)$ are
$$h^{(k)}(t) = \begin{cases} (-1)^{\frac{k}{2}}(N \sin t + N^k \sin Nt) & \text{if } k \equiv 0 \pmod 2, \\ (-1)^{\frac{k-1}{2}}(N \cos t + N^k \cos Nt) & \text{if } k \equiv 1 \pmod 2. \end{cases}$$
We also observe that N is even and $\log N = \log n + O(1/n)$.

We now suppose $t \in I(a)$.

Suppose first that $a = \pi/2$ or $a = 3\pi/2$. Then we have $h(a) = \pm N$, $|h'(t)| \leqslant 2N$, $|t - a| \leqslant 1/n$ and the Taylor expansion of $h(t)$ about a is
$$h(t) = \pm N + h'(\theta)(t - a)$$
for some $\theta \in I(a)$. It follows that
$$\log |h(t)| = \log N + \log \left(1 + \frac{h'(\theta)(t - a)}{N} \right) = \log n + O(1/n).$$

For $a = 0$, we have $h(0) = h''(0) = 0$, $h'(0) = 2N$ and $|h'''(t)| < 2N^3$. Hence we have
$$h(t) = 2Nt + \frac{h'''(\theta)t^3}{6} = 2Nt \left(1 + \frac{h'''(\theta)t^2}{12N} \right)$$
for some $\theta \in I(0)$. Now since
$$\left| \frac{h'''(\theta)t^2}{12N} \right| < \frac{1}{6} \left(\frac{N}{n} \right)^2 = \frac{1}{6} \left(1 + \frac{1}{n} \right)^2 \leq \frac{2}{3},$$
so
$$\frac{1}{3} < 1 - \left| \frac{h'''(\theta)t^2}{12N} \right| \leq \left| 1 + \frac{h'''(\theta)t^2}{12N} \right| \leq 1 + \left| \frac{h'''(\theta)t^2}{12N} \right| < \frac{5}{3}.$$
Hence
$$\log |h(t)| = \log N + \log |t| + O(1).$$

Finally, for $a = \pi$, we have $h(a) = h'(a) = h''(a) = h^{(4)}(a) = 0$, $h'''(t) = N - N^3$ and $|h^{(5)}(t)| < 2N^5$. Hence we get
$$h(t) = \frac{N - N^3}{3!}(t - \pi)^3 + \frac{h^{(5)}(\theta)}{5!}(t - \pi)^5 = \frac{N - N^3}{6}(t - \pi)^3 \left(1 + \frac{h^{(5)}(\theta)(t - \pi)^2}{20(N - N^3)} \right)$$
for some $\theta \in I(\pi)$. Now since
$$\left| \frac{h^{(5)}(\theta)(t - \pi)^2}{20(N - N^3)} \right| < \frac{N^5}{10(N^3 - N)n^2} = \frac{1}{10} \left(\frac{(n+1)^4}{n^3(n+2)} \right) < \frac{1}{10} \left(1 + \frac{1}{n} \right)^3 \leq \frac{4}{5},$$
so
$$\frac{1}{5} < \left| 1 + \frac{h^{(5)}(\theta)(t - \pi)^2}{20(N - N^3)} \right| < \frac{9}{5}.$$
Hence
$$\log |h(t)| = \log(N^3 - N) + \log |t - \pi|^3 + O(1) = 3 \log n + 3 \log |t - \pi| + O(1).$$
This completes the proof. $\qquad\square$

Lemma 3.4. *Let $n \geq 2$. If $a \in \{0, \pi\}$, then we have*

$$\int_{a-2/n}^{a+2/n} \log |\sin t| dt = -\frac{4}{n} \log n + O\left(\frac{1}{n}\right),$$

and

$$\int_{a-1/n}^{a+1/n} \log |\sin (t + \beta(n,t))| dt = -\frac{2}{n} \log n + O\left(\frac{1}{n}\right)$$

where

$$\beta(n,t) = \tan^{-1}\left(\frac{\sin(nt)}{n+1+\cos(nt)}\right).$$

If $a \in \{\pi/2, 3\pi/2\}$, then we have

$$\int_{a-2/n}^{a+2/n} \log |\sin t| dt \ll \frac{1}{n} \quad and \quad \int_{a-1/n}^{a+1/n} \log |\sin (t + \beta(n,t))| dt \ll \frac{1}{n^2}.$$

Proof. If $a = \frac{\pi}{2}$ or $a = \frac{3\pi}{2}$, then $0 < \sin\left(\frac{\pi}{2} - 1\right) \leqslant |\sin t| \leqslant 1$ in the interval $[a - 2/n, a + 2/n]$. Hence

$$\left|\int_{a-2/n}^{a+2/n} \log |\sin t| dt\right| \leq \left(\frac{4}{n}\right) \left|\log \sin\left(\frac{\pi}{2} - 1\right)\right| \ll \frac{1}{n}.$$

If $a = 0$ or $a = \pi$, then we write

$$\log |\sin t| = \log \left|\frac{\sin(t-a)}{t-a}\right| + \log |t - a|.$$

Note that function $|\sin(t - a)/(t - a)|$ is continuous and strictly positive in $[a - 2/n, a + 2/n]$. Hence

$$\int_{a-2/n}^{a+2/n} \log \left|\frac{\sin(t-a)}{t-a}\right| dt \ll \frac{1}{n}.$$

On the other hand,

$$\int_{a-2/n}^{a+2/n} \log |t - a| dt = \frac{4}{n}\left(\log \frac{2}{n} - 1\right) = -\frac{4}{n} \log n + O\left(\frac{1}{n}\right)$$

and hence

$$\int_{a-2/n}^{a+2/n} \log |\sin t| dt = -\frac{4}{n} \log n + O\left(\frac{1}{n}\right).$$

From Lemma 3.2 with $a = n + 1, b = 1, x = t$ and $\alpha = nt$, we have

$$\sin (t + \beta(n,t)) = \frac{(n+1)\sin t + \sin(n+1)t}{c(n,t)}$$

where $c(n,t) = \sqrt{(n+1)^2 + 1 + 2(n+1)\cos(nt)}$. Let $h(t) = (n+1)\sin t + \sin(n+1)t$. By Lemma 3.3, if $a = \pi/2$ or $3\pi/2$, then $\log |h(t)| = \log n + O(1/n)$ for all $t \in [a - 1/n, a + 1/n]$. Hence

$$\int_{a-1/n}^{a+1/n} \log |h(t)| dt = 2\frac{\log n}{n} + O(1/n^2).$$

In addition, $\log c(n,t) = \log n + O(1/n)$. Hence

$$\int_{a-1/n}^{a+1/n} \log |\sin (t + \beta(n,t))| dt \ll \frac{1}{n^2}.$$

If $a = 0$, then $\log |h(t)| = \log n + \log |t| + O(1)$ for all $t \in [a - 1/n, a + 1/n]$ and hence

$$\int_{a-1/n}^{a+1/n} \log |h(t)| dt = \frac{2 \log n}{n} + \int_{-1/n}^{1/n} \log |t| dt + O(1/n) = O(1/n).$$

If $a = \pi$, then $\log |h(t)| = 3 \log n + 3 \log |t - \pi| + O(1)$ for all $t \in [a - 1/n, a + 1/n]$ and hence

$$\int_{a-1/n}^{a+1/n} \log|h(t)|dt = \frac{6\log n}{n} + 3\int_{\pi-1/n}^{\pi+1/n} \log|t-\pi|dt + O(1/n) = O(1/n).$$

Hence

$$\int_{a-1/n}^{a+1/n} \log|\sin(t+\beta(n,t))|dt = -\frac{2}{n}\log n + O\left(\frac{1}{n}\right). \qquad \square$$

4. Proofs of the propositions and theorems

Proof of Proposition 2.1. Let $P \in \mathcal{L}P_n$. Then

$$P(z) = (n+1) + \sum_{k=-n,\ k\text{ odd}}^{n} c_k z^k = (n+1) + \sum_{k=1,\ k\text{ odd}}^{n} c_k(z^k + z^{-k}).$$

For $z = e^{it}$, $t \in [0, 2\pi)$,

$$P(e^{it}) = (n+1) + \sum_{\substack{k=1 \\ k-\text{ odd}}}^{n} c_k(e^{ikt} + e^{-ikt}) = (n+1) + \sum_{\substack{k=1 \\ k-\text{ odd}}}^{n} 2c_k \cos kt \geqslant (n+1) - (n+1) = 0,$$

since n is odd and $c_k = \pm 1$. The equality is possible if and only if all $c_k \cos(kt) = -1$. In particular, for $k = 1$, $\cos t = 1$ or $\cos t = -1$. In the first case, $t = 0$, hence all the coefficients $c_k = -1$. In the second case $t = \pi$ and all $c_k = 1$. This proves that $P(e^{it}) \geqslant 0$ for all $P \in \mathcal{L}P_n$ and $P(e^{it}) = 0$ only if $P(z) = R_n(\pm z)$. Thus for any $P \in \mathcal{L}P_n$

$$\|P\|_1 = \frac{1}{2\pi}\int_0^{2\pi}|P(e^{it})|dt = \frac{1}{2\pi}\int_0^{2\pi}P(e^{it})dt = (n+1) + \sum_{\substack{k=1 \\ k-\text{ odd}}}^{n}\frac{2c_k}{2\pi}\int_0^{2\pi}\cos kt\,dt = (n+1).$$

By Parserval's formula,

$$\|P\|_2^2 = (n+1)^2 + \sum_{\substack{k=1 \\ k-\text{ odd}}}^{n} 2c_k^2 = (n+1)^2 + (n+1).$$

It remains to compute L_4 norm. By Parserval's formula, $\|P\|_4^4$ is equal to the sum of squares of coefficients of the polynomial P^2. Clearly, if all $c_k = 1$, then all the coefficients of P^2 are positive and achieve maximal values. Thus maximal L_4 norm is achieved by the polynomial R_n. By direct calculation,

$$
\begin{aligned}
R_n(z)^2 &= \left((n+1) + \sum_{k=-n,k\text{ odd}}^{n} z^k\right)^2 \\
&= (n+1)^2 + 2(n+1)\sum_{k=-n,k\text{ odd}}^{n} z^k + \left(\sum_{k=-n,k\text{ odd}}^{n} z^k\right)^2 \\
&= (n+1)^2 + 2(n+1)\sum_{k=-n,k\text{ odd}}^{n} z^k + \sum_{\substack{m=-2n \\ m\text{ even}}}^{2n}\left(\sum_{\substack{k_1,k_2=-n,k_i\text{ odd} \\ k_1+k_2=m}}^{n}\right)z^m \\
&= \sum_{m=-2n}^{2n} d_m z^m
\end{aligned}
$$

where

$$d_m := \begin{cases} 2(n+1) & \text{if } m \text{ is odd and } |m| \leq n, \\ 0 & \text{if } m \text{ is odd and } |m| > n, \\ (n+1)(n+2) & \text{if } m = 0, \\ n+1-\frac{|m|}{2} & \text{if } m \text{ is even } 0 < |m| \leq 2n. \end{cases}$$

Therefore,

$$
\begin{aligned}
\|R_n(z)\|_4^4 &= \sum_{m=-2n}^{2n} d_m^2 \\
&= ((n+1)(n+2))^2 + (n+1)(2(n+1))^2 + 2\sum_{m=1}^{n}(n+1-m)^2 \\
&= \frac{1}{3}(n+1)\left(3n^3 + 29n^2 + 49n + 24\right). \qquad \square
\end{aligned}
$$

Proof of Theorem 2.3. As in the proof of Proposition 2.1, for $z = e^{it}$, $t \in [0, 2\pi)$

$$
P(e^{it}) = \sum_{k=1,\ k\ \text{odd}}^{n} 2(1 + c_k \cos kt).
$$

Now, let A be the set of indices k for which $c_k = 1$ and B the set of indices k for which $c_k = -1$. Then

$$
\begin{aligned}
1 + c_k \cos kt &= \begin{cases} 1 + \cos kt & \text{if } k \in A, \\ 1 - \cos kt & \text{if } k \in B, \end{cases} \\
&= \begin{cases} 2\cos^2(kt/2) & \text{if } k \in A, \\ 2\sin^2(kt/2) & \text{if } k \in B. \end{cases}
\end{aligned}
$$

Then

$$
P(e^{it}) = \sum_{k \in A} 4\cos^2(kt/2) + \sum_{k \in B} 4\sin^2(kt/2).
$$

Since every summand on the right hand side of this identity is non-negative, in view of AM-GM inequality, we obtain

$$
P(e^{it}) \geqslant \frac{n+1}{2}\left(\prod_{k \in A} 4\cos^2(kt/2) \prod_{k \in B} 4\sin^2(kt/2)\right)^{\frac{2}{n+1}}.
$$

Note that there may be at most finitely many points $t \in [0, 2\pi]$, where the expressions on both sides of the last inequality vanish. For all the other t, we can take the logarithm of both sides and get

$$
\log|P(e^{it})| \geqslant \log\frac{n+1}{2} + \frac{2}{n+1}\left(\sum_{k \in A} \log 4\cos^2(kt/2) + \sum_{k \in B} \log 4\sin^2(kt/2)\right).
$$

The summands on both sides now are integrable for $t \in [0, 2\pi]$. By Lemma 3.1, the integrals of $\log 4\cos^2(kt/2)$ and $\log 4\sin^2(kt/2)$ vanish. Thus we have

$$
\log M(P) > \log\frac{n+1}{2}
$$

by (2.1). Observe that the inequality must be strict, since the AM-GM equality can hold at most on finite number of points e^{it}. This completes the proof. $\qquad \square$

Proof of Theorem 2.6. It suffices to consider the "+" case in R_n, since the "−" case is $R_n(-z)$. We may also assume $n \geq 3$. Let $z := e^{it}$. Then

$$
\begin{aligned}
R_n(z) &= (n+1) + \sum_{k=-n,\ k\ \text{odd}}^{n} z^k \\
&= (n+1) + \frac{z^{n+1} - \bar{z}^{(n+1)}}{z - \bar{z}} \\
&= \frac{(n+1)\sin t + \sin(n+1)t}{\sin t}.
\end{aligned}
$$

Now using Lemma 3.2 with $a = n + 1$, $b = 1$, $x = t$ and $\alpha = nt$, we have

$$R_n(e^{it}) = c(n,t) \frac{\sin(t + \beta(n,t))}{\sin t}$$

with

$$c(n,t) = \sqrt{(n+1)^2 + 1 + 2(n+1)\cos(nt)}, \qquad \beta(t) = \tan^{-1}\left(\frac{\sin(nt)}{n + 1 + \cos(nt)}\right),$$

since $a + b\cos\alpha = (n+1) + \cos nt \geqslant n > 0$. Observe that

(4.1) $$n \leqslant c(n,t) \leqslant n + 2,$$

and

(4.2) $$|\beta(t)| \leq \tan^{-1}\left(\frac{1}{n}\right) < \frac{1}{n}.$$

The last inequality follows from $\tan^{-1}|t| \leq |t|$ for all $t \in \mathbb{R}$. By Jensen's formula, we have

$$2\pi \log M(R_n) - \int_0^{2\pi} \log c(n,t)dt \;\; = \;\; \int_0^{2\pi} \left(\log|R_n(e^{it})| - \log c(n,t)\right) dt$$

(4.3) $$= \;\; \int_0^{2\pi} \log|\sin(t + \beta(n,t))| - \log|\sin t|dt.$$

We now estimate the integral of $\log|\sin(t + \beta(n,t))|$ by using Lemma 3.4. Let

$$I = \int_0^{2\pi} \log|\sin(t + \beta(n,t))|dt = \int_{-1/n}^{2\pi - 1/n} \log|\sin(t + \beta(n,t))|dt$$

because the integral is periodic with period 2π. We now write

(4.4) $$I = \int_{I(0) \cup I(\pi/2) \cup I(\pi) \cup I(3\pi/2)} \log|\sin(t + \beta(n,t))|dt + \int_J \log|\sin(t + \beta(n,t))|dt$$

where $I(a) = [a - 1/n, a + 1/n]$ as in Lemma 3.3 and J is the complement of the disjoint union $I(0) \cup I(\pi/2) \cup I(\pi) \cup I(3\pi/2)$ in $[-1/n, 2\pi - 1/n)$.

In view of Lemma 3.4, we first have

(4.5) $$\int_{I(0) \cup I(\pi/2) \cup I(\pi) \cup I(3\pi/2)} \log|\sin(t + \beta(n,t))|dt = -\frac{4}{n}\log n + O\left(\frac{1}{n}\right).$$

Since the function $|\sin t|$ is increasing for $t \in [0, \pi/2]$ and $t \in [\pi, 3\pi/2]$, we find

$$\int_{J \cap ([0,\pi/2] \cup [\pi,3\pi/2])} \log|\sin(t + \beta(n,t))|dt \;\; \geq \;\; \int_{J \cap ([0,\pi/2] \cup [\pi,3\pi/2])} \log|\sin(t - 1/n)|dt$$

$$= \;\; \left(\int_0^{\pi/2 - 2/n} + \int_\pi^{3\pi/2 - 2/n}\right) \log|\sin t|dt.$$

Similarly, since the function $|\sin t|$ is decreasing in intervals $[\pi/2, \pi]$ and $[3\pi/2, 2\pi]$, we find

$$\int_{J \cap ([\pi/2,\pi] \cup [3\pi/2,2\pi])} \log|\sin(t + \beta(n,t))|dt \;\; \geq \;\; \int_{J \cap ([\pi/2,\pi] \cup [3\pi/2,2\pi])} \log|\sin(t + 1/n)|dt$$

$$= \;\; \left(\int_{\pi/2 + 2/n}^\pi + \int_{3\pi/2 + 2/n}^{2\pi}\right) \log|\sin t|dt.$$

Therefore, we have

$$\int_J \log|\sin(t + \beta(n,t))|dt \;\; \geq \;\; \int_0^{2\pi} \log|\sin t|dt - \left(\int_{\pi/2 - 2/n}^{\pi/2 + 2/n} + \int_{3\pi/2 - 2/n}^{3\pi/2 + 2/n}\right) \log|\sin t|dt$$

(4.6) $$= \;\; \int_0^{2\pi} \log|\sin t|dt + O\left(\frac{1}{n}\right)$$

by Lemma 3.4. In view of (4.4), (4.5) and (4.6), we have

$$\int_0^{2\pi} \log|\sin(t+\beta(n,t))|dt \geq \int_0^{2\pi} \log|\sin t|dt - \frac{4}{n}\log n + O\left(\frac{1}{n}\right).$$

Hence from (4.3),

$$2\pi \log M(R_n) > \int_0^{2\pi} \log c(n,t)dt - \frac{4}{n}\log n + O\left(\frac{1}{n}\right).$$

Finally, since

$$c(n,t) = \sqrt{(n+1)^2 + 1 + (2n+1)\cos t} = n\sqrt{1 + O(1/n)}$$

so $\log c(n,t) = \log n + O(1/n)$. Hence

$$\log M(R_n) > \log n - \frac{2}{n\pi}\log n + O\left(\frac{1}{n}\right).$$

Applying the exponent on both sides of the above inequality and using $e^{-t} > 1 - t$ for $t > 0$ one obtains

$$M(R_n) > n\left(1 - \frac{2}{n\pi}\log n + O\left(\frac{1}{n}\right)\right) = n - \frac{2}{\pi}\log n + O(1),$$

as claimed. $\square$

References

[1] R.H. Barker, *Group synchronizing of binary digital systems*, Communication theory, 273–287, Butterworths Sci. Pub., London, 1953.

[2] P. Borwein, E. Kaltofen, M.J. Mossinghoff *Irreducible polynomials and Barker sequences*, ACM Commun. Comput. Algebra **41** (2007), no. 3-4, 118–121.

[3] P. Borwein, M.J. Mossinghoff, *Number theory and polynomials*, 71–88, London Math. Soc. Lecture Note Ser., 352, Cambridge Univ. Press, Cambridge, 2008.

[4] S. Eliahou, M. Kervaire, *Barker sequences and difference sets*, Enseign. Math. (2) **38** (1992), no. 3–4, 345–382.

[5] S. Eliahou, M. Kervaire, B. Saffari, *A new restriction on the lengths of Golay complementary sequences*, J. Combin. Theory Ser. A **55** (1990), no. 1, 49–59.

[6] P. Fan, M. Darnell, *Sequence design for communications applications*, Research Studies Press, Somerset, England, 1996.

[7] J. Jedwab, *A survey of the merit factor problem for binary sequences*, Sequences and Their Applications, Proceedings of SETA 2004, Lecture Notes in Computer Science **3486**, 30–55, Springer Verlag, Berlin, 2005.

[8] J. Jedwab, S. Lloyd, *A note on the nonexistence of Barker sequences*, Des. Codes Cryptogr. **2** (1992), no. 1, 93–97.

[9] M. J. Mossinghoff, *Wieferich Pairs and Barker Sequences*, Des. Codes Cryptogr., **53** (2009), no. 3, 149-163.

[10] D.J. Newman, *Norms of polynomials*, Amer. Math. Monthly **67** (1960), 778–779.

[11] D. J. Newman, *An L_1 extremal problem for polynomials*, Proc. Amer. Math. Soc. **16** (1965), 1287–1290.

[12] K.H. Leung, B. Schmidt, *The field descent method*, Des. Codes Cryptogr. **36** (2005), no. 2, 171–188.

[13] B. Saffari, *Barker sequences and Littlewood two–sided conjectures on polynomials with ±1 coefficients*, Séminaire d'Analyse Harmonique, Année 1989/90, 139–151, Univ. Paris XI, Orsay, 1990.

[14] C. Smyth, *Mahler measure of algebraic numbers: a survey*, Number theory and polynomials, 322??349, London Math. Soc. Lecture Note Ser., 352, Cambridge Univ. Press, Cambridge, 2008.

[15] R. Turyn *On Barker codes of even length*, IEEE Trans. Inform. Theory **51** (1963), no. 9, 1256.

[16] R. Turyn *Character sums and difference sets*, Pacific J. Math. **15** (1965), 319–346.

[17] R. Turyn, *Sequences with small correlation*, Error Correcting Codes (Proc. Sympos. Math. Res. Center, Madison, Wis.), 195-228, John Wiley, New York, 1968.

[18] R. Turyn, J. Storer, *On binary sequences*, Proc. Amer. Math. Soc. **12** (1961), 394–399.

[19] J.E. Littlewood, *On polynomials $\sum^n \pm z^m, \sum^n e^{\alpha_m i} z^m$, $z = e^{\Theta i}$*, J. London Math. Soc. **41** (1966), 367–376.

[20] J.E. Littlewood, *Some Problems in real and complex analysis*, D.C. Heath and Co., Lexington, Mass., 1968.

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