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On a problem of Bourgain concerning the L_p norms of exponential sums

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Abstract For $n \geq 1$ let

$$\mathcal{A}_n := \left\{ P : P(z) = \sum_{j=1}^{n} z^{k_j} : 0 \leq k_1 < k_2 < \cdots < k_n, k_j \in \mathbb{Z} \right\},$$

that is, $\mathcal{A}_n$ is the collection of all sums of n distinct monomials. These polynomials are also called Newman polynomials. Let

$$M_p(Q) := \left(\int_0^1 \left| Q(e^{i2\pi t}) \right|^p \, dt \right)^{1/p}, \qquad p > 0.$$

We define

$$S_{n,p} := \sup_{Q \in \mathcal{A}_n} \frac{M_p(Q)}{\sqrt{n}} \qquad \text{and} \qquad S_p := \liminf_{n \to \infty} S_{n,p} \leq \Sigma_p := \limsup_{n \to \infty} S_{n,p}.$$

We show that

$$\Sigma_p \geq \Gamma(1 + p/2)^{1/p}, \qquad p \in (0, 2).$$

The special case $p = 1$ recaptures a recent result of Aistleitner [1], the best known lower bound for Σ_1.

Keywords L_p norms · Constrained coefficients · Littlewood polynomials · Newman polynomials · Sums of monomials · Bourgain's problem

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1 Introduction

Let

$$M_p(Q) := \left(\int_0^1 \left| Q(e^{i2\pi t}) \right|^p \, dt \right)^{1/p}, \qquad p > 0.$$

For $n \ge 1$ let

$$\mathcal{A}_n := \left\{ P : P(z) = \sum_{j=1}^{n} z^{k_j} : 0 \le k_1 < k_2 < \cdots < k_n, k_j \in \mathbb{Z} \right\},$$

that is, $\mathcal{A}_n$ is the collection of all sums of n distinct monomials. We define

$$S_{n,p} := \sup_{Q \in \mathcal{A}_n} \frac{M_p(Q)}{\sqrt{n}} \qquad \text{and} \qquad S_p := \liminf_{n \to \infty} S_{n,p} \le \Sigma_p := \limsup_{n \to \infty} S_{n,p}.$$

We also define

$$I_{n,p} := \inf_{Q \in \mathcal{A}_n} \frac{M_p(Q)}{\sqrt{n}} \qquad \text{and} \qquad I_p := \limsup_{n \to \infty} I_{n,p} \ge \Omega_p := \liminf_{n \to \infty} I_{n,p}.$$

The problem of calculating Σ_1 appears in a paper of Bourgain [5]. Deciding whether $\Sigma_1 < 1$ or $\Sigma_1 = 1$ would be a major step toward confirming or disproving other important conjectures. Karatsuba [7] observed that $\Sigma_1 \ge 1/\sqrt{2} \ge 0.707$. Indeed, taking, for instance,

$$P_n(z) = \sum_{k=0}^{n-1} z^{2^k}, \qquad n = 1, 2, \ldots,$$

it is easy to see that

$$M_4(P_n)^4 = 2n(n-1) + n, \tag{1.1}$$

and as Hölder's inequality implies

$$n = M_2(P_n)^2 \le M_1(P_n)^{2/3} M_4(P_n)^{4/3},$$

we conclude

$$M_1(P_n) \ge \sqrt{\frac{n^2}{2n-1}} \ge \frac{\sqrt{n}}{\sqrt{2}}. \tag{1.2}$$

Similarly, if $S_n := \{a_1 < a_2 < \cdots < a_n\}$ is a Sidon set (that is, S_n is a subset of integers such that no integer has two essentially distinct representations as the sum of two elements of S_n), then the polynomials

$$P_n(z) = \sum_{a \in S_n} z^a, \qquad n = 1, 2, \ldots,$$

satisfy (1.1) and (1.2). In fact, it was observed in [4] that

$$\min_{P \in \mathcal{A}_n} M_4(P)^4 = 2n(n-1) + n, \tag{1.3}$$

and such minimal polynomials in $\mathcal{A}_n$ are precisely constructed by Sidon sets as above.

Improving Karatsuba's result, by using a probabilistic method Aistleitner [1] proved that $\Sigma_1 \ge \sqrt{\pi}/2 \ge 0.886$. We note that Borwein and Lockhart [3] investigated the asymptotic behavior of the mean value of normalized L_p norms of Littlewood polynomials for arbitrary $p > 0$. Using the Lindeberg Central Limit Theorem and dominated convergence, they proved that

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f\in\mathcal{L}_n} \frac{(M_p(f))^p}{n^{p/2}} = \Gamma(1+p/2)$$

where $\mathcal{L}_n := \left\{P : P(z) = \sum_{j=0}^{n} a_j z^j, a_j \in \{-1,+1\}\right\}$. It follows simply from the $p = 1$ case of the the result in [3] quoted above that $\Sigma_1 \geq \sqrt{\pi/8} \geq 0.626$. Moreover, this can be achieved by taking the sum of approximately half of the monomials of $\{x^0, x^1, \ldots, x^{2n}\}$ and letting n tend to ∞.

In this note we show that

$$\Sigma_p \geq S_p \geq \Gamma(1+p/2)^{1/p}, \qquad p \in (0,2),$$

and

$$\Omega_p \leq I_p \leq \Gamma(1+p/2)^{1/p}, \qquad p \in (2,\infty).$$

The special case $p = 1$ recaptures a recent result of Aistleitner [1], the best known lower bound for Σ_1. Observe that Parseval's formula gives $\Omega_2 = \Sigma_2 = 1$.

2 New results

Theorem 2.1 *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > k_j\left(1 + \frac{c_j}{j^{1/2}}\right), \qquad j = 1,2,\ldots,$$

where $\lim_{j\to\infty} c_j = \infty$. Let

$$P_n(z) = \sum_{j=1}^{n} z^{k_j}, \qquad n = 1,2,\ldots.$$

We have

$$\lim_{n\to\infty} \frac{M_p(P_n)}{\sqrt{n}} = \Gamma(1+p/2)^{1/p}$$

for every $p \in (0,2)$.

Theorem 2.2 *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > qk_j, \qquad j = 1,2,\ldots,$$

where $q > 1$. Let

$$P_n(z) = \sum_{j=1}^{n} z^{k_j}, \qquad n = 1,2,\ldots.$$

We have

$$\lim_{n\to\infty} \frac{M_p(P_n)}{\sqrt{n}} = \Gamma(1+p/2)^{1/p}$$

for every $p \in [1,\infty)$.

Corollary 2.3 *We have* $\Sigma_p \ge S_p \ge \Gamma(1+p/2)^{1/p}$ *for all* $p \in (0,2)$.

The special case $p = 1$ of Corollary 2.3 recaptures a recent result of Aistleitner, and it is the best known lower bound in the problem of Bourgain mentioned in the introduction.

Corollary 2.4 *We have* $\Sigma_1 \ge S_1 \ge \sqrt{\pi/2}$.

Corollary 2.5 *We have* $\Omega_p \le I_p \le \Gamma(1+p/2)^{1/p}$ *for all* $p \in (2,\infty)$.

We remark here that the same results also hold for the polynomials $\sum_{j=1}^n a_j z^{k_j}$ with coefficients a_j if a general form of Salem–Zygmund theorem is used (e.g see (2) in [6]).

Our final result shows that the upper bound $\Gamma(1+p/2)^{1/p}$ in Corollary 2.5 is optimal at least for even integers.

Corollary 2.6 *For any even integer* $p = 2m \ge 2$, *we have*

$$\lim_{n\to\infty} \min_{P\in\mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} = \Gamma(1+p/2)^{1/p}.$$

Observe that a standard way to prove a Nikolskii-type inequality for trigonometric polynomials [2, p. 394] applies to the classes $\mathcal{A}_n$. Indeed,

$$\begin{aligned} M_p(P) &= \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{it})|^p\,dt\right)^{1/p} \\ &\le \left(\left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{it})|^2\,dt\right)\left(\max_{t\in[0,2\pi]} |P(e^{it})|\right)^{p-2}\right)^{1/p} \\ &= (nn^{p-2})^{1/p} = n^{1-1/p}, \end{aligned}$$

for every $P \in \mathcal{A}_n$ and $p \ge 2$, and the Dirichlet kernel $D_n(z) := 1 + z + \cdots + z^n$ shows the sharpness of this upper bound up to a multiplicative factor constant $c > 0$. So if we study the original Bourgain problem in the case of $p > 2$, we should normalize by dividing by $n^{1-1/p}$ rather than $n^{1/2}$.

3 Proofs

Let $m(A)$ denote the Lebesgue measure of measurable sets $A \subset [0,1]$. To prove Theorem 2.1 we need the complex-valued analogue of the following result of Erdős [6] (note that there is a typo in (4) in [6], the term $N^{1/2}$ should be $(N/2)^{1/2}$).

Theorem 3.1 *Let* (k_j) *be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > k_j\left(1 + \frac{c_j}{j^{1/2}}\right), \qquad j = 1, 2, \ldots,$$

where $\lim_{j\to\infty} c_j = \infty$. *Let*

$$Q_n(t) = \sum_{j=1}^n \cos(2\pi k_j(t - \theta_j)) \qquad \theta_j \in \mathbb{R}.$$

Then

$$\lim_{n\to\infty} m(\{t \in [0,1] : Q_n(t) < x(n/2)^{1/2}\}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\,dt$$

for every $x \in \mathbb{R}$.

Following the proof of Theorem 1 in Erdős's paper [6], we can calculate the moments of $|P_n(e^{i2\pi t})|^2$ on $[0, 1]$ in the same way, and the limit distribution function

$$F(x) := \lim_{n\to\infty} m(\{t \in [0,1] : |P_n(e^{i2\pi t})|^2 < xn\})$$

can be identified as $F(x) = 1 - e^{-x}$ on $[0, \infty)$. Hence the following complex-valued version of Erdős's result can be obtained. While Erdős could have easily claimed it in [6], our Theorem 3.2 below seems to be a new result.

Theorem 3.2 *Let* (k_j) *be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > k_j \left(1 + \frac{c_j}{j^{1/2}}\right), \qquad j = 1, 2, \ldots,$$

where $\lim_{j\to\infty} c_j = \infty$. *Let*

$$P_n(z) = \sum_{j=1}^{n} z^{k_j}, \qquad n = 1, 2, \ldots .$$

Then

$$\lim_{n\to\infty} m(\{t \in [0,1] : |P_n(e^{i2\pi t})|^2 < xn\}) = 1 - e^{-x}$$

for every $x \in [0, \infty)$.

We also need the following result from [8, p. 215].

Theorem 3.3 *Let* (k_j) *be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > qk_j, \qquad j = 1, 2, \ldots,$$

where $q > 1$. *Let*

$$Q_n(t) = \sum_{j=1}^{n} \cos(2\pi k_j (t - \theta_j)).$$

Then for every $r > 0$ *there are constants (depending only on* r *and* q*)* $A_{r,q} > 0$ *and* $B_{r,q} > 0$ *such that*

$$A_{r,q}\sqrt{n} \le M_r(Q_n) \le B_{r,q}\sqrt{n}$$

for every $n \in \mathbb{N}$ *and* $r > 0$.

Proof of Theorem 2.1 Let

$$Z_n(t) := \frac{1}{\sqrt{n}} P_n(e^{i2\pi t}), \qquad n = 1, 2, \ldots .$$

Observe that the functions $|Z_n|^p$, $n = 1, 2, \ldots,$ are uniformly integrable on $[0, 1]$. To see this let $a \ge 1$,

$$E := E_{n,p} := \{t \in [0,1] : |Z_n(t)|^p \ge a\}$$

and

$$F := F_n := \{t \in [0,1] : |Z_n(t)|^2 \ge a\}.$$

Using $p \in (0,2)$, we have $E \subset F$. This, together with

$$\int_0^1 |Z_n(t)|^2 \, dt = 1$$

gives $m(E) \le m(F) \le a^{-1}$ for every $a > 1$, $p \in (0,2)$, and $n \in \mathbb{N}$. Using Hölder's inequality we obtain that

$$\int_E |Z_n(t)|^p \, dt \le \left(\int_E |Z_n(t)|^2 \, dt \right)^{p/2} (m(E))^{(2-p)/2} \le a^{(p-2)/2}$$

for every $a > 1$, $p \in (0,2)$, and $n \in \mathbb{N}$, which shows that the functions $|Z_n|^p$, $n = 1, 2, \dots,$ are uniformly integrable on $[0,1]$, indeed.

By Theorem 3.2 we have

$$U_n(x) := m(\{t \in [0,1] : |Z_n(t)|^2 < x\})$$

converges to $F(x) := 1 - e^{-x}$ pointwise on $[0, \infty)$ as $n \to \infty$. Combining this with the uniform integrability of $|Z_n|^p$, $n = 1, 2, \dots,$ on $[0,1]$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_0^1 |Z_n(t)|^p \, dt &= \int_0^\infty x^{p/2} \, dF(x) = \int_0^\infty x^{p/2} F'(x) \, dx = \int_0^\infty x^{p/2} e^{-x} \, dx \\ &= \Gamma(1 + p/2).\end{aligned}$$

□

Proof of Theorem 2.2 Let, as before,

$$Z_n(t) := \frac{1}{\sqrt{n}} P_n(e^{i2\pi t}), \qquad n = 1, 2, \dots.$$

Introducing

$$X_n(t) := \frac{1}{\sqrt{n}} \operatorname{Re}(P_n(e^{i2\pi t})) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \cos(2\pi k_j t),$$

and

$$Y_n(t) := \frac{1}{\sqrt{n}} \operatorname{Im}(P_n(e^{i2\pi t})) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \sin(2\pi k_j t).$$

we have $Z_n(t) := X_n(t) + iY_n(y)$. Observe that the functions $|Z_n|^p, n = 1, 2, \dots,$ are uniformly integrable on $[0,1]$. To see this let $a > 0$ and

$$E := E_{n,p} := \{t \in [0,1] : |Z_n(t)|^{2p} \ge a\}.$$

By Theorem 3.3 (recall that $p \ge 1$) we have

$$\begin{aligned} m(E)a \le \int_0^1 |Z_n(t)|^{2p} \, dt &= \int_0^1 \left(|X_n(t)|^2 + |Y_n(t)|^2 \right)^p dt \\ &\le 2^{p-1} \int_0^1 \left(|X_n(t)|^{2p} + |Y_n(t)|^{2p} \right) dt \le 2^p B_{2p,q}^{2p}. \end{aligned}$$

Hence $m(E) \le 2^p B_{2p,q}^{2p} a^{-1}$ for every $a > 0$, $p \ge 1$, and $n \in \mathbb{N}$. Combining this with the Cauchy–Schwarz inequality and Theorem 3.3, we obtain

$$\begin{aligned}\int_E |Z_n(t)|^p\,dt &\le \left(\int_E |Z_n(t)|^{2p}\,dt\right)^{1/2} (m(E))^{1/2} \le B_{2p,q}^p (2^{p/2} B_{2p,q}^p a^{-1/2})\\ &= 2^{p/2} B_{2p,q}^{2p} a^{-1/2}\end{aligned}$$

for every $a > 0$, $p \ge 1$, and $n \in \mathbb{N}$, which shows that the functions $|Z_n|^p$, $n = 1, 2, \dots,$ are uniformly integrable on $[0, 1]$, indeed.

By Theorem 3.2 we have

$$U_n(x) := m(\{t \in [0,1] : |Z_n(t)|^2 < x\})$$

converges to $F(x) := 1 - e^{-x}$ pointwise on $[0, \infty)$ as $n \to \infty$. Combining this with the uniform integrability of $|Z_n|^p$, $n = 1, 2, \dots,$ on $[0, 1]$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_0^1 |Z_n(t)|^p\,dt &= \int_0^\infty x^{p/2}\,dF(x) = \int_0^\infty x^{p/2} F'(x)\,dx = \int_0^\infty x^{p/2} e^{-x}\,dx\\ &= \Gamma(1 + p/2).\end{aligned}$$

□

Proof of Corollary 2.6 Let $P \in \mathcal{A}_n$ be of the form $P(z) = \sum_{j=1}^n z^{k_j}$ with some integers $0 \le k_1 < k_2 < \dots < k_n$. We have

$$(P(z))^m = \left(\sum_{j=1}^n z^{k_j}\right)^m = \sum_{k=0}^\infty \left(\sum_{\substack{1\le j_1, j_2, \dots, j_m \le n,\\ k_{j_1}+k_{j_2}+\dots+k_{j_m}=k}} 1\right) z^k.$$

Hence

$$M_p(P)^p = M_2(P^m)^2 = \sum_{k=0}^\infty \left(\sum_{\substack{1\le j_1, j_2, \dots, j_m \le n,\\ k_{j_1}+k_{j_2}+\dots+k_{j_m}=k}} 1\right)^2 \ge \sum_{k=0}^\infty \left(\sum_{\substack{1\le j_1, j_2, \dots, j_m \le n,\, j_\ell \ne j_i\\ k_{j_1}+k_{j_2}+\dots+k_{j_m}=k}} 1\right)^2.$$

Now, as the number of permutations of distinct values $k_{j_1}, k_{j_2}, \dots, k_{j_m}$ is $m!$, it follows that

$$M_p(P)^p \ge (m!)^2 \binom{n}{m} = m!n(n-1)\cdots(n-m+1).$$

Hence, we have

$$\min_{P\in\mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} \ge \Gamma(1+p/2)^{1/p} \left(1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right)\right)^{1/p}.$$

Therefore

$$\liminf_{n\to\infty} \min_{P\in\mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} \ge \Gamma(1+p/2)^{1/p}. \tag{3.1}$$

By Theorem 2.2, there are polynomials $P_n \in \mathcal{A}_n$ such that

$$\lim_{n\to\infty} \frac{M_p(P_n)}{\sqrt{n}} = \Gamma(1+p/2)^{1/p}.$$

Hence

$$\limsup_{n\to\infty} \min_{P\in\mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} \le \Gamma(1+p/2)^{1/p}. \tag{3.2}$$

The corollary now follows (3.1) and (3.2). □

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