SMALL PRIME SOLUTIONS OF QUADRATIC EQUATIONS II

KWOK-KWONG STEPHEN CHOI AND JIANYA LIU

Abstract. Let $b_1, \ldots, b_5$ be non-zero integers and n any integer. Suppose that $b_1 + \cdots + b_5 \equiv n \pmod{24}$ and $(b_i, b_j) = 1$ for $1 \le i < j \le 5$. In this paper we prove that (i) if b_j are not all of the same sign, then the above quadratic equation has prime solutions satisfying $p_j \ll \sqrt{|n|} + \max\{|b_j|\}^{25/2+\varepsilon}$; and (ii) if all b_j are positive and $n \gg \max\{|b_j|\}^{26+\varepsilon}$, then the quadratic equation $b_1p_1^2 + \cdots + b_5p_5^2 = n$ is soluble in primes p_j. Our previous results are $\max\{|b_j|\}^{20+\varepsilon}$ and $\max\{|b_j|\}^{41+\varepsilon}$ in place of $\max\{|b_j|\}^{25/2+\varepsilon}$ and $\max\{|b_j|\}^{26+\varepsilon}$ above, respectively.

For any integer n, we consider the quadratic equations in the form

$$b_1p_1^2 + \cdots + b_5p_5^2 = n, \tag{1}$$

where p_j are prime variables and the coefficients b_j are non-zero integers. A necessary condition for the solubility of (1) is

$$b_1 + \cdots + b_5 \equiv n \pmod{24}. \tag{2}$$

We also suppose

$$(b_i, b_j) = 1, \quad 1 \le i < j \le 5, \tag{3}$$

and write $B = \max\{2, |b_1|, \ldots, |b_5|\}$. The main results in this note are the following two theorems.

Theorem 1. *Suppose* (2) *and* (3). *If* $b_1, \ldots, b_5$ *are not all of the same sign, then* (1) *has solutions in primes* p_j *satisfying*

$$p_j \ll \sqrt{|n|} + B^{25/2+\varepsilon},$$

where the implied constant depends only on ε.

Theorem 2. *Suppose* (2) *and* (3). *If* $b_1, \ldots, b_5$ *are all positive, then* (1) *is soluble whenever*

$$n \gg B^{26+\varepsilon},$$

where the implied constant depends only on ε.

Theorem 2 with $b_1 = \ldots = b_5 = 1$ is a classical result of Hua [3] in 1938. Theorems 1 and 2 improve our previous results in [1] with the bounds $B^{20+\varepsilon}$ and $B^{41+\varepsilon}$ in the place of $B^{25/2+\varepsilon}$ and $B^{26+\varepsilon}$ respectively.

Recently, the second author introduced in [4] an iterative procedure to deal with the enlarged major arcs in the Waring-Goldbach problem which can be used to improve the previous results substantially. In this note, we will demonstrate how to use this iterative procedure to improve our previous results in [1]. Most of the argument are similar to that in [1] and we therefore only sketch the proof here. We refer all the details to [1] and only emphasize the main difference between the arguments.

Denote by $r(n)$ the weighted number of solutions of (1), i.e.

$$r(n) = \sum_{\substack{n=b_1p_1^2+\cdots+b_5p_5^2 \\ M<|b_j|p_j^2\le N}} (\log p_1)\cdots(\log p_5),$$

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where $M = N/200$. We will investigate $r(n)$ by the circle method. To this end, we set

$$P = (N/B)^{1/5-\varepsilon}, \quad Q = N/(PL^{9000}), \quad \text{and} \quad L = \log N. \tag{4}$$

We should remark that the previous choice of P in [1] is $P = (N/B)^{1/8-\varepsilon}$. The improvement in our theorems is due to choice of larger P in (4).

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1 + 1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \quad |\lambda| \le 1/(qQ) \tag{5}$$

for some integers a, q with $1 \le a \le q \le Q$ and $(a, q) = 1$. We denote by $\mathfrak{M}(a, q)$ the set of α satisfying (5), and define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(a, q), \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \backslash \mathfrak{M}. \tag{6}$$

It follows from $2P \le Q$ that the major arcs $\mathfrak{M}(a, q)$ are mutually disjoint. Let

$$S_j(\alpha) = \sum_{M < |b_j| p^2 \le N} (\log p) e(b_j p^2 \alpha),$$

where $e(x) := e^{2\pi i x}$. Then we have

$$r(n) = \int_0^1 S_1(\alpha) \cdots S_5(\alpha) e(-n\alpha) d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}. \tag{7}$$

For χ mod q, we define

$$C(\chi, a) = \sum_{h=1}^{q} \bar{\chi}(h) e\left(\frac{ah^2}{q}\right), \quad C(q, a) = C(\chi^0, a).$$

Here χ^0 is the principal character mod q. If $\chi_1, \ldots, \chi_5$ are characters mod q, then we write

$$B(n, q, \chi_1, \ldots, \chi_5) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) C(\chi_1, b_1 a) \cdots C(\chi_5, b_5 a),$$

and

$$\mathfrak{S}(n, x) = \sum_{q \le x} \frac{B(n, q, \chi^0, \ldots, \chi^0)}{\varphi^5(q)} \tag{8}$$

where $\varphi(q)$ is Euler totient function. The integral on the major arcs $\mathfrak{M}$ causes the main difficulty, which is solved by the following

Theorem 3. *Assume* (3). *Let $\mathfrak{M}$ be as in* (6) *with P and Q determined by* (4). *If $N \ge P^{5+\varepsilon} B$, then we have*

$$\int_{\mathfrak{M}} S_1(\alpha) \cdots S_5(\alpha) e(-n\alpha) d\alpha = \frac{1}{32} \mathfrak{S}(n, P) \mathfrak{J}(n) + O\left(\frac{N^{3/2}}{|b_1 \cdots b_5|^{1/2} L}\right),$$

where $\mathfrak{S}(n, P)$ is defined in (8) *and*

$$\mathfrak{J}(n) := \sum_{\substack{b_1 m_1 + \cdots + b_5 m_5 = n \\ M < |b_j| m_j \le N}} (m_1 \cdots m_5)^{-1/2}.$$

As shown in [1], the integral on $\mathfrak{m}$ satisfies

$$\left|\int_{\mathfrak{m}}\right| \ll \frac{N^{3/2+\varepsilon}}{|b_1\cdots b_5|^{1/4}P^{1/4}}. \tag{9}$$

The contribution from the major arcs can be handled by Theorem 3, which together with (7) and (9) gives

$$r(n) = \frac{1}{32}\mathfrak{S}(n,P)\mathfrak{J}(n) + O\left(\frac{N^{3/2}}{|b_1\cdots b_5|^{1/2}L} + \frac{N^{3/2+\varepsilon}}{|b_1\cdots b_5|^{1/4}P^{1/4}}\right).$$

The lower bounds for $\mathfrak{S}(n,P)$ and $\mathfrak{J}(n)$ were estimated in [1]. The following are Lemmas 2.1 and 2.2 in [1].

Lemma 4. *Assuming* (2)*, we have* $\mathfrak{S}(n,P) \gg (\log\log B)^{-c_1}$ *for some constant* $c_1 > 0$.

Lemma 5. *Suppose* (3) *and either (i)* b_j*'s are not all of the same sign and* $N \geq 10|n|$*; or (ii) all* b_j*'s are positive and* $n = N$*. Then we have*

$$\mathfrak{J}(n) \asymp \frac{N^{3/2}}{|b_1\cdots b_5|^{1/2}}.$$

Now assume the conditions (i) or (ii) in Lemma 5. Applying Lemmas 4 and 5 to the above formula, we conclude that

$$r(n) \gg |b_1\cdots b_5|^{-1/2}N^{3/2}(\log\log B)^{-c_1}$$

provided that $P \gg N^{\varepsilon}|b_1\cdots b_5|$, or equivalently $N \gg B^{1+\varepsilon}|b_1\cdots b_5|^5$. This proves Theorems 1 and 2.

Therefore, it remains to prove Theorem 3.

For $j = 1,\ldots,5$, set

$$V_j(\lambda) = \sum_{M<|b_j|m^2\leq N} e(b_jm^2\lambda),$$

and

$$W_j(\chi,\lambda) = \sum_{M<|b_j|p^2\leq N}(\log p)\chi(p)e(b_jp^2\lambda) - \delta_{\chi}\sum_{M<|b_j|m^2\leq N} e(b_jm^2\lambda), \tag{10}$$

where $\delta_{\chi} = 1$ or 0 according as χ is principal or not. We can rewrite the exponential sum $S_j(\alpha)$ as (see for example [2], §26, (2))

$$S_j\left(\frac{h}{q}+\lambda\right) = \frac{C(q,b_jh)}{\varphi(q)}V_j(\lambda) + \frac{1}{\varphi(q)}\sum_{\chi \bmod q} C(\chi,b_jh)W_j(\chi,\lambda) =: T_j + U_j,$$

say. Thus,

$$\int_{\mathfrak{M}} S_1(\alpha)\cdots S_5(\alpha)e(-n\alpha)d\alpha = I_0 + \cdots + I_5,$$

where I_ν denotes the contribution from those products with ν pieces of U_j and $5-\nu$ pieces of T_j, i.e.,

$$I_\nu = \sum_{q\leq P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{an}{q}\right)\int_{-1/(qQ)}^{1/(qQ)}(U_1\cdots U_\nu T_{\nu+1}\cdots T_5 + \text{s.t.})e(-n\lambda)d\lambda.$$

where "s.t." means similar terms. For example, "$A_1B_2C_3D_4E_5+$s.t." means the sum of all possible terms $A_\alpha B_\beta C_\gamma D_\delta E_\iota$ with $(\alpha,\ldots,\iota)$ being any permutation of $(1,\ldots,5)$.

We will prove that I_0 gives the main term and $I_1, \ldots, I_5$ the error term. The estimation of I_0 is the same as that in [1] and we have

$$I_0 = \frac{1}{32}\mathfrak{S}(n, P)\mathfrak{I}(n) + O\left(\frac{N^{3/2}}{|b_1 \cdots b_5|^{1/2} L}\right).$$

It remains to show that $|I_i| \ll N^{3/2}|b_1 \cdots b_5|^{-1/2}L^{-1}$ for $1 \le i \le 5$. To this end, we define, for any $g \ge 1$,

$$J_j(g) = \sum_{r \le P} [g, r]^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \max_{|\lambda| \le 1/(rQ)} |W_j(\chi, \lambda)|,$$

and

$$K_j(g) = \sum_{r \le P} [g, r]^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \left(\int_{-1/(rQ)}^{1/(rQ)} |W_j(\chi, \lambda)|^2 d\lambda\right)^{1/2},$$

where $\sum_{\chi \bmod r}{}^{*}$ is over all the primitive characters modulo r and $[g, r]$ is the least common multiple of g and r.

Our Theorem 3 depends on the following three main lemmas.

Lemma 6. *For P, Q satisfying (4), we have*

$$J_j(g) \ll g^{-1+2\varepsilon} N^{1/2} |b_j|^{-1/2} L^c$$

for some constant $c > 0$.

Lemma 7. *Let P, Q satisfy (4). For $g = 1$, Lemma 6 can be improved to*

$$J_j(1) \ll N^{1/2} |b_j|^{-1/2} L^{-A},$$

where $A > 0$ is arbitrary.

Lemma 8. *For P, Q satisfying (4), we have*

$$K_j(g) \ll g^{-1+2\varepsilon} |b_j|^{-1/2} L^c$$

for some constant $c > 0$.

We omit the proof of Lemmas 6-8, since they can be proved by combining the corresponding arguments in [4] and [1]. In fact, Lemmas 6-8 with $b_j = 1$ can be established in exactly the same way as Lemmas 3.1-3.3 of [4], which depend Lemma 2.1 of [4], a hybrid estimate for Dirichlet polynomials. Lemmas 6-8 are essential in our iterative argument below; another application of the iterative method appears in [5].

For example, following the same proof of Lemmas 3.1 of [4], one can show that our Lemma 6 is a consequence of the following two estimates: For $R \le P$ and $0 < T_1 \le T_0$, we have

$$\sum_{r \sim R} [g, r]^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \int_{T_1}^{2T_1} \left|F\left(\frac{1}{2} + it, \chi\right)\right| dt \ll g^{-1+2\varepsilon} N_j^{1/4} (T_1 + 1)^{1/2} L^c, \tag{11}$$

while for $R \le P$ and $T_0 < T_2 \le T$, we have

$$\sum_{r \sim R} [g, r]^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \int_{T_2}^{2T_2} \left|F\left(\frac{1}{2} + it, \chi\right)\right| dt \ll g^{-1+2\varepsilon} N_j^{1/4} T_2 L^c. \tag{12}$$

Here $T_0 = 8\pi N/(RQ)$, $N_j = N/|b_j|$, and $F(s, \chi)$ is as in §2 of [4] with $X = N_j^{1/2}$ and $Y = (N_j/200)^{1/2}$. To show (11), we note that $[g, r](g, r) = gr$. Then the left-hand side of (11) is

$$\ll g^{-1+\varepsilon} \sum_{\substack{d \mid g \\ d \le R}} \left(\frac{R}{d}\right)^{-1+\varepsilon} \sum_{\substack{r \sim R \\ d \mid r}} \sum_{\chi \bmod r}{}^{*} \int_{T_1}^{2T_1} \left|F\left(\frac{1}{2} + it, \chi\right)\right| dt.$$

Let $\tau(g)$ be the divisor function. By Lemma 2.1 in [4], the above quantity can be estimated as

$$
\begin{aligned}
&\ll g^{-1+\varepsilon}\sum_{\substack{d|g\\ d\le R}}\left(\frac{R}{d}\right)^{-1+\varepsilon}\left(\frac{R^2}{d}T_1+\frac{R}{d^{1/2}}T_1^{1/2}N_j^{3/20}+N_j^{1/4}\right)L^c\\
&\ll g^{-1+\varepsilon}\tau(g)(R^{1+\varepsilon}T_1+R^{1/2+\varepsilon}T_1^{1/2}N_j^{3/20}+N_j^{1/4})L^c\\
&\ll g^{-1+2\varepsilon}N_j^{1/4}(T_1+1)^{1/2}L^c,
\end{aligned}
$$

provided that $R\le N_j^{1/5-\varepsilon}$. This requirement is necessary, since otherwise $R^{1/2+\varepsilon}T_1^{1/2}N_j^{3/20}$ cannot be bounded from above by $N_j^{1/4}(T_1+1)^{1/2}$. This establishes (11). Similarly we can prove (12). Therefore $P=(N/B)^{1/5-\varepsilon}$ in (4) is the optimal choice. The general assertions Lemmas 6-8 can be obtained as in Lemmas 4.1, 4.2, and 5.1 of [1].

We demonstrate by estimating I_5 here, and the treatment of the other I_i are similar. We first reduce the characters in I_5 into primitive characters, to get

$$
\begin{aligned}
|I_5| &= \left|\sum_{q\le P}\sum_{\chi_1 \bmod q}\cdots\sum_{\chi_5 \bmod q}\frac{B(n,q,\chi_1,\ldots,\chi_5)}{\varphi^5(q)}\int_{-1/(qQ)}^{1/(qQ)}W_1(\chi_1,\lambda)\cdots W_5(\chi_5,\lambda)e(-n\lambda)d\lambda\right|\\
&\le \sum_{r_1\le P}\cdots\sum_{r_5\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\cdots\sum_{\chi_5 \bmod r_5}{}^{*}\sum_{\substack{q\le P\\ r_0|q}}\frac{|B(n,q,\chi_1\chi^0,\ldots,\chi_5\chi^0)|}{\varphi^5(q)}\\
&\quad\times\int_{-1/(qQ)}^{1/(qQ)}|W_1(\chi_1\chi^0,\lambda)|\cdots|W_5(\chi_5\chi^0,\lambda)|d\lambda,
\end{aligned}
$$

where $r_0=[r_1,\ldots,r_5]$. For $q\le P$ and $M<|b_j|p^2\le N$, we have $(q,p)=1$. Using this and (10), we have $W_j(\chi_j\chi^0,\lambda)=W_j(\chi_j,\lambda)$ for the primitive characters χ_j above. Consequently by Lemma 3.1 in [1], we obtain

$$
\begin{aligned}
|I_5| &\le \sum_{r_1\le P}\cdots\sum_{r_5\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\cdots\sum_{\chi_5 \bmod r_5}{}^{*}\int_{-1/(r_0Q)}^{1/(r_0Q)}|W_1(\chi_1,\lambda)|\cdots|W_5(\chi_5,\lambda)|d\lambda\\
&\quad\times\sum_{\substack{q\le P\\ r_0|q}}\frac{|B(n,q,\chi_1\chi^0,\ldots,\chi_5\chi^0)|}{\varphi^5(q)}\\
&\ll L^{c_2}\sum_{r_1\le P}\cdots\sum_{r_5\le P}r_0^{-1+\varepsilon}\sum_{\chi_1 \bmod r_1}{}^{*}\cdots\sum_{\chi_5 \bmod r_5}{}^{*}\int_{-1/(r_0Q)}^{1/(r_0Q)}|W_1(\chi_1,\lambda)|\cdots|W_5(\chi_5,\lambda)|d\lambda.
\end{aligned}
$$

The previous estimate of I_5 was using the trivial inequality $r_0^{-1+\varepsilon}\le r_1^{-1/5+\varepsilon}\cdots r_5^{-1/5+\varepsilon}$. Instead of using this inequality which is responsible for a weaker result, we employ an iterative argument introduced in [4] to bound the above sums over r_1,r_2,r_3,r_4,r_5 consecutively. By Cauchy's inequality, we get

$$\begin{aligned}
|I_5| \ll\ & L^{c_2} \sum_{r_1 \le P} \sum_{\chi_1 \bmod r_1}{}^{*} \max_{|\lambda| \le 1/(r_1 Q)} |W_1(\chi_1, \lambda)| \\
& \times \cdots \times \sum_{r_3 \le P} \sum_{\chi_3 \bmod r_3}{}^{*} \max_{|\lambda| \le 1/(r_3 Q)} |W_3(\chi_3, \lambda)| \\
& \times \sum_{r_4 \le P} \sum_{\chi_4 \bmod r_4}{}^{*} \left(\int_{-1/(r_4 Q)}^{1/(r_4 Q)} |W_4(\chi_4, \lambda)|^2 d\lambda \right)^{1/2} \\
& \times \sum_{r_5 \le P} r_0^{-1+\varepsilon} \sum_{\chi_5 \bmod r_5}{}^{*} \left(\int_{-1/(r_5 Q)}^{1/(r_5 Q)} |W_5(\chi_5, \lambda)|^2 d\lambda \right)^{1/2}. \qquad (13)
\end{aligned}$$

The summation over r_5 on the last line is $K_5([r_1, r_2, r_3, r_4])$. Therefore, by Lemma 8,

$$K_5([r_1, r_2, r_3, r_4]) \ll [r_1, r_2, r_3, r_4]^{-1+2\varepsilon} |b_5|^{-1/2} L^{c_3}.$$

The contribution of the above quantity to the summation over r_4 in (13) is, by Lemma 8 again,

$$\begin{aligned}
& \ll |b_5|^{-1/2} L^{c_3} \sum_{r_4 \le P} [r_1, r_2, r_3, r_4]^{-1+2\varepsilon} \sum_{\chi_4 \bmod r_4}{}^{*} \left(\int_{-1/(r_4 Q)}^{1/(r_4 Q)} |W_4(\chi_4, \lambda)|^2 d\lambda \right)^{1/2} \\
& = |b_5|^{-1/2} L^{c_3} K_4([r_1, r_2, r_3]) \\
& \ll [r_1, r_2, r_3]^{-1+4\varepsilon} |b_4 b_5|^{-1/2} L^{c_4}.
\end{aligned}$$

Using Lemma 6, we can compute the contribution of the above quantity to the sum over r_3 in (13) as follows:

$$\begin{aligned}
& \ll |b_4 b_5|^{-1/2} L^{c_4} \sum_{r_3 \le P} [r_1, r_2, r_3]^{-1+4\varepsilon} \sum_{\chi_3 \bmod r_3}{}^{*} \max_{|\lambda| \le 1/(r_3 Q)} |W_3(\chi_3, \lambda)| \\
& = |b_4 b_5|^{-1/2} L^{c_4} J_3([r_1, r_2]) \\
& \ll |b_3 b_4 b_5|^{-1/2} [r_1, r_2]^{-1+8\varepsilon} N^{1/2} L^{c_5}.
\end{aligned}$$

Inserting this into (13) and applying Lemma 7, we have

$$\begin{aligned}
I_5 & \ll N^{1/2} |b_3 b_4 b_5|^{-1/2} L^{c_5} \sum_{r_1 \le P} \sum_{\chi_1 \bmod r_1}{}^{*} \max_{|\lambda| \le 1/(r_1 Q)} |W_1(\chi_1, \lambda)| J_2(r_1) \\
& \ll N |b_2 b_3 b_4 b_5|^{-1/2} L^{c_6} J_1(1) \\
& \ll N^{3/2} |b_1 b_2 b_3 b_4 b_5|^{-1/2} L^{-A}
\end{aligned}$$

for arbitrary $A > 0$ by applying Lemma 6 to J_2 and Lemma 7 to J_1. Similarly we have $|I_4|, \dots, |I_1| \ll N^{3/2} |b_1 b_2 b_3 b_4 b_5|^{-1/2} L^{-A}$. This completes our proof.

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Department of Mathematics, Simon Fraser University, Burnaby B.C., Canada
E-mail address: kkchoi@cecm.sfu.ca

Department of Mathematics, Shandong University, Jinan, Shandong 250100, P. R. China
E-mail address: jyliu@sdu.edu.cn