Title: Conditional Bounds for Small Prime Solutions of Linear Equations

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ABSTRACT: Let a_1, a_2, a_3 be non-zero integers with $\gcd(a_1, a_2, a_3) = 1$ and let b be an arbitrary integer satisfying $\gcd(b, a_i, a_j) = 1$ for $i \neq j$ and $b \equiv a_1 + a_2 + a_3 \pmod 2$. In a previous paper [3] which completely settled a problem of A. Baker, the 2nd and 3rd authors proved that if a_1, a_2, a_3 are not all of the same sign, then the equation $a_1p_1 + a_2p_2 + a_3p_3 = b$ has a solution in primes p_j satisfying

$$\max_{1\le j\le 3} p_j \le 3|b| + (3 \max_{1\le j\le 3} |a_j|)^A$$

where $A > 0$ is an absolute constant. In this paper, under the generalized Riemann hypothesis, the authors obtain a much more precise bound for the solutions p_j. In particular they obtain $A < 4 + \varepsilon$ for some $\varepsilon > 0$. In connection with Linnik's constant they also prove that if ℓ, q are coprime positive integers and ℓ is odd, then there is a prime $p \ll q^2 \log^{10} q$ of the form $p = \ell + (p' + p'')q$ where p' and p'' are also primes.

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CONDITIONAL BOUNDS FOR SMALL PRIME SOLUTIONS OF LINEAR EQUATIONS

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Let a_1, a_2, a_3 be non-zero integers with $\gcd(a_1, a_2, a_3) = 1$ and let b be an arbitrary integer satisfying $\gcd(b, a_i, a_j) = 1$ for $i \neq j$ and $b \equiv a_1+a_2+a_3 \pmod 2$. In a previous paper [3] which completely settled a problem of A. Baker, the 2nd and 3rd authors proved that if a_1, a_2, a_3 are not all of the same sign, then the equation $a_1p_1 + a_2p_2 + a_3p_3 = b$ has a solution in primes p_j satisfying

$$\max_{1\le j\le 3} p_j \le 3|b| + (3 \max_{1\le j\le 3} |a_j|)^A$$

where $A > 0$ is an absolute constant. In this paper, under the Generalized Riemann Hypothesis, the authors obtain a more precise bound for the solutions p_j. In particular they obtain $A < 4+\varepsilon$ for some $\varepsilon > 0$. An immediate consquence of the main result is that the Linnik's courtant is less than or equal to 2.

1. Introduction

In previous papers [3], [4], [5] the second and third authors completely settled a problem of A. Baker [1, Lemma 6] (see also the introduction in [3]) and studied the solubility and insolubility of some additive equations in prime variables. In particular, they considered [3] Vinogradov's type of equations

$$a_1p_1 + a_2p_2 + a_3p_3 = b \tag{1.1}$$

in prime variables p_1, p_2, p_3. Here a_1, a_2, a_3 are non-zero integral coefficients such that

$$(a_1, a_2, a_3) = 1 \tag{1.2}$$

and b is an arbitrary integer satisfying the conditions:

$$(1.3) \qquad (b, a_i, a_j) = 1 \quad \text{for} \quad 1 \le i < j \le 3 \ ,$$

$$(1.4) \qquad b \equiv a_1 + a_2 + a_3 \pmod 2 \ .$$

Here and in the sequel, for integers $n_1, \cdots, n_s$ not all equal to zero, $(n_1, \cdots, n_s)$ denotes their greatest common divisor. In [3], the second and third authors proved

Theorem LT. *Subject to the conditions (1.2)-(1.4), there exist effective absolute constants* $A_1, A_2 > 0$ *such that*
(i) *if* a_1, a_2, a_3 *are all positive, then equation (1.1) is soluble in primes* p_1, p_2, p_3 *whenever*

$$(1.5) \qquad b \ge (3 \max\{a_1, a_2, a_3\})^{A_1} \ ;$$

(ii) *if* a_1, a_2, a_3 *are not all of the same sign, then equation (1.1) has a prime solution* p_1, p_2, p_3 *satisfying*

$$(1.6) \qquad \max\{p_1, p_2, p_3\} \le 3|b| + (3 \max\{|a_1|, |a_2|, |a_3|\})^{A_2} \ .$$

This theorem includes many interesting special cases. For instance, when $a_1 = a_2 = a_3 = 1$ and b is a positive odd integer, then Theorem LT(i) is the classical three primes theorem of Vinogradov. Another interesting example is that, for any pair of coprime positive integers ℓ and q, $\ell \le q$, if we take in Theorem LT(ii) $a_1 = 1$, $a_2 = -q$, $a_3 = q$ and $b = \ell$ or $\ell + q$ according as ℓ is odd or even, then there is a prime p_1 in the arithmetic progression $\ell + kq$, $k = 0, 1, \cdots$ such that $p_1 \ll q^{A_2}$. If L denotes the infimum of the c's such that every such arithmetic progression contains a prime $p \ll q^c$, the so-called *Linnik's constant*, this example also shows that $A_2 \ge L$. Analogous example, namely, $a_1 = a_2 = q$, $a_3 = q + 1$ and $b = \ell + kq$ for some suitable k so that (1.3) to (1.5) hold, shows that $A_1 \ge L + 1$. So the bounds given in (1.5) and (1.6) are of the right order of infinity. However, the upper bounds we can assign to A_1 and A_2 are still far too large. Recently, the first author has shown that $A_1, A_2 < 4191$. In this paper, we shall replace (1.5) and (1.6) by much more precise estimates under the assumption of the Generalized Riemann Hypothesis (GRH). We shall prove

Theorem 1. *Assuming the GRH, we can replace the bounds in (1.5) and (1.6) repsectively by*

$$(1.7) \qquad b \gg f(a_1, a_2, a_3)$$

and

$$\max_{1\le j\le 3}\{|a_j|p_j\} \ll |b| + f(a_1,a_2,a_3) \tag{1.8}$$

where

$$f(a_1,a_2,a_3) := \frac{|a_1a_2a_3|}{(a_1,a_2)(a_1,a_3)(a_2,a_3)}\left\{\frac{|a_1a_2|}{(a_1,a_3)(a_2,a_3)} + (a_1,a_3) + (a_2,a_3)\right\}\log^{10}(3\max_{1\le j\le 3}|a_j|)\ .$$

In particular, $f(a_1,a_2,a_3) \ll (a_1a_2)^2|a_3|\log^{10}(3\max_{1\le j\le 3}|a_j|)$ *and the* A_1, A_2 *in Theorem LT can be taken respectively as* $5+\varepsilon$ *and* $4+\varepsilon$ *for any* $\varepsilon > 0$.

Remarks. (I) To appreciate the precision of the above bounds, let us consider, say, the estimate (1.8) in the situation where a_1, and a_2 are bounded and a_3 varies. Then (1.8) asserts that equation (1.1) has solutions p_1, p_2, p_3 in which $p_3 \ll |a_3|^{-1}b + \log^{10}|a_3|$, a function which grows much slower than $|a_3|$. Such information on the location of the solutions p_1, p_2, p_3 is not obtainable from the type of results like (1.6) even when the best possible value of $A_2(\ge 1)$ is obtained.
(II) Let $1\le \ell \le q$, $(\ell,q)=1$ and consider the equation

$$p - qp' - qp'' = \ell \quad \text{or} \quad \ell + q$$

according as ℓ is odd or even. Our Theorem 1 (the bound in (1.8)) ensures that, under the GRH there is always a prime p in the arithmetic progression $\ell + kq$, $k = 0,1,2,\cdots$, such that $p \ll q^2\log^{10} q$. [(*)] Furthermore, k can be specified as a sum of two primes if ℓ is odd or of the form $p'+p''+1$ if ℓ is even.

We shall reduce our Theorem 1 to the following system of 4 linear equations:

$$\alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 = \kappa, \quad r_j n_j - u_j = p_j \quad (j=1,2,3)\ , \tag{1.9}$$

where n_1,n_2,n_3 vary over the natural numbers and p_1,p_2,p_3 are prime variables. Suppose $\alpha_1,\alpha_2,\alpha_3$ are pairwise coprime non-zero integers and r_j,u_j $(j=1,2,3)$ are integers such that

$$0 \le u_j < r_j, \quad (r_j,u_j) = (r_j,\alpha_j) = 1\ . \tag{1.10}$$

Further, let κ be any integer such that

$$r_1r_2r_3(\alpha_1(u_1+1) + \alpha_2(u_2+1) + \alpha_3(u_3+1) - \kappa) \quad \text{is even}\ . \tag{1.11}$$

(*) It has been conjectured that the bound for p may be $q\log^2 q$. Our estimate here is only slightly weaker than the well-known bound $q^2\log^2 q$ obtained directly under the GRH.

We have

Theorem 2. (i) *If $\alpha_1, \alpha_2, \alpha_3$ are all positive, then the system (1.9) is soluble whenever*

$$\kappa \gg \alpha_1\alpha_2\alpha_3(r_1 + r_2 + \alpha_1\alpha_2 r_3^2) \log^{10}(3 \max_{1\le j\le 3} \alpha_j r_j) \; ; \tag{1.12}$$

(ii) *if $\alpha_1, \alpha_2, \alpha_3$ are not all of the same sign, then the system (1.9) has a solution which satisfies*

$$\max_{1\le j\le 3}\{|\alpha_j| n_j\} \ll |\kappa| + |\alpha_1\alpha_2\alpha_3|(r_1 + r_2 + |\alpha_1\alpha_2| r_3^2) \log^{10}(3 \max_{1\le j\le 3} |\alpha_j r_j|) \; . \tag{1.13}$$

To recover Theorem 1, we take in Theorem 2 $r_1 = (a_2, a_3)$, $r_2 = (a_1, a_3)$ and $r_3 = (a_1, a_2)$. Since $(a_1, a_2, a_3) = 1$, we have $r_2 r_3 = (a_1, a_2 a_3)|a_1$. Take $\alpha_1 = a_1(r_2 r_3)^{-1}$ and similarly $\alpha_2 = a_2(r_1 r_3)^{-1}$, $\alpha_3 = a_3(r_1 r_2)^{-1}$. Then clearly, α_1, α_2, α_3 are pairwise coprime and $(r_j, \alpha_j) = 1$ for $j = 1, 2, 3$. Equation (1.1) implies $p_j \equiv a_j^{-1} b \pmod{r_j}$, so we let $u_j \equiv -a_j^{-1} b \pmod{r_j}$, $0 \le u_j < r_j$ for $j = 1, 2, 3$. Condition (1.3) then ensures that $(r_j, u_j) = 1$. Also, we see that $a_1 u_1 + a_2 u_2 + a_3 u_3 \equiv -b \pmod{k}$ holds for $k = r_1 r_2 r_3$, since it holds for $k = r_1, r_2, r_3$ which are pairwise coprime. Finally, take $\kappa = (a_1 u_1 + a_2 u_2 + a_3 u_3 + b)(r_1 r_2 r_3)^{-1}$, then

$$r_1 r_2 r_3\Big(\sum_{j=1}^{3} \alpha_j (u_j + 1) - \kappa\Big) = \sum_{j=1}^{3} a_j (r_j - 1)(u_j + 1) + a_1 + a_2 + a_3 - b$$

and condition (1.11) is clearly equivalent to (1.4). The primes p_1, p_2, p_3 in (1.9) satisfy the equation $a_1 p_1 + a_2 p_2 + a_3 p_3 = b$. Since $b = r_1 r_2 r_3 \kappa - (a_1 u_1 + a_2 u_2 + a_3 u_3) \le r_1 r_2 r_3 \kappa$ for $a_1, a_2, a_3 > 0$, hypothesis (1.12) is now a consequence of hypothesis (1.7) and the first part of Theorem 1 follows from Theorem 2(i). When a_1, a_2, a_3 are not all of the same sign, we have $|a_j| p_j \le r_1 r_2 r_3 |\alpha_j| n_j$ and $r_1 r_2 r_3 |\kappa| \le |b| + 3 \max_{1\le j\le 3}\{|a_j| r_j\} = |b| + 3 r_1 r_2 r_3 \max_{1\le j\le 3} |\alpha_j|$. Whence (1.8) follows from (1.13).

Like Theorem LT, the proof of our Theorem 2 is also built on the circle method. While the techniques we developed in [3] have successful applications in some other related problems (see [4], [5]), they are not efficient in giving good upper bounds to A_1 and A_2. Our aim in this present article is to obtain the sharpest possible estimates in (1.5) and (1.6) under the GRH. To achieve this, we have to modify much of the previous arguments in [3] by injecting new techniques and ideas.

One novel idea is a further averaging of some error terms over the major arcs. This consideration leads to the following mean value result (see Lemma 2(ii) below)

$$\sum_{\chi,\chi'(\mathrm{mod}\, rq)} \left|\sum_{h=1}^{q} C_\chi(nh,q,r,u)C_{\chi'}(-nh,q,r,u)\right| = q\phi(rq)^2$$

involving the generalised Gaussian sums $C_\chi(m,q,r,u)$ defined in (2.8). It is remarkable that we actually obtain an exact evaluation for the above sum on the left side.

In addition, the major arcs in our arguments are also very specialized and delicate.

2. Notation and some preliminary lemmas

We shall use the standard arithmetic functions, $\mu(n)$-the Möbius function, $\Lambda(n)$-the von Mangoldt function and $\phi(n)$-the Euler totient function. The symbol p always denotes a prime. By $p^\sigma \| n$, we shall mean $p^\sigma | n$, $p^{\sigma+1} \nmid n$. We write, as usual, $e(x) = e^{2\pi i x}$ and $e_q(x) = e(x/q)$. The constants implied in the symbols $\ll$, $\gg$ and O are effectively computable.

Let α_j, r_j, u_j $(j = 1,2,3)$ and κ be as given in (1.9)–(1.11). We assume, without loss of generality, that $\alpha_3 \geq 1$. Let $N \geq N_0(\varepsilon)$ be a large parameter which satisfies

$$N \log^{-10} N \geq \varepsilon^{-4}|\alpha_1\alpha_2\alpha_3|(r_1 + r_2 + |\alpha_1\alpha_2|r_3^2) \tag{2.1}$$

for a sufficiently small $\varepsilon > 0$. Furthermore, we assume that

$$\begin{cases} N = 2\kappa/3, & \text{in case (i) of Theorem 2 ,} \\ N \geq |\kappa|/12, & \text{in case (ii) of Theorem 2 .} \end{cases} \tag{2.2}$$

Let

$$\lambda(n) := \begin{cases} \log p, & \text{if } n = p, \text{ a prime ,} \\ 0, & \text{otherwise ,} \end{cases}$$

and define for $j = 1,2,3$ the generating functions

$$S_j(x) := \sum_{N_j' < n \leq N_j} \lambda(r_j n - u_j)e(\alpha_j n x)$$

where

$$N_j := c_j N|\alpha_j|^{-1}, \quad N_j' := c_j' N|\alpha_j|^{-1} \tag{2.3}$$

and the constants $c_j > c_j' > 0$ are being determined in §3 (following (3.2)). Set

$$\tau := N_3^{-1/2} \log N \tag{2.4}$$

and define

$$I(N) := \int_{\tau/\alpha_3}^{1+\tau/\alpha_3} e(-\kappa x) S_1(x) S_2(x) S_3(x) dx \ .$$

Then plainly,

$$\begin{aligned} I(N) = & \sum_{\substack{\alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 = \kappa \\ N_j' \le n_j \le N_j, j=1,2,3}} \prod_{j=1}^{3} \lambda(r_j n_j - u_j) \\ & \ll (\log N)^3 \times \text{card}\{n_1, n_2, n_3 : \sum_{j=1}^{3} \alpha_j n_j = \kappa, \ N_j' < n_j \le N_j \quad \text{and} \\ & \qquad r_j n_j - u_j \text{ is a prime for } j = 1,2,3\} \ . \end{aligned}$$

Our objective is to show that $I(N) \gg N^2 \prod_{j=1}^{3} (|\alpha_j|^{-1} r_j \phi(r_j)^{-1})$ whereby, in view of (2.1) and (2.2), our Theorem 2 follows. For convenience, set

$$\Omega := N^2 \prod_{j=1}^{3} (|\alpha_j|^{-1} r_j \phi(r_j)^{-1}) \ , \tag{2.5}$$

$$Q := |\alpha_1 \alpha_2|^{1/2} \log^{3/2} N \ . \tag{2.6}$$

We begin by defining $\mathcal{M}$, the *major arcs*. For coprime positive integers h and q such that $h \le \alpha_3 q$ and $q \le Q$, let $m(h,q) := [(h-\tau)(\alpha_3 q)^{-1}, (h+\tau)(\alpha_3 q)^{-1})$. These intervals are pairwise disjoint and are all lying inside $[\tau/\alpha_3, 1+\tau/\alpha_3)$, since, by (2.6), (2.4) and (2.1), we have $2\tau Q < 1$. The union of these intervals $m(h,q)$ then forms our $\mathcal{M}$ and, as usual, its complement in $[\tau/\alpha_3, 1+\tau/\alpha_3)$ is the *minor arcs* $\mathcal{M}'$. Accordingly, we have the decomposition

$$\begin{aligned} I(N) = \alpha_3^{-1} & \sum_{q \le Q} \sum_{\substack{h=1 \\ (h,q)=1}}^{\alpha_3 q} \int_{-\tau/q}^{\tau/q} e(-\kappa \alpha_3^{-1}(hq^{-1}+\theta)) \prod_{j=1}^{3} S_j(\alpha_3^{-1}(hq^{-1}+\theta)) d\theta \\ & + \int_{\mathcal{M}'} e(-\kappa x) \prod_{j=1}^{3} S_j(x) dx := I_1(N) + I_2(N) \ , \end{aligned} \tag{2.7}$$

say.

For any integers m, q, r, u such that $r, q \ge 1$ and $(r,u) = 1$, we define

$$C_\chi(m,q,r,u) := \sum_{\ell=1}^{q} \chi(r\ell - u) e_q(\ell m) \tag{2.8}$$

for any Dirichlet character $\chi \pmod{rq}$. When $r=1$, $u=0$, this recovers the usual Gaussian sum $C_\chi(m) := \sum_{1\le \ell \le q} \chi(\ell) e_q(\ell m)$. When $\chi=\chi_0$, the principal character, we write

$$\text{(2.9)}\quad C_0(m,q,r,u) := C_{\chi_0}(m,q,r,u) = \sum_{\substack{\ell=1\\(r\ell-u,q)=1}}^{q} e_q(\ell m) \text{ and } C_q(m) := C_{\chi_0}(m)\ .$$

Lemma 1. (i) *If q_1, q_2 are coprime positive integers, then*

$$C_0(m, q_1q_2, r, u) = C_0(m\overline{q}_2, q_1, r, u)C_0(m\overline{q}_1, q_2, r, u)$$

where $q_1\overline{q}_1 + q_2\overline{q}_2 = 1$.

(ii) *Let $(h,q)=1$. Then $C_0(mh,q,r,u)=0$ if $(r,q)\not| m$ and $C_0(mh,q,r,u) = e_q(\overline{r}umh)C_q(m)$ if $(r,q)=1$. Here $r\overline{r}\equiv 1 \pmod{q}$.*

(iii) *Let $(h,q)=1$ and $j=1$ or 2. If $(q,\alpha_3 r_j)>1$ then $C_0(\alpha_j h, \alpha_3 q, r_j, u_j)=0$.*

Proof. (i) Write $\ell = \ell_1 q_2 + \ell_2 q_1$, $\ell_j = 1,\cdots,q_j$ $(j=1,2)$, then the defining sum for $C_0(m,q_1q_2,r,u)$ in (2.9) can be rearranged as

$$\sum_{\substack{\ell_1=1\\(rq_2\ell_1-u,q_1)=1}}^{q_1} e_{q_1}(\ell_1\, m) \sum_{\substack{\ell_2=1\\(rq_1\ell_2-u,q_2)=1}}^{q_2} e_{q_2}(\ell_2\, m) = C_0(m\overline{q}_2, q_1, r, u)C_0(m\overline{q}_1, q_2, r, u)\ .$$

This proves part (i).

(ii), (iii) First of all, if $(r,q)=1$ then

$$\text{(2.10)}\qquad C_0(mh,q,r,u) = \sum_{\substack{\ell=1\\(\ell,q)=1}}^{q} e_q((\ell+u)\overline{r}mh) = e_q(\overline{r}umh)C_q(\overline{r}mh)$$
$$= e_q(\overline{r}umh)C_q(m)\ .$$

If $(r,q)\not| m$, let $p^\sigma \| q$ such that $p|r$, $p^\sigma \not| m$. By (i), $C_0(mh,q,r,u)$ has a factor of the form

$$C_0(mhw, p^\sigma, r, u) = \sum_{\ell=1}^{p^\sigma} e_{p^\sigma}(\ell mhw)$$

where $p\not| w$. Since $p^\sigma \not| mhw$, the last sum vanishes and so does $C_0(mh,q,r,u)$.

Similarly, we have $C_0(\alpha_j h, \alpha_3 q, r_j, u_j)=0$ if $(q,r_j)>1$ since, by (1.9), $p\not| \alpha_j h$ for any $p|(q,r_j)$. Finally, if $(q,r_j)=1$ and $(q,\alpha_3)>1$, let $p|(q,\alpha_3)$. Then $p\not| r_j$ and $p^\sigma \| \alpha_3 q$ for some $\sigma\ge 2$. As above, we consider the factor $C_0(\alpha_j hw, p^\sigma, r_j, u_j)$ where $p\not| w$. However, by (2.10), $|C_0(\alpha_j hw, p^\sigma, r_j, u_j)| = |C_{p^\sigma}(\alpha_j)|$. Since $(\alpha_3,\alpha_j)=1$, $p|\alpha_3$ and $\sigma\ge 2$, we infer from (2.16) below that $C_{p^\sigma}(\alpha_j)=0$, and so does $C_0(\alpha_j h, \alpha_3 q, r_j, u_j)$. This proves Lemma 1.

The following lemma, especially its part (ii), provides the most crucial estimate we need in our argument in §3.

Lemma 2. *We have* (i) $\sum\limits_{\chi(\mathrm{mod}\, rq)} |C_\chi(m,q,r,u)| \leq q^{1/2}\phi(rq)$ *and*

(ii) $\sum\limits_{\chi,\chi'(\mathrm{mod}\, rq)} |\sum\limits_{h=1}^{q} C_\chi(nh,q,r,u)C_{\chi'}(-nh,q,r,u)| = q\phi(rq)^2$

for any integer n coprime to r.

Proof. (i) It is quite easy to establish (i). In fact, from (2.8), we have

$$\sum_{\chi(\mathrm{mod}\, rq)} \left|C_\chi(m,q,r,u)\right|^2 = \sum_{\ell_1=1}^{q}\sum_{\ell_2=1}^{q} e_q((\ell_1-\ell_2)m) \sum_{\chi(\mathrm{mod}\, rq)} \chi(r\ell_1-u)\overline{\chi}(r\ell_2-u)$$
$$= \phi(rq) \sum_{\substack{\ell=1 \\ (r\ell-u,q)=1}}^{q} 1 \ .$$

Hence $\sum\limits_{\chi(\mathrm{mod}\, rq)} |C_\chi(m,q,r,u)|^2 \leq q\phi(rq)$ and the desired inequality follows by applying Cauchy's inequality.

(ii) First, we write $q = q_1 q_2$ such that $(r,q_2) = 1$ and every prime factor of q_1 appears in r. Accordingly, each χ (mod rq) is factorizable as χ_1 (mod rq_1)$\cdot$ χ_2 (mod q_2). Writing $\ell = \ell_1 q_2 + \ell_2 q_1$, $\ell_j = 1,\cdots,q_j$ $(j=1,2)$, we find that

$$C_\chi(nh,q,r,u) = \sum_{\ell_1=1}^{q_1} \chi_1(rq_2\ell_1-u)e_{q_1}(\ell_1 nh) \sum_{\ell_2=1}^{q_2} \chi_2(rq_1\ell_2-u)e_{q_2}(\ell_2 nh)$$
$$= C_{\chi_1}(\overline{q}_2 nh, q_1, r, u)e_{q_2}(\overline{r}\,\overline{q}_1 unh)C_{\chi_2}(\overline{r}\,\overline{q}_1 nh) \ .$$

Here $q_2\overline{q}_2 \equiv 1 \pmod{q_1}$ and $q_1\overline{q}_1$, $r\overline{r} \equiv 1 \pmod{q_2}$. Analogously, χ' (mod rq) $=$ χ'_1 (mod rq_1)χ'_2 (mod q_2) and we have

$$C_\chi(nh,q,r,u)C_{\chi'}(-nh,q,r,u)$$
$$= C_{\chi_1}(\overline{q}_2 nh,q_1,r,u)C_{\chi'_1}(-\overline{q}_2 nh,q_1,r,u)C_{\chi_2}(\overline{r}\,\overline{q}_1 nh)C_{\chi'_2}(-\overline{r}\,\overline{q}_1 nh) \ .$$

Summing both sides for $h = h_1 q_2 + h_2 q_1$ such that $h_j = 1,\cdots,q_j$ $(j=1,2)$, we obtain, after some simplifications,

$$\sum\nolimits_{\chi,\chi' \ (\mathrm{mod}\ rq)} \left|\sum\nolimits_{h=1}^{q} C_\chi(nh,q,r,u)C_{\chi'}(-nh,q,r,u)\right| \tag{2.11}$$
$$= \left(\sum\nolimits_{\chi_1,\chi'_1 \ (\mathrm{mod}\ rq_1)} \left|\sum\nolimits_{h=1}^{q_1} C_{\chi_1}(nh,q_1,r,u)C_{\chi'_1}(-nh,q_1,r,u)\right|\right)\times$$
$$\times\left(\sum\nolimits_{\chi_2,\chi'_2 \ (\mathrm{mod}\ q_2)} \left|\sum\nolimits_{h=1}^{q_2} C_{\chi_2}(\overline{r}nh)C_{\chi'_2}(-\overline{r}nh)\right|\right) := \sum\nolimits_1 \sum\nolimits_2 \ ,$$

say. Let us first examine $\sum_2$. For any integer m coprime to q_2, we have

$$\sum_{h=1}^{q_2} C_{\chi_2}(nh\overline{r})C_{\chi_2'}(-nh\overline{r}) = \sum_{h=1}^{q_2} C_{\chi_2}(nmh\overline{r})C_{\chi_2'}(-nmh\overline{r}) = \overline{\chi_2\chi_2'}(m)\sum_{h=1}^{q_2} C_{\chi_2}(nh\overline{r})C_{\chi_2'}(-nh\overline{r}) \ . \tag{2.12}$$

The first equality holds because h and mh run over the same set of integers modulo q_2. The second equality follows from the fact that $C_{\chi_2}(mn) = \overline{\chi}_2(m)C_{\chi_2}(n)$. If $\chi_2\chi_2' \neq \chi_0$ then there exists an m such that $\chi_2\chi_2'(m) \neq 1$ and whence the sum $\sum\limits_{1\le h\le q_2}$ in (2.12) must vanish. Thus,

$$\sum\nolimits_2 = \sum_{\chi_2 (\mathrm{mod} q_2)} \sum_{h=1}^{q_2} |C_{\chi_2}(nh\overline{r})|^2 = \sum_{h=1}^{q_2} \sum_{\chi_2 (\mathrm{mod} q_2)} |C_{\chi_2}(nh\overline{r})|^2 = \sum_{h=1}^{q_2} \phi(q_2)^2 = q_2\phi(q_2)^2 \ , \tag{2.13}$$

by the same argument in (i). Next we consider $\sum_1$. The inner sum over h is equal to

$$\sum_{k=1}^{q_1}\sum_{\ell=1}^{q_1} \chi_1(rk-u)\chi_1'(r\ell-u)\sum_{h=1}^{q_1} e_{q_1}((k-\ell)nh) = q_1 \sum_{\substack{k=1 \\ q_1|n(k-\ell)}}^{q_1}\sum_{\ell=1}^{q_1} \chi_1(rk-u)\chi_1'(r\ell-u) = q_1\sum_{\ell=1}^{q_1}\chi_1\chi_1'(r\ell-u) \tag{2.14}$$

since $(n,r) = 1$ implies $(n,q_1) = 1$. Suppose $\chi_1\chi_1'$ is induced by the primitive character ψ (mod ν) with $\nu|rq_1$. Since r and rq_1 have the same set of prime factors, we have $(r\ell-u, rq_1) = 1$ so that $\chi_1\chi_1'(r\ell-u) = \psi(r\ell-u)$. Recall the well-known formula $\psi(s) = C_{\overline{\psi}}(1)^{-1}\sum\limits_{1\le k\le\nu}\overline{\psi}(k)e_\nu(sk)$ for primitive ψ [2, p.65], we have

$$\sum_{\ell=1}^{q_1}\chi_1\chi_1'(r\ell-u) = C_{\overline{\psi}}(1)^{-1}\sum_{k=1}^{\nu}\overline{\psi}(k)e_\nu(-ku)\sum_{\ell=1}^{q_1}e_\nu(\ell rk) \ . \tag{2.15}$$

Now $\overline{\psi}(k)\sum\limits_{1\le\ell\le q_1} e_\nu(\ell rk)$ vanishes if $\nu \not| rk$ or $(\nu,k) > 1$. So the above double sum is non-zero only if $\nu|rk$ for some k coprime to ν, that is, only if $\nu|r$. In that case, the sum in (2.15) is equal to $q_1\psi(-u)$. Since there are precisely $\phi(r)$ primitive ψ

with modulus dividing r, we find from (2.14) that

$$\sum\nolimits_1 = \sum_{\substack{\psi(\mathrm{mod}\,\nu),\nu|r \\ \psi \text{ primitive}}} \sum_{\substack{\chi_1(\mathrm{mod}\, rq_1) \\ \chi_1' = \overline{\chi}_1\psi}} |q_1^2\psi(-u)| = \phi(r)\phi(rq_1)q_1^2 = q_1\phi(rq_1)^2 .$$

Combining this with (2.13) and (2.11), we prove part (ii) of Lemma 2.

Lemma 3. *For any integers m, q, ν with $q, \nu \geq 1$, we have*

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{\nu q} e_{\nu q}(hm) = \begin{cases} 0, & \text{if } \nu \nmid m , \\ \nu C_q(m/\nu), & \text{if } \nu | m . \end{cases}$$

Proof. Let $\nu = \nu_1\nu_2$ such that $(\nu_2, q) = 1$ and every prime factor of ν_1 appears in q. By the same argument in Lemma 1(i), we show that

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{\nu q} e_{\nu q}(hm) = \sum_{\substack{h=1 \\ (h,q\nu_1)=1}}^{q\nu_1\nu_2} e_{q\nu_1\nu_2}(hm) = \sum_{\ell=1}^{\nu_2} e_{\nu_2}(\ell m) \sum_{\substack{k=1 \\ (k,q\nu_1)=1}}^{q\nu_1} e_{q\nu_1}(km)$$
$$= \Big(\sum_{\ell=1}^{\nu_2} e_{\nu_2}(\ell m)\Big) C_{q\nu_1}(m) .$$

The sum over ℓ will vanish if $\nu_2 \nmid m$. Also, it is well-known that

$$C_q(m) = \mu(q/(q,m))\phi(q)\phi(q/(q,m))^{-1} . \tag{2.16}$$

Hence $C_{q\nu_1}(m) = 0$ if there is a prime power $p^\sigma \| \nu_1$ such that $p^\sigma \nmid m$. Since $(\nu_1, \nu_2) = 1$, the first case of the lemma follows readily. The second case is straightforward.

We come now to establish the key for estimating the generating functions $S_j(x)$.

Lemma 4. *Let h, q, r, u be integers such that $(r, u) = 1$, $0 \leq u < r$, $1 \leq q$ and $r, q \leq Z^{10}$. Let $Z \ll Y < Z$ and*

$$W(\theta) := \sum_{Y<n\leq Z} \lambda(rn-u)e(n(hq^{-1}+\theta)) .$$

Then under the GRH, we have
(i) $W(\theta) = r\phi(rq)^{-1}C_0(h,q,r,u)\int_Y^Z e(t\theta)dt + O((rqZ)^{1/2}((Z|\theta|)^{1/2} + \log Z)\log Z)$ *for* $|\theta| \ll r\log^{-2} Z$,
(ii) $W(0) = r\phi(rq)^{-1}(Z-Y)C_0(h,q,r,u) + \phi(rq)^{-1}\sum_{\chi(\mathrm{mod}\, rq)} C_{\overline{\chi}}(h,q,r,u)\Phi_\chi +$ $O(\log Z)$, *where Φ_χ does not depend on h and it satisfies*

$$\Phi_\chi \ll (rZ)^{1/2}\log^2 Z\ . \tag{2.17}$$

Proof. Since $\lambda(n)=0$ if n is not a prime, we see that

$$W(\theta)=\sum_{\substack{Y<n\le Z\\ (rn-u,q)=1}} \lambda(rn-u)e(n(hq^{-1}+\theta))+O(\log q)\ .$$

Using the orthogonality relation of the characters $\chi \pmod{rq}$, we group the numbers $rn-u$ according to $n\equiv \ell \pmod q$. This leads to

$$\begin{aligned}W(\theta)&=\phi(rq)^{-1}\sum_{\substack{\ell=1\\(r\ell-u,q)=1}}^{q} e_q(\ell h)\sum_{\chi(\mathrm{mod}\, rq)}\overline{\chi}(r\ell-u)\times\\ &\quad\times\sum_{rY-u<m\le rZ-u}\lambda(m)\chi(m)e_r((m+u)\theta)+O(\log Z)\\ &=\phi(rq)^{-1}e_r(u\theta)\sum_{\chi(\mathrm{mod}\, rq)}\Big(\sum_{\ell=1}^{q}\overline{\chi}(r\ell-u)e_q(\ell h)\Big)\times\\ &\quad\times\Big(\sum_{rY-u<m\le rZ-u}\lambda(m)\chi(m)e_r(m\theta)\Big)+O(\log Z)\\ &=\phi(rq)^{-1}e_r(u\theta)\sum_{\chi(\mathrm{mod}\, rq)}C_{\overline{\chi}}(h,q,r,u)H_\chi(\theta)+O(\log Z)\ ,\end{aligned} \tag{2.18}$$

where $H_\chi(\theta):=\sum_{rY-u<m\le rZ-u}\lambda(m)\chi(m)e_r(m\theta)$.

It is well-known that [2, Chapter 19], for $t\ge T\ge 2$,

$$\psi(t,\chi):=\sum_{n\le t}\Lambda(n)\chi(n)=\delta_\chi t-\sum_{|\gamma|\le T}t^\rho\rho^{-1}+O(tT^{-1}\log^2(rqt))\ . \tag{2.19}$$

Here $\rho=\beta+i\gamma$ are the non-trivial zeros of the L-function $L(s,\chi)$ and $\delta_\chi=1$ or 0 according as χ equals to the principal character or not. Set $T:=rY-u$. Since $\lambda(n)$ and $\Lambda(n)$ differ only at $n=p^\sigma$, $\sigma\ge 2$, we find that

$$\begin{aligned}H_\chi(\theta)&=\sum_{rY-u<m\le rZ-u}\Lambda(m)\chi(m)e_r(m\theta)+O((rZ)^{1/2})\\ &=\int_T^{rZ-u}e_r(t\theta)d\psi(t,\chi)+O((rZ)^{1/2})\\ &=\delta_\chi\int_T^{rZ-u}e_r(t\theta)dt-\sum_{|\gamma|\le T}\int_T^{rZ-u}t^{\beta-1+i\gamma}e_r(t\theta)dt+\\ &\quad+O((1+|\theta|Z)\log^2 Z+(rZ)^{1/2})\ .\end{aligned} \tag{2.20}$$

Now

$$\int_T^{rZ-u} t^{\beta-1+i\gamma} e_r(t\theta)dt = r^{\beta+i\gamma} \int_{Y-ur^{-1}}^{Z-ur^{-1}} t^{\beta-1+i\gamma} e(t\theta)dt \ .$$

As shown in Lemma 3.2 of [3], the integral on the right side is

$$\ll \begin{cases} Z^{\beta}|\gamma|^{-1}, & \text{if } |\theta| \le |\gamma|(4\pi Z)^{-1} \ , \\ Z^{\beta}|\gamma|^{-1/2}, & \text{if } |\gamma|(4\pi Z)^{-1} < |\theta| \le |\gamma|(\pi Y)^{-1} \ , \\ Z^{\beta-1}|\theta|^{-1}, & \text{if } |\gamma|(\pi Y)^{-1} < |\theta| \ . \end{cases} \tag{2.21}$$

Under the GRH, $\beta = 1/2$. Hence, for $Y|\theta| \ge 2$, we have

$$\begin{aligned} &\sum_{|\gamma|\le T} \int_T^{rZ-u} t^{\beta-1+i\gamma} e_r(t\theta)dt \\ &\ll r^{1/2}\Big(\sum_{|\gamma|<\pi Y|\theta|} Z^{-1/2}|\theta|^{-1} + \sum_{\pi Y|\theta|<|\gamma|\le 4\pi Z|\theta|} Z^{1/2}|\gamma|^{-1/2} + \sum_{4\pi Z|\theta|<|\gamma|\le T} Z^{1/2}|\gamma|^{-1}\Big) \\ &\ll r^{1/2}(YZ^{-1/2}\log Z + Z^{1/2}(Y|\theta|)^{-1/2} Z|\theta|\log Z + Z^{1/2}\log^2 Z) \\ &\ll (rZ)^{1/2}((Z|\theta|)^{1/2} + \log Z)\log Z \ , \end{aligned}$$

by using the well-known zero counting formula:

$$\sum_{|\gamma|\le t} 1 = \frac{t}{\pi}\log\Big(\frac{rqt}{2\pi}\Big) + O(t\log rq) \quad \text{for} \quad t \ge 2 \ .$$

For the case $Y|\theta| < 2$, the same estimate still holds, by using the first case in (2.21) and the trivial estimate $\int_T^{rZ-u} t^{-1/2+iY} e_r(t\theta)dt \ll (rZ)^{1/2}$. Applying these in (2.20), we have

$$H_\chi(\theta) = re_r(-u\theta)\delta_\chi \int_Y^Z e(t\theta)dt + O((rZ)^{1/2}((Z|\theta|)^{1/2} + \log Z)\log Z) \ .$$

Substitution in (2.18) then yields

$$\begin{aligned} W(\theta) &= r\phi(rq)^{-1} C_0(h,q,r,u) \int_Y^Z e(t\theta)dt + O(\log Z) \\ &\quad + O(\phi(rq)^{-1} \sum_{\chi (\mathrm{mod}\, rq)} |C_{\overline{\chi}}(h,q,r,u)|(rZ)^{1/2}((Z|\theta|)^{1/2} + \log Z)\log Z) \ . \end{aligned}$$

Finally, we estimate the sum in the last O-term by Lemma 2(i). This proves part (i).

When $\theta = 0$, we take $\Phi_\chi = H_\chi(0) - \delta_\chi r(Z-Y)$, which is independent of h. By (2.19) (with $T = rY - u$) and the GRH,

$$\begin{aligned}\Phi_\chi &= \sum_{rY-u<m\le rZ-u} \Lambda(m)\chi(m) + O((rZ)^{1/2}) - \delta_\chi r(Z-Y) \\ &= \sum_{|\gamma|\le T} \{(rY-u)^\rho - (rZ-u)^\rho\}\rho^{-1} + O((rZ)^{1/2}) \\ &\ll (rZ)^{1/2} \sum_{|\gamma|\le T} |\frac{1}{2} + i\gamma|^{-1} \ll (rZ)^{1/2}\log^2 Z\ .\end{aligned}$$

Part (ii) now follows from (2.18). This completes the proof of Lemma 4.

Verifying the hypotheses by means of (2.4), (2.6), (2.3) and (2.1), we obtain from Lemma 4 the following formulas which will be used in the next section:

$$\begin{aligned}S_3(\alpha_3^{-1}(hq^{-1}+\theta)) = r_3\phi(r_3q)^{-1}C_0(h,q,r_3,u_3)\int_{N_3'}^{N_3} e(t\theta)dt \\ + O((r_3qN_3)^{1/2}((N_3|\theta|)^{1/2} + \log N)\log N)\end{aligned} \tag{2.22}$$

for $q \le \tau^{-1}$, $|\theta| \le \tau q^{-1}$.

$$\begin{aligned}S_j(h(\alpha_3q)^{-1}) = r_j\phi(r_j\alpha_3q)^{-1}(N_j - N_j')C_0(\alpha_jh,\alpha_3q,r_j,u_j) \\ + O(\phi(r_j\alpha_3q)^{-1}|\sum_{\chi(\mathrm{mod} r_j\alpha_3q)} C_{\overline{\chi}}(\alpha_jh,\alpha_3q,r_j,u_j)\Phi_\chi| + \log N)\end{aligned} \tag{2.23}$$

for $q \le Q$, $j = 1,2$. Furthermore, with $h = \theta = 0$, $q = 1$, Lemma 4(i) together with (2.1) gives

$$S_j(0) \ll N_jr_j\phi(r_j)^{-1} \quad \text{for} \quad j = 1,2\ . \tag{2.24}$$

3. Proof of Theorem 2

We now proceed to prove Theorem 2 by establishing that $I_1(N) \gg \Omega$ and $I_2(N) \ll \varepsilon\Omega$. The constants implied in the symbols $\ll$, $\gg$ and O are independent of ε.

Let us consider $I_2(N)$ first. Take any $x \in \mathcal{M}'$. By Dirichlet's theorem on diophantine approximation, there exist coprime integers h and q such that $1 \le q \le \tau^{-1}$ and $\theta := \alpha_3x - hq^{-1}$ satisfies $|\theta| < \tau q^{-1}$. Since $x \ge \tau/\alpha_3$, we have $1 \le h$. If $q \le Q$, then the two facts: $2\tau Q < 1$ and $\alpha_3x < \alpha_3 + \tau$ together imply $h \le \alpha_3q$ and hence $x \in m(h,q)$ for some $q \le Q$. This contradicts that $x \in \mathcal{M}'$. Thus, it must be $q > Q$. Now, with the help of Lemma 1 (ii), (2.16) and (2.4), we deduce from (2.22) that

$$\begin{aligned} S_3(x) &= S_3(\alpha_3^{-1}(hq^{-1}+\theta)) \\ &\ll r_3\phi(r_3q)^{-1}N_3 + (r_3qN_3)^{1/2}\log^2 N + N_3(r_3q|\theta|)^{1/2}\log N \\ &\ll Q^{-1}N_3\log\log N + (r_3\tau^{-1}N_3)^{1/2}\log^2 N + N_3(r_3\tau)^{1/2}\log N \\ &\ll Q^{-1}N_3\log\log N + r_3^{1/2}N_3^{3/4}\log^{3/2} N \end{aligned}$$

for any $x \in \mathcal{M}'$. Hence

$$\begin{aligned} (3.1)\quad I_2(N) &= \int\limits_{\mathcal{M}'} e(-\kappa x)\prod_{j=1}^{3} S_j(x)dx \\ &\ll (Q^{-1}N_3\log\log N + r_3^{1/2}N_3^{3/4}\log^{3/2} N)\int_{\tau/\alpha_3}^{1+\tau/\alpha_3} |S_1(x)S_2(x)|dx\ . \end{aligned}$$

Applying Cauchy's inequality, the last integral is

$$\begin{aligned} &\leq \prod_{j=1}^{2}\Big(\int_{\tau/\alpha_3}^{1+\tau/\alpha_3} |S_j(x)|^2dx\Big)^{1/2} = \prod_{j=1}^{2}\Big(\sum_{N_j'<n\leq N_j} \lambda(r_jn-u_j)^2\Big)^{1/2} \\ &\ll (\log N)\prod_{j=1}^{2} S_j(0)^{1/2} \ll (\log N)\prod_{j=1}^{2}(N_jr_j\phi(r_j)^{-1})^{1/2}\ , \end{aligned}$$

by (2.24). Hence, from (3.1), (2.6), (2.3), (2.1) and (2.5), we find that $I_2(N) \ll \varepsilon\Omega$ as desired.

Next, we turn to $I_1(N)$, the main contribution in (2.7). As above, we substitute $S_3(\alpha_3^{-1}(hq^{-1}+\theta))$ from (2.22). The O-term there contributes $\ll \varepsilon\Omega$ to $I_1(N)$, as it can be seen by the same argument for $I_2(N)$ in (3.1). Therefore, we can now write

$$\begin{aligned} (3.2)\quad I_1(N) &= O(\varepsilon\Omega) + \\ &+ r_3\alpha_3^{-1}\sum_{q\leq Q}\phi(r_3q)^{-1}\sum_{\substack{N_j'<n_j\leq N_j \\ j=1,2}}\sum\ \prod_{j=1}^{2}\lambda(r_jn_j-u_j)\sum_{\substack{h=1 \\ (h,q)=1}}^{\alpha_3q} C_0(h,q,r_3,u_3)\times \\ &\times e_{\alpha_3q}(h(\alpha_1n_1+\alpha_2n_2-\kappa))\int_{N_3'}^{N_3}\int_{-\tau/q}^{\tau/q} e(\alpha_3^{-1}(\alpha_1n_1+\alpha_2n_2+\alpha_3t-\kappa)\theta)d\theta dt\ . \end{aligned}$$

The double integral $\int_{N_3'}^{N_3}\int_{-\tau/q}^{\tau/q}$ is equal to $\pi^{-1}\int_{\xi_1}^{\xi_2} z^{-1}\sin z\ dz$, where

$$\xi_1 := \frac{2\pi\tau}{\alpha_3q}(\alpha_1n_1+\alpha_2n_2+\alpha_3N_3'-\kappa),\ \ \xi_2 := \frac{2\pi\tau}{\alpha_3q}(\alpha_1n_1+\alpha_2n_2+\alpha_3N_3-\kappa)\ .$$

We now choose the constants c_j, c'_j as follow: In case (i) of Theorem 3, that is, when all $\alpha_1, \alpha_2, \alpha_3$ are positive, we set $c_1 = c_2 = 1/2$, $c_3 = 5/4$ and $c'_j = 1/4$ for $j = 1, 2, 3$. In case (ii), that is, not both α_1, α_2 are positive, say $\alpha_1 < 0$, we take $c_1 = 32$, $c'_1 = 28$, $c_2 = 2$, $c'_2 = 1$, $c_3 = 48$ and $c'_3 = 12$. In view of (2.2), we find, in both cases, that $\xi_2 \geq \pi\tau N(2\alpha_3 q)^{-1}$ and $\xi_1 \leq -\pi\tau N(2\alpha_3 q)^{-1}$. Hence

$$\pi^{-1}\int_{\xi_1}^{\xi_2} z^{-1}\sin z\, dz = 1 + O(|\xi_1|^{-1} + \xi_2^{-1}) = 1 + O((\alpha_3 q)(\tau N)^{-1}). \tag{3.3}$$

Let E_1 be the contribution of the above O-term to $I_1(N)$ in (3.2). Then plainly

$$E_1 \ll r_3(\tau N)^{-1}\sum_{q\leq Q} q\phi(r_3 q)^{-1}\sum_{\substack{N'_j<n_j\leq N_j \\ j=1,2}}\sum \prod_{j=1}^{2}\lambda(r_j n_j - u_j)\times$$

$$\times\left|\sum_{\substack{h=1 \\ (h,q)=1}}^{\alpha_3 q} C_0(h,q,r_3,u_3)e_{\alpha_3 q}(h(\alpha_1 n_1 + \alpha_2 n_2 - \kappa))\right| .$$

Denote by $\sum_{(h)}$ the above inner sum over h. By Lemma 1(ii) and (2.16) we see that $C_0(h,q,r_3,u_3) = 0$ if $(q, r_3) > 1$ and $C_0(h,q,r_3,u_3) = \mu(q)e_q(hu_3\overline{r}_3)$ if $(q, r_3) = 1$. Here $r_3\overline{r}_3 \equiv 1 \pmod q$. Thus, for $(q, r_3) = 1$, we have

$$\sum_{(h)} = \mu(q)\sum_{\substack{h=1 \\ (h,q)=1}}^{\alpha_3 q} e_{\alpha_3 q}(h(\alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 u_3\overline{r}_3 - \kappa)) .$$

According to Lemma 3, the last sum vanishes if $\alpha_3 \nmid \alpha_1 n_1 + \alpha_2 n_2 - \kappa$. Otherwise, we have $\sum_{(h)} \ll \alpha_3\phi(q)$. Consequently,

$$\begin{aligned} E_1 &\ll r_3\alpha_3(\tau N\phi(r_3))^{-1}\sum_{q\leq Q} q \sum_{\substack{n_1 \quad n_2 \\ \alpha_3|\alpha_1 n_1+\alpha_2 n_2-\kappa}}\sum \prod_{j=1}^{2}\lambda(r_j n_j - u_j) \\ &\ll r_3\alpha_3(\tau N\phi(r_3))^{-1}Q^2\sum_{n_1}\lambda(r_1 n_1 - u_1)\sum_{\substack{n_2 \\ \alpha_2 n_2\equiv\kappa-\alpha_1 n_1 \pmod{\alpha_3}}}\log N \\ &\ll r_3\alpha_3(\tau N\phi(r_3))^{-1}Q^2(N_2\alpha_3^{-1}S_1(0)\log N) \ll \varepsilon\Omega , \end{aligned}$$

by (2.4), (2.6), (2.24), (2.3), (2.5) and (2.1). Hence, when (3.3) is substituted in (3.2), we have

$$I_1(N) = O(\varepsilon\Omega)+$$

$$+ r_3\alpha_3^{-1}\sum_{q\leq Q}\phi(r_3 q)^{-1}\sum_{\substack{h=1 \\ (h,q)=1}}^{\alpha_3 q} C_0(h,q,r_3,u_3)e_{\alpha_3 q}(-h\kappa)S_1(h(\alpha_3 q)^{-1})S_2(h(\alpha_3 q)^{-1}) .$$

The next step is to approximate $S_j(h(\alpha_3 q)^{-1})$ $(j=1,2)$ by the formula in (2.23). For simplicity, let us rewrite the formula as $S_j = M_j + O(R_j)$ where M_j is the main term and R_j is the remainder. Then we have

$$(3.4) \qquad I_1(N) = r_3\alpha_3^{-1}\sum_{q\le Q}\phi(r_3 q)^{-1}\sum_{\substack{h=1\\(h,q)=1}}^{\alpha_3 q} C_0(h,q,r_3,u_3)e_{\alpha_3 q}(-h\kappa)(M_1M_2+ \\ + O(|M_1|R_2+|M_2|R_1+R_1R_2)) + O(\varepsilon\Omega)\ .$$

The three terms inside the O-symbol are to be estimated by similar arguments. Consider, for instance, the term $|M_1|R_2$ and let E_2 be its total contribution to the right side of (3.4). Then by Cauchy's inequality, we have

$$(3.5) \quad E_2 \ll r_3\alpha_3^{-1}\sum_{q\le Q}\phi(r_3 q)^{-1}\Big(\sum_{\substack{h=1\\(h,q)=1}}^{\alpha_3 q}|C_0(h,q,r_3,u_3)M_1|^2\Big)^{1/2}\Big(\sum_{h=1}^{\alpha_3 q}|R_2|^2\Big)^{1/2}\ .$$

Let $j=1,2$. With reference to (2.23), we see that

$$C_0(h,q,r_3,u_3)M_j = r_j\phi(r_j\alpha_3 q)^{-1}(N_j-N_j')C_0(h,q,r_3,u_3)C_0(\alpha_j h,\alpha_3 q,r_j,u_j)$$

and Lemma 1 (ii), (iii) show that this will vanish if $(q,r_jr_3\alpha_3)>1$. So we consider only those q's such that $(q,r_jr_3\alpha_3)=1$. In this case, applying Lemma 1 (i), (ii), we have

$$|C_0(h,q,r_3,u_3)M_j| \ll r_j\phi(r_j\alpha_3 q)^{-1}N_j|C_q(\alpha_j)||C_0(\alpha_j h\overline{q},\alpha_3,r_j,u_j)|$$

where $q\overline{q}\equiv 1 \pmod{\alpha_3}$. Since $|C_q(\alpha_j)|\le (q,\alpha_j)$, we deduce that

$$\begin{aligned}
&\Big(\sum_{\substack{h=1\\(h,q)=1}}^{\alpha_3 q}|C_0(h,q,r_3,u_3)M_j|^2\Big)^{1/2}\\
&\ll r_j\phi(r_j\alpha_3 q)^{-1}N_j(q,\alpha_j)\phi(q)^{1/2}\Big(\sum_{h=1}^{\alpha_3}|C_0(\alpha_j h\overline{q},\alpha_3,r_j,u_j)|^2\Big)^{1/2}\\
&= r_j\phi(r_j\alpha_3 q)^{-1}N_j(q,\alpha_j)(\alpha_3\phi(q))^{1/2}\Big(\sum_{\substack{\ell=1\\(r_j\ell-u_j,\alpha_3)=1}}^{\alpha_3} 1\Big)^{1/2}\\
&\ll r_j\alpha_3(\phi(r_j)\phi(\alpha_3))^{-1}\phi(q)^{-1/2}(q,\alpha_j)N_j\ .
\end{aligned}$$

Next, in view of (2.23), we have

$$\sum_{h=1}^{\alpha_3 q} |R_j|^2 \ll \phi(r_j\alpha_3 q)^{-2} \sum_{h=1}^{\alpha_3 q} |\sum_{\chi(\mathrm{mod} r_j\alpha_3 q)} C_{\overline{\chi}}(\alpha_j h, \alpha_3 q, r_j, u_j)\Phi_\chi|^2 + \alpha_3 q \log^2 N$$

$$= \phi(r_j\alpha_3 q)^{-2} \sum_{\chi,\chi'(\mathrm{mod} r_j\alpha_3 q)} \Phi_\chi \overline{\Phi}_{\chi'} \sum_{h=1}^{\alpha_3 q} C_{\overline{\chi}}(\alpha_j h, \alpha_3 q, r_j, u_j) C_{\chi'}(-\alpha_j h, \alpha_3 q, r_j, u_j) +$$
$$+ \alpha_3 q \log^2 N \ .$$

Invoking (2.17) and then Lemma 2(ii), we see that the last expression is

$$\ll \phi(r_j\alpha_3 q)^{-2} r_j N_j \alpha_3 q \phi(r_j \alpha_3 q)^2 \log^4 N = \alpha_3 q r_j N_j \log^4 N \ .$$

Collecting these estimates into (3.5), we find that

$$E_2 \ll r_1 r_3 \alpha_3 (\phi(r_1)\phi(r_3)\phi(\alpha_3))^{-1} N_1 (N_2 r_2 \alpha_3^{-1})^{1/2} \log^2 N (\log\log Q)^{3/2} \sum_{q \le Q} q^{-1}(q, \alpha_1) \ .$$

The last sum over q is

$$\le \sum_{k|\alpha_1} k \sum_{\substack{q\le Q \\ k|q}} q^{-1} = \sum_{k|\alpha_1} \sum_{q \le Qk^{-1}} q^{-1} \ll \sum_{k|\alpha_1} \log Q \ll |\alpha_1|^{1/2} \log Q \ .$$

Hence, $E_2 \ll N^{3/2} r_2^{1/2} |\alpha_1\alpha_2\alpha_3|^{-1/2} \log^4 N \ll \varepsilon\Omega$, by (2.3), (2.5) and (2.1). The same bound holds for the contributions from $|M_2|R_1$ and $R_1 R_2$ in (3.4). We therefore have

$$(3.6)\ I_1(N) = r_3 \alpha_3^{-1} \sum_{q\le Q} \phi(r_3 q)^{-1} \sum_{\substack{h=1 \\ (h,q)=1}}^{\alpha_3 q} C_0(h, q, r_3, u_3) M_1 M_2 e_{\alpha_3 q}(-h\kappa) + O(\varepsilon\Omega)$$

$$= (c_1 - c_1')(c_2 - c_2') N^2 |\alpha_1\alpha_2\alpha_3|^{-1} r_1 r_2 r_3 \sum_{q \le Q} \{\phi(r_3 q)\phi(r_1\alpha_3 q)\phi(r_2\alpha_3 q)\}^{-1} T(q)$$
$$+ O(\varepsilon\Omega)$$

where

$$T(q) := \sum_{\substack{h=1 \\ (h,q)=1}}^{\alpha_3 q} C_0(h, q, r_3, u_3) C_0(\alpha_1 h, \alpha_3 q, r_1, u_1) C_0(\alpha_2 h, \alpha_3 q, r_2, u_2) e_{\alpha_3 q}(-h\kappa) \ .$$

Let us first examine $T(q)$. Using Lemma 1(ii), (iii), we see that the summand in $T(q)$ will vanish if $(q, r_1 r_2 r_3 \alpha_3) > 1$. For those q's such that $(q, r_1 r_2 r_3 \alpha_3) = 1$, we find by Lemma 1 (i), (ii) that

$$C_0(h,q,r_3,u_3)\prod_{j=1}^{2}C_0(\alpha_j h,\alpha_3 q,r_j,u_j)=$$

$$=C_q(1)e_q(\overline{r}_3u_3h)\prod_{j=1}^{2}C_q(\alpha_j)e_q(\overline{r}_ju_j\alpha_j\overline{\alpha}_3h)C_0(\alpha_jh\overline{q},\alpha_3,r_j,u_j)$$

where $q\overline{q}\equiv 1 \pmod{\alpha_3}$. If we write $h=h_1\alpha_3+h_2q$ with $h_1=1,\cdots,q,(h_1,q)=1$ and $h_2=1,\cdots,\alpha_3$, then for $(q,r_1r_2r_3\alpha_3)=1$ we have

$$\begin{aligned}T(q)&=C_q(1)C_q(\alpha_1)C_q(\alpha_2)\sum_{\substack{h_1=1\\(h_1,q)=1}}^{q}e_q(h_1(\alpha_1\overline{r}_1u_1+\alpha_2\overline{r}_2u_2+\alpha_3\overline{r}_3u_3-\kappa))\times\\&\quad\times\sum_{h_2=1}^{\alpha_3}e_{\alpha_3}(-h_2\kappa)\prod_{j=1}^{2}C_0(\alpha_jh_2,\alpha_3,r_j,u_j)\\&=\mu(q)C_q(\alpha_1)C_q(\alpha_2)C_q(\eta)F(\alpha_3)\ ,\end{aligned}$$

where $\eta:=\alpha_1\overline{r}_1u_1+\alpha_2\overline{r}_2u_2+\alpha_3\overline{r}_3u_3-\kappa$, $r_j\overline{r}_j\equiv 1 \pmod q$ and

$$F(n):=\sum_{h=1}^{n}e_n(-h\kappa)\prod_{j=1}^{2}C_0(\alpha_jh,n,r_j,u_j)$$

for any positive integer n. Substituting this into (3.6), we have

$$\begin{aligned}(3.7)\ I_1(N)&=(c_1-c_1')(c_2-c_2')N^2|\alpha_1\alpha_2\alpha_3|^{-1}r_1r_2r_3\{\phi(r_3)\phi(r_1\alpha_3)\phi(r_2\alpha_3)\}^{-1}\times\\&\quad\times F(\alpha_3)\sum_{\substack{q\le Q\\(q,r_1r_2r_3\alpha_3)=1}}\mu(q)C_q(\alpha_1)C_q(\alpha_2)C_q(\eta)\phi(q)^{-3}+O(\varepsilon\Omega)\ .\end{aligned}$$

Since $|C_q(\alpha_1)C_q(\alpha_2)C_q(\eta)|\le\alpha_1\alpha_2\phi(q)$, we see that the above sum over q converges absolutely. Let ξ be its limit. Then

$$\begin{aligned}\xi&=\sum_{\substack{q=1\\(q,r_1r_2r_3\alpha_3)=1}}^{\infty}\mu(q)C_q(\alpha_1)C_q(\alpha_2)C_q(\eta)\phi(q)^{-3}\\&=\prod_{p\nmid r_1r_2r_3\alpha_3}(1-C_p(\alpha_1)C_p(\alpha_2)C_p(\eta)\phi(p)^{-3})\end{aligned}$$

since $C_q(m)$ is multiplicative in q. For each $p\nmid r_1r_2r_3\alpha_3$, let $\omega(p)$ be the number of the integers α_1,α_2,η which are divisible by p. Then

$$\xi=\prod_{p\nmid r_1r_2r_3\alpha_3}(1-(-\phi(p))^{\omega(p)-3})\ .$$

Furthermore, $(\alpha_1, \alpha_2) = 1$ shows that $0 \le \omega(p) \le 2$. Thus, $1 - (-\phi(p))^{\omega(p)-3} \ge 1 - \phi(p)^{-2} \ge 1 - 3p^{-2}$ for $p \ge 3$. For $p = 2$, the condition (1.11) ensures that $\omega(2) \ne 1$ if $2 \not| r_1 r_2 r_3 \alpha_3$. Hence $1 - (-\phi(2))^{\omega(2)-3} = 2$ and

$$\xi \ge \prod_{p \not| 2r_1r_2r_3\alpha_3} (1 - 3p^{-2}) \gg 1 \ . \tag{3.8}$$

We turn now to $F(n)$. Similar to Lemma 1(i), we can show that $F(n)$ is multiplicative in n. Furthermore, if $p^\sigma \| \alpha_3$ then

$$F(p^\sigma) = \sum_{\substack{\ell_1=1 \\ (r_j\ell_j - u_j, p)=1}}^{p^\sigma} \sum_{\ell_2=1}^{p^\sigma} \sum_{h=1}^{p^\sigma} e_{p^\sigma}(h(\alpha_1\ell_1 + \alpha_2\ell_2 - \kappa))$$

$$= p^\sigma \times \text{card}\{\ell_1, \ell_2 : 1 \le \ell_1, \ell_2 \le p^\sigma, p \not| r_j\ell_j - u_j, \alpha_1\ell_1 + \alpha_2\ell_2 \equiv \kappa \pmod{p^\sigma}\} \ .$$

Since $p|\alpha_3$ and $(\alpha_2, \alpha_3) = 1$, each given ℓ_1 determines one $\ell_2 \pmod{p^\sigma}$ by the congruence $\alpha_1\ell_1 + \alpha_2\ell_2 \equiv \kappa \pmod{p^\sigma}$. Hence

$$\begin{aligned} F(p^\sigma) &= p^\sigma \times \text{card}\{\ell : 1 \le \ell \le p^\sigma,\ p \not| r_1\ell - u_1,\ p \not| r_2(\kappa - \alpha_1\ell) - \alpha_2 u_2\} \\ &= p^{2\sigma-1} \times \text{card}\{\ell : 1 \le \ell \le p,\ p \not| r_1\ell - u_1,\ p \not| r_2(\kappa - \alpha_1\ell) - \alpha_2 u_2\} \\ &= p^{2\sigma-1}(p - \nu(p)) \ , \end{aligned}$$

where $\nu(p) = 0$ if $p|(r_1, r_2)$, $\nu(p) = 1$ if p divides exactly one of r_1, r_2, and $\nu(p) = 1$ or 2 if $p \not| r_1 r_2$. The case $\nu(p) = 2$ occurs only if $p \not| r_1 r_2$ and $\alpha_1 r_2 u_1 + \alpha_2 r_1 u_2 \not\equiv r_1 r_2 \kappa \pmod p$. In particular, condition (1.11) ensures that such situation will not occur for $p = 2$, that is, $\nu(2) \le 1$ if $2|\alpha_3$. Hence

$$\begin{aligned} F(\alpha_3) &= \prod_{p^\sigma \| \alpha_3} p^{2\sigma-1}(p - \nu(p)) = \alpha_3^2 \prod_{\substack{p|(\alpha_3, r_1r_2) \\ p \not| (r_1, r_2)}} (1 - \frac{\nu(p)}{p}) \prod_{\substack{p|\alpha_3 \\ p \not| r_1r_2}} (1 - \frac{\nu(p)}{p}) \\ &\gg \alpha_3^2 \prod_{\substack{p|(\alpha_3, r_1r_2) \\ p \not| (r_1, r_2)}} (1 - p^{-1}) \prod_{\substack{p|\alpha_3 \\ p \not| 2r_1r_2}} (1 - 2p^{-1}) \\ &\gg \alpha_3^2 \prod_{\substack{p|(\alpha_3, r_1r_2) \\ p \not| (r_1, r_2)}} (1 - p^{-1}) \prod_{\substack{p|\alpha_3 \\ p \not| r_1r_2}} (1 - p^{-1})^2 = \alpha_3^2 \prod_{\substack{p|\alpha_3 \\ p \not| r_1}} (1 - \frac{1}{p}) \prod_{\substack{p|\alpha_3 \\ p \not| r_2}} (1 - \frac{1}{p}) \ . \end{aligned}$$

Applying this and (3.8) in (3.7), we conclude that, when Q is sufficiently large

$$
\begin{aligned}
I_1(N) \gg & N^2|\alpha_1\alpha_2\alpha_3|^{-1} r_1 r_2 r_3 \{\phi(r_3)\phi(r_1\alpha_3)\phi(r_2\alpha_3)\}^{-1}\alpha_3^2 \times \\
& \times \prod_{\substack{p|\alpha_3 \\ p \nmid r_1}} (1-p^{-1}) \prod_{\substack{p|\alpha_3 \\ p \nmid r_2}} (1-p^{-1})\xi + O(\varepsilon\Omega) \\
& \gg N^2|\alpha_1\alpha_2\alpha_3|^{-1} r_1 r_2 r_3 \{\phi(r_1)\phi(r_2)\phi(r_3)\}^{-1} + O(\varepsilon\Omega) \ .
\end{aligned}
$$

In view of (2.5) and (2.1), when ε is sufficiently small, we have $I(N) = I_1(N) + O(\varepsilon\Omega) \gg \Omega$, as desired. This completes the proof of Theorem 2.

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