ON DIRICHLET SERIES FOR SUMS OF SQUARES

JONATHAN MICHAEL BORWEIN AND KWOK-KWONG STEPHEN CHOI

Abstract. In [14], Hardy and Wright recorded elegant closed forms for the generating functions of the divisor functions $\sigma_k(n)$ and $\sigma_k^2(n)$ in the terms of Riemann Zeta function $\zeta(s)$ only. In this paper, we explore other arithmetical functions enjoying this remarkable property. In Theorem 2.1 below, we are able to generalize the above result and prove that if f_i and g_i are completely multiplicative, then we have

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)}{n^s} = \frac{L_{f_1 f_2}(s) L_{g_1 g_2}(s) L_{f_1 g_2}(s) L_{g_1 f_2}(s)}{L_{f_1 f_2 g_1 g_2}(2s)}$$

where $L_f(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$ is the Dirichlet series corresponding to f. Let $r_N(n)$ be the number of solutions of $x_1^2 + \cdots + x_N^2 = n$ and $r_{2,P}(n)$ be the number of solutions of $x^2 + Py^2 = n$. One of the applications of Theorem 2.1 is to obtain closed forms, in terms of $\zeta(s)$ and Dirichlet L-functions, for the generating functions of $r_N(n), r_N^2(n), r_{2,P}(n)$ and $r_{2,P}(n)^2$ for certain N and P. We also use these generating functions to obtain asymptotic estimates of the average values for each function for which we obtain a Dirichlet series.

1. Introduction

Let σ_k denote the sum of kth powers of the divisors of n. It is also quite usual to write d for σ_0 and τ for σ_1. There is a beautiful formula for the generating functions of $\sigma_k(n)$ (see Theorem 291 in Chapter XVII of [14])

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k), \qquad \Re(s) > \max\{1, k+1\} \tag{1.1}$$

which is in terms of only the Riemann Zeta function $\zeta(s)$. Following Hardy and Wright, by standard techniques, one can prove the following remarkable identity due to Ramanujan (see [21]) (also see Theorem 305 in Chapter XVII of [14])

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} \tag{1.2}$$

for $\Re(s) > \max\{1, a+1, b+1, a+b+1\}$. In this paper, we identify other arithmetical functions enjoying similarly explicit representations. In Theorem 2.1 of §2 below,

---

Date: March 13, 2012.

1991 *Mathematics Subject Classification.* Primary 11M41, 11E25.

Key words and phrases. Dirichlet Series, Sums of Squares, Closed Forms, Binary Quadratic Forms, Disjoint Discriminants, L-functions.

Research supported by NSERC and by the Canada Research Chair Programme.

CECM Preprint 01:167.

we are able to generalize the above result and prove that if f_i and g_i are completely multiplicative, then we have

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)}{n^s} = \frac{L_{f_1 f_2}(s) L_{g_1 g_2}(s) L_{f_1 g_2}(s) L_{g_1 f_2}(s)}{L_{f_1 f_2 g_1 g_2}(2s)}$$

where $L_f(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$ is the Dirichlet series corresponding to f. As we shall see, this result recovers Hardy and Wright's formulae (1.1) and (1.2) immediately.

More generally, for certain classes of Dirichlet series, $\sum_{n=1}^{\infty} A(n) n^{-s}$, our Theorem 2.1 can be applied to obtain closed forms for the series $\sum_{n=1}^{\infty} A^2(n) n^{-s}$. In particular, if the generating function $L_f(s)$ of an arithmetic function f is expressible as a sum of products of two L-functions:

$$L_f(s) = \sum_{\chi_1, \chi_2} a(\chi_1, \chi_2) L_{\chi_1}(s) L_{\chi_2}(s)$$

for certain coefficients $a(\chi_1, \chi_2)$ and Dirichlet characters χ_i, then we are able to find a simple closed form (in term of L-functions) for the generating function $L_f^2(s) := \sum_{n=1}^{\infty} f^2(n) n^{-s}$.

One of our central applications is to the study of the number of representations as a sum of squares. Let $r_N(n)$ be the number of solutions to $x_1^2 + x_2^2 + \cdots + x_N^2 = n$ (counting permutations and signs). Hardy and Wright record a classical closed form, due to Lorenz, of the generating function for $r_2(n)$ in the terms of $\zeta(s)$ and a Dirichlet L-function, namely,

$$\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = 4\zeta(s) L_{-4}(s)$$

where $L_\mu(s) = \sum_{n=1}^{\infty} \left(\frac{\mu}{n}\right) n^{-s}$ is the *primitive L-function* corresponding to the *Kronecker symbol* $\left(\frac{\mu}{n}\right)$. Define

$$\mathcal{L}_N(s) := \sum_{n=1}^{\infty} \frac{r_N(n)}{n^s} \qquad \text{and} \qquad \mathcal{R}_N(s) := \sum_{n=1}^{\infty} \frac{r_N^2(n)}{n^s}.$$

Simple closed forms for $\mathcal{L}_N(s)$ are known for $N = 2, 4, 6$ and 8; indeed the corresponding $q-$series were known to Jacobi. The entity $\mathcal{L}_3(s)$ in particular is still shrouded in mystery, as a series relevant to the study of lattice sums in the physical sciences. Lately there has appeared a connection between $\mathcal{L}_3$ and a modern theta-cubed identity of G. Andrews [1] which we list in (6.7), R. Crandall [6] and p.301 of [3]. In §3, we shall obtain simple closed forms for $\mathcal{R}_N(s)$ for these N from the corresponding $\mathcal{L}_N(s)$, via Theorem 2.1. Since the generating functions are accessible, by an elementary convolution argument, see §3 below, we are also able to deduce

$$\sum_{n \le x} r_N^2(n) = W_N x^{N-1} + O(x^{N-2})$$

for $N = 6, 8$ and for $N = 4$ with an error term $O(x^2 \log^5 x)$ where

$$W_N := \frac{1}{(N-1)(1-2^{-N})} \frac{\pi^N}{\Gamma^2(\frac{1}{2}N)} \frac{\zeta(N-1)}{\zeta(N)}, \qquad (N \ge 3). \tag{1.3}$$

This technique can be adjusted to handle all $N \geq 2$ except $N = 3$, see Theorem 3.3, and so to establish all but the most difficult case of the following general conjecture due to Wagon:

Wagon's Conjecture. *For* $N \geq 3, \sum_{n\leq x} r_N^2(n) \sim W_N x^{N-1}$ *as* $x \longrightarrow \infty$.

Now from (3.14) below, one has $\sum_{n\leq x} r_2^2(n) \sim 4x \log x$ so that Wagon's conjecture holds only for $N \geq 3$. This conjecture motivated our interest in such explicit series representations. Recently, it has been proved by Crandall and Wagon in [8]. In fact, they show that

$$\lim_{x\to\infty} x^{1-N} \sum_{n\leq x} r_N^2(n) = W_N,$$

with various rates of convergence (those authors found the $N = 3$ case especially difficult, with relevant computations revealing very slow convergence to the above limit). In their treatment of the Wagon conjecture and related matters, they needed to evaluate the following Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\phi(n)\sigma_0(n^2)}{n^s}$$

and we have established, by an easier version of what follows, that it is

$$\sum_{n=1}^{\infty} \frac{\phi(n)\sigma_0(n^2)}{n^s} = \zeta^3(s-1) \prod_p \left(1 - \frac{3}{p^s} - \frac{1}{p^{2s-2}} + \frac{4}{p^{2s-1}} - \frac{1}{p^{3s-2}}\right)$$

where the product is over all primes. A word is in order concerning the importance of first- and second-order summatories. In a theoretical work [7] and a computational one [8] it is explained that the Wagon conjecture implies that *sums of three squares have positive density*. This interesting research connection is what inspired Wagon to posit his computationally motivated conjecture. Though it is known that the density of the set $S = \{x^2 + y^2 + z^2\}$ is exactly 5/6 due to Landau (e.g [18] or [11]), there are intriguing signal-processing and analytic notions that lead more easily at least to positivity of said density. Briefly, the summatory connection runs as follows: from the Cauchy-Schwarz inequality we know

$$\#\{n < x; n \in S\} > \frac{\left(\sum_{n<x} r_3(n)\right)^2}{\sum_{n<x} r_3^2(n)},$$

so the Wagon conjecture even gives an explicit numerical lower bound on the density of S. Of course, the density for sums of more than 3 squares is likewise positive, and boundable, yet the Lagrange theorem that sums of four squares comprise *all* nonnegative integers dominates in the last analysis. Still, the signal-processing and computational notions of Crandall and Wagon forge an attractive link between these L-series of our current interest and additive number theory.

In §4 and §5, we similarly study the number of representations by a binary quadratic forms. Let $r_{2,P}(n)$ be the number of solutions of the binary quadratic form $x^2 + Py^2 = n$. Define

$$\mathcal{L}_{2,P}(s) := \sum_{n=1}^{\infty} \frac{r_{2,P}(n)}{n^s} \qquad \text{and} \qquad \mathcal{R}_{2,P}(s) := \sum_{n=1}^{\infty} \frac{r_{2,P}(n)^2}{n^s}.$$

The closed forms of $\mathcal{L}_{2,P}(s)$ has been studied by a number of people, particular by Glasser, Zucker and Robertson (see [10] and [23]). In finding the exact evaluation of lattice sums, they are interested in expressing a multiple sum, such as the generating functions of $r_{2,P}(n)$, as a product of simple sums. As a result, plenty of closed forms of Dirichlet series $\sum_{(n,m)\neq(0,0)}(am^2+bmn+cn^2)^{-s}$ in terms of L-functions have been found. One of the most interesting cases is when the binary quadratic forms have *disjoint discriminants*, i.e, have only one form per genus. Then there are simple closed forms for the corresponding $\mathcal{L}_{2,P}(s)$ (see (4.1) below). By applying Theorem 2.1, we obtain closed forms for $\mathcal{R}_{2,P}(s)$ and from this we also deduce asymptotic estimates for $r_{2,P}(n)$ and $r_{2,P}(n)^2$.

In the last section, we shall discuss $\mathcal{L}_N(s)$ for some other less tractable cases. In particular, we collect some representations of the generating function for $r_3(n), r_N(n)$, and discuss $r_{12}(n)$ and $r_{24}(n)$.

Throughout, our notation is consistent with that in [14, 15] and [16]. We should also remark that we were lead to the structures exhibited herein by a significant amount of numeric and symbolic computation: leading to knowledge of the formulae for $\mathcal{R}_2, \mathcal{R}_4, \mathcal{R}_8, \mathcal{R}_{2,2}$ and $\mathcal{R}_{2,3}$ before finding our general results. And indeed R. Crandall triggered our interest by transmitting his formula for $\mathcal{R}_4$.

2. Basic Results

Let $\sigma(f)$ be the *abscissa* of absolute convergence of the Dirichlet series

$$L_f(s) := \sum_{n=1}^{\infty} f(n)n^{-s}.$$

For any two arithmetic functions f and g, define

$$f * g(n) := \sum_{d|n} f(d)g(n/d)$$

to be the *convolution* of f and g.

Theorem 2.1. *Suppose f_1, f_2 and g_1, g_2 are completely multiplicative arithmetic functions. Then for $\Re(s) \geq \max\{\sigma(f_i), \sigma(g_i)\}$, we have*

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)}{n^s} = \frac{L_{f_1f_2}(s)L_{g_1g_2}(s)L_{f_1g_2}(s)L_{g_1f_2}(s)}{L_{f_1f_2g_1g_2}(2s)}. \tag{2.1}$$

Proof. Since $(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)$ is multiplicative, we only need to consider its values at the prime powers. For any prime p and any $l \geq 0$,

$$(f_i * g_i)(p^l) = \sum_{d|p^l} f_i(d)g_i(p^l/d) = \frac{f_i(p)^{l+1} - g_i(p)^{l+1}}{f_i(p) - g_i(p)},$$

as each of f_1, f_2, g_1, g_2 is completely multiplicative. We intend above that if both $f_i(p)$ and $g_i(p)$ are zero, then

$$(f_i * g_i)(p^l) = \begin{cases} 1 & \text{if } l = 0; \\ 0 & \text{if } l \geq 1. \end{cases}$$

Thus, we have

$$\begin{aligned}\Sigma_p &:= \sum_{l=0}^{\infty}(f_1*g_1)(p^l)(f_2*g_2)(p^l)p^{-ls}\\ &= \sum_{l=0}^{\infty}\frac{(f_1(p)^{l+1}-g_1(p)^{l+1})(f_2(p)^{l+1}-g_2(p)^{l+1})}{(f_1(p)-g_1(p))(f_2(p)-g_2(p))}p^{-ls}\\ &= \frac{\sum_{l=0}^{\infty}\left\{(f_1f_2)(p)^{l+1}p^{-ls}+(g_1g_2)(p)^{l+1}p^{-ls}-(f_1g_2)(p)^{l+1}p^{-ls}-(g_1f_2)(p)^{l+1}p^{-ls}\right\}}{(f_1(p)-g_1(p))(f_2(p)-g_2(p))}.\end{aligned}$$

On summing up all the geometric series, we arrive at

$$\begin{aligned}\Sigma_p &:= \frac{\frac{(f_1f_2)(p)}{1-(f_1f_2)(p)p^{-s}}+\frac{(g_1g_2)(p)}{1-(g_1g_2)(p)p^{-s}}-\frac{(f_1g_2)(p)}{1-(f_1g_2)(p)p^{-s}}-\frac{(g_1f_2)(p)}{1-(g_1f_2)(p)p^{-s}}}{(f_1(p)-g_1(p))(f_2(p)-g_2(p))}\\ &= \frac{1-(f_1f_2g_1g_2)(p)p^{-2s}}{(1-(f_1f_2)(p)p^{-s})(1-(g_1g_2)(p)p^{-s})(1-(f_1g_2)(p)p^{-s})(1-(g_1f_2)(p)p^{-s})}.\end{aligned}$$

In view of the Euler product form for a Dirichlet series, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(f_1*g_1)(n)\cdot(f_2*g_2)(n)}{n^s} &= \prod_p\left\{\sum_{l=0}^{\infty}\frac{(f_1*g_1)(p^l)(f_2*g_2)(p^l)}{p^{ls}}\right\}\\ &= \frac{L_{f_1f_2}(s)L_{g_1g_2}(s)L_{f_1g_2}(s)L_{g_1f_2}(s)}{L_{f_1f_2g_1g_2}(2s)}.\end{aligned}$$

This proves our theorem. □

A first easy application of Theorem 2.1 is to evaluate the Dirichlet series $\sum_{n=1}^{\infty}\sigma_k(n)n^{-s}$ and $\sum_{n=1}^{\infty}\sigma_a(n)\sigma_b(n)n^{-s}$. If we let $f_1(n) := n^k$, $f_2(n) := \delta(n)$ and $g_1(n) = g_2(n) := 1$ where $\delta(n)$ is 1 if $n = 1$ and 0 otherwise, then

$$\begin{aligned}L_{f_1f_2}(s) = L_{g_1f_2}(s) = L_{f_1f_2g_1g_2}(s) = 1,\\ L_{f_1g_2}(s) = \zeta(s-k), \quad L_{g_1g_2}(s) = \zeta(s).\end{aligned}$$

Thus Theorem 2.1 recovers the identity (1.1)

Similarly, if we let $f_1(n) := n^a$, $f_2(n) := n^b$ and $g_1(n) = g_2(n) := 1$, then

$$\begin{aligned}L_{f_1f_2}(s) = L_{f_1f_2g_1g_2}(s) = \zeta(s-(a+b)), \quad L_{g_1g_2}(s) = \zeta(s),\\ L_{f_1g_2}(s) = \zeta(s-a), \quad L_{f_2g_1}(s) = \zeta(s-b).\end{aligned}$$

and Theorem 2.1 gives (1.2).

In particular, for any real λ,

$$\sum_{n=1}^{\infty}\sigma_\lambda^2(n)n^{-s} = \frac{\zeta(s-2\lambda)\zeta(s-\lambda)^2\zeta(s)}{\zeta(2(s-\lambda))}. \tag{2.2}$$

We shall discuss more elaborate applications of Theorem 2.1 in the latter sections. Before doing this, we give the following example here to explain why Theorem 2.1 cannot in general be extended nicely to higher order.

We are interested in obtaining the generating functions for the *kth moment* of $r_2(n)$. For any $n \geq 1$ and $|x| < 1$, in view of

$$\sum_{l=0}^{\infty} lx^l = x(1-x)^{-2}$$

and

$$x\frac{d}{dx}\sum_{l=0}^{\infty} l^n x^l = \sum_{l=0}^{\infty} l^{n+1} x^l \tag{2.3}$$

it is immediate that

$$\sum_{l=0}^{\infty} l^n x^l = \frac{xE_n(x)}{(1-x)^{n+1}}, \qquad n = 1, 2, \cdots \tag{2.4}$$

for a certain polynomial $E_n(x)$ of degree $n-1$. $E_n(x)$ is known as the nth *Euler polynomial* [4] and it is easy to see that (2.3) implies the recursion

$$E_{n+1}(x) = (1+nx)E_n(x) + x(1-x)E_n'(x).$$

Explicitly, the first few Euler polynomials are $E_1(x) = 1, E_2(x) = 1+x, E_3(x) = 1+4x+x^2$ and $E_4(x) = 1+11x+11x^2+x^3$. Equation (2.4) enables us to obtain the generating functions for the higher moments of $r_2(n)$ as follows: for $\mu \equiv 0$ or 1 (mod 4), we let $\left(\frac{\mu}{n}\right)$ be the *Jacobi-Legendre-Kronecker symbol* and again consider

$$L_\mu(s) := \sum_{n=1}^{\infty} \left(\frac{\mu}{n}\right) n^{-s}$$

the L-function corresponding to $\left(\frac{\mu}{n}\right)$. It is known (e.g. p. 291 in [3]) that

$$\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = 4\zeta(s)L_{-4}(s) = \sum_{n=1}^{\infty} \frac{4(1 * \left(\frac{-4}{n}\right))(n)}{n^s}$$

and $r_2(n) = 4(1 * \left(\frac{-4}{n}\right))(n)$ for any $n \geq 1$. A simple calculation shows that for any $l \geq 0$,

$$\left(1 * \left(\frac{-4}{n}\right)\right)(p^l) = \begin{cases} 1 & \text{if } p = 2; \\ l+1 & \text{if } p \geq 3 \text{ and } \left(\frac{-1}{p}\right) = 1; \\ \frac{(-1)^l + 1}{2} & \text{if } p \geq 3 \text{ and } \left(\frac{-1}{p}\right) = -1. \end{cases}$$

We now have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{r_2^N(n)}{n^s} &= 4^N \sum_{n=1}^{\infty} \frac{\left\{(1 * \left(\frac{-4}{n}\right))(n)\right\}^N}{n^s} \\ &= 4^N \prod_p \sum_{l=0}^{\infty} \frac{\left\{(1 * \left(\frac{-4}{n}\right))(p^l)\right\}^N}{p^{ls}} \\ &= \frac{4^N}{1-2^{-s}} \left\{ \prod_{\left(\frac{-1}{p}\right)=-1} \sum_{l=0}^{\infty} \left(\frac{(-1)^l+1}{2}\right)^N p^{-ls} \right\} \left\{ \prod_{\left(\frac{-1}{p}\right)=1} \sum_{l=0}^{\infty} (l+1)^N p^{-ls} \right\} \\ &= \frac{4^N}{1-2^{-s}} \prod_{\left(\frac{-1}{p}\right)=-1} \frac{1}{1-p^{-2s}} \prod_{\left(\frac{-1}{p}\right)=1} \frac{E_N(p^{-s})}{(1-p^{-s})^{N+1}} \end{aligned}$$

on using (2.4). [Here $\prod_p$ denotes the infinite product over all primes.] Firstly, when $N = 2$, we have most pleasingly,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{r_2^2(n)}{n^s} &= \frac{16}{1-2^{-s}}\prod_{(\frac{-1}{p})=-1}\frac{1}{1-p^{-2s}}\prod_{(\frac{-1}{p})=1}\frac{1+p^{-s}}{(1-p^{-s})^3}\\ &= \frac{16}{1+2^{-s}}\left\{\frac{1}{1-2^{-s}}\prod_{(\frac{-1}{p})=-1}\frac{1}{1-p^{-2s}}\prod_{(\frac{-1}{p})=1}\frac{1}{(1-p^{-s})^2}\right\}^2\prod_p(1-p^{-2s})\\ &= \frac{\left(4\zeta(s)L_{-4}(s)\right)^2}{(1+2^{-s})\zeta(2s)}. \qquad (2.5)\end{aligned}$$

However, when $N \geq 3$, the generating functions cannot be expressed in terms of L-functions as completely as in formula (2.5). For example, when $N = 3$

$$\sum_{n=1}^{\infty}\frac{r_2^3(n)}{n^s} = \frac{64}{1-2^{-s}}\prod_{(\frac{-1}{p})=-1}\frac{1}{1-p^{-2s}}\prod_{(\frac{-1}{p})=1}\frac{1+4p^{-s}+p^{-2s}}{(1-p^{-s})^4},$$

and when $N = 4$

$$\sum_{n=1}^{\infty}\frac{r_2^4(n)}{n^s} = \frac{256}{1-2^{-s}}\prod_{(\frac{-1}{p})=-1}\frac{1}{1-p^{-2s}}\prod_{(\frac{-1}{p})=1}\frac{1+11p^{-s}+11p^{-2s}+p^{-3s}}{(1-p^{-s})^5}.$$

This helps explain why our Theorem 2.1 has no 'closed-form' extension to higher order. For the detailed asymptotic estimate of the generating function of the kth moment of $r_2(n)$, we refer the reader to [5].

3. Sums of a Small Even Number of Squares

In view of Theorem 2.1, whenever a Dirichlet series is expressible as a sum of two-fold products of L-functions:

$$L_f(s) = \sum_{\chi_1,\chi_2} a(\chi_1,\chi_2)L_{\chi_1}(s)L_{\chi_2}(s),$$

we are able to provide a closed form (in terms of L-functions) of the Dirichlet series $L_{f^2}(s) = \sum_{n=1}^{\infty} f^2(n)n^{-s}$, on using (2.1).

In particular, let $r_N(n)$ be the number of solutions to $x_1^2 + x_2^2 + \cdots + x_N^2 = n$ (counting permutations and signs) and let

$$\mathcal{L}_N(s) := \sum_{n=1}^{\infty} r_N(n)n^{-s}, \qquad \mathcal{R}_N(s) := \sum_{n=1}^{\infty} r_N^2(n)n^{-s}$$

be the Dirichlet series corresponding to $r_N(n)$ and $r_N^2(n)$. Closed forms are obtainable for $\mathcal{L}_N(s)$ for certain even N from the explicit formulae known for $r_N(n)$. For example, we have

$$\mathcal{L}_2(s) = 4\zeta(s)L_{-4}(s), \qquad (3.1)$$

$$\mathcal{L}_4(s) = 8(1-4^{1-s})\zeta(s)\zeta(s-1), \qquad (3.2)$$

$$\mathcal{L}_6(s) = 16\zeta(s-2)L_{-4}(s) - 4\zeta(s)L_{-4}(s-2), \qquad (3.3)$$

$$\mathcal{L}_8(s) = 16(1-2^{1-s}+4^{2-s})\zeta(s)\zeta(s-3). \qquad (3.4)$$

The derivation of (3.1) and (3.3) from the formulas for $r_2(n)$ and $r_6(n)$ (e.g. §91 in [20]) is immediate if we write those formulas in the form

$$\begin{aligned} r_2(n) &= 4\sum_{\substack{m,d\ge 1\\ md=n}} \chi(d) \\ r_6(n) &= 16\sum_{\substack{m,d\ge 1\\ md=n}} \chi(m)d^2 - 4\sum_{\substack{m,d\ge 1\\ md=n}} \chi(d)d^2 \end{aligned}$$

where χ denotes the non-principal character modulo 4. For derivation of (3.2) and (3.4) from the formulas for $r_4(n)$ and $r_8(n)$ (e.g. §91 in [20]) is immediate if we write those formulas in the form

$$\begin{aligned} r_4(n) &= 8\sigma_1(n) - 32\sigma_1(n/4) \\ r_8(n) &= 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4) \end{aligned}$$

where it is understood that $\sigma_k(n) = 0$ if n is not a positive integer.

In this section, we shall demonstrate how to use our Theorem 2.1 to obtain counterpart closed forms for $\mathcal{R}_N(s)$ from the above expressions for $\mathcal{L}_N(s)$.

Let us start with $\mathcal{R}_2(s)$. It has already been shown in (2.5) that

$$\mathcal{R}_2(s) = \sum_{n=1}^{\infty} \frac{r_2^2(n)}{n^s} = \frac{(4\zeta(s)L_{-4}(s))^2}{(1+2^{-s})\zeta(2s)}$$

but it can also be deduced directly from our Theorem 2.1 and (3.1) by taking $f_1(n) = f_2(n) = 1$ and $g_1(n) = g_2(n) = \left(\frac{-4}{n}\right)$.

We shall consider $\mathcal{R}_4(s)$ and $\mathcal{R}_8(s)$ later. For $\mathcal{R}_6(s)$, we first write

$$\begin{aligned} \mathcal{L}_6(s) &= 16\zeta(s-2)L_{-4}(s) - 4\zeta(s)L_{-4}(s-2) \\ &= 16\sum_{n=1}^{\infty}\left(\sum_{d|n} d^2\left(\frac{-4}{n/d}\right)\right)n^{-s} - 4\sum_{n=1}^{\infty}\left(\sum_{d|n} d^2\left(\frac{-4}{d}\right)\right)n^{-s} \\ &= \sum_{n=1}^{\infty}\left(16(f_1*g_1)(n) - 4(f_2*g_2)(n)\right)n^{-s} \end{aligned}$$

where $f_1(n) = n^2$, $g_1(n) = \left(\frac{-4}{n}\right)$, $f_2(n) = 1$ and $g_2(n) = \left(\frac{-4}{n}\right)n^2$. It follows from our Theorem 2.1 and (3.3) that

$$\begin{aligned} \mathcal{R}_6(s) &= \sum_{n=1}^{\infty}\left(16(f_1*g_1)(n) - 4(f_2*g_2)(n)\right)^2 n^{-s} \\ &= 16^2\sum_{n=1}^{\infty}(f_1*g_1)^2(n)n^{-s} - 128\sum_{n=1}^{\infty}(f_1*g_1)(n)(f_2*g_2)(n)n^{-s} \\ &\qquad +16\sum_{n=1}^{\infty}(f_2*g_2)^2(n)n^{-s} \\ &= 16^2\frac{L_{f_1^2}(s)L_{g_1^2}(s)L_{f_1g_1}(s)^2}{L_{f_1^2g_1^2}(2s)} - 128\frac{L_{f_1f_2}(s)L_{g_1g_2}(s)L_{f_1g_2}(s)L_{g_1f_2}(s)}{L_{f_1f_2g_1g_2}(2s)} \\ &\qquad +16\frac{L_{f_2^2}(s)L_{g_2^2}(s)L_{f_2g_2}(s)^2}{L_{f_2^2g_2^2}(2s)}. \end{aligned} \tag{3.5}$$

It remains to evaluate the component L-functions and they are

$$L_{f_1^2}(s) = \zeta(s-4), \qquad L_{g_1^2}(s) = (1-2^{-s})\zeta(s),$$

$$L_{f_2^2}(s) = \zeta(s), \qquad L_{g_2^2}(s) = (1-16\cdot 2^{-s})\zeta(s-4),$$

$$L_{f_1g_1}(s) = L_{-4}(s-2), \qquad L_{f_1f_2}(s) = \zeta(s-2), \qquad L_{g_1g_2}(s) = (1-4\cdot 2^{-s})\zeta(s-2),$$

$$L_{f_1g_2}(s) = L_{-4}(s-4), \qquad L_{g_1f_2}(s) = L_{-4}(s), \qquad L_{f_2g_2}(s) = L_{-4}(s-2),$$

$$L_{f_1^2g_1^2}(s) = L_{f_2^2g_2^2}(s) = L_{f_1f_2g_1g_2}(s) = (1-16\cdot 2^{-s})\zeta(s-4).$$

Now from (3.5), we have

$$\mathcal{R}_6(s) = 16\frac{(17-32\cdot 2^{-s})}{(1-16\cdot 2^{-2s})}\frac{\zeta(s-4)L_{-4}^2(s-2)\zeta(s)}{\zeta(2s-4)} - \frac{128}{(1+4\cdot 2^{-s})}\frac{L_{-4}(s-4)\zeta^2(s-2)L_{-4}(s)}{\zeta(2s-4)}.$$

For $\mathcal{R}_4(s)$ and $\mathcal{R}_8(s)$, we need the following companion lemma:

Lemma 3.1. *Suppose $f(n)$ is a multiplicative function. Let p be a prime and let the Dirichlet series*

$$\sum_{n=1}^{\infty}\frac{A(n)}{n^s} := \sum_{m=0}^{\infty}\frac{a_m}{p^{ms}}\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

be the product of $L_f(s)$ and a power series in p^{-s}. Then

$$\sum_{n=1}^{\infty}\frac{A^2(n)}{n^s} = L_{f^2}(s)\sum_{m=0}^{\infty}\frac{a_m^2}{p^{ms}} + 2L_{f^2}(s)\left(\sum_{l=0}^{\infty}\frac{f^2(p^l)}{p^{ls}}\right)^{-1} \times\sum_{k=1}^{\infty}\left\{\sum_{m=0}^{\infty}\frac{a_{m+k}a_m}{p^{ms}}\right\}\left\{\sum_{l=0}^{\infty}\frac{f(p^l)f(p^{l+k})}{p^{ls}}\right\}p^{-ks}. \tag{3.6}$$

Proof. Since

$$\sum_{n=1}^{\infty}A(n)n^{-s} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}a_m f(n)(p^m n)^{-s} = \sum_{n=1}^{\infty}\left\{\sum_{\substack{m=0\\ p^m|n}}^{\infty}a_m f\left(\frac{n}{p^m}\right)\right\}n^{-s},$$

we deduce

$$\begin{aligned}\sum_{n=1}^{\infty}A^2(n)n^{-s} &= \sum_{n=1}^{\infty}\left\{\sum_{\substack{m=0\\ p^m|n}}^{\infty}a_m f\left(\frac{n}{p^m}\right)\right\}^2 n^{-s}\\ &= \sum_{m_1,m_2=0}^{\infty}a_{m_1}a_{m_2}\sum_{\substack{n=1\\ p^{m_1},p^{m_2}|n}}^{\infty}f\left(\frac{n}{p^{m_1}}\right)f\left(\frac{n}{p^{m_2}}\right)n^{-s}. \qquad (3.7)\end{aligned}$$

For any $m_1, m_2 \geq 1$ we let $M := \max(m_1, m_2)$ and $m := \min(m_1, m_2)$. Then the last summation (over n) in (3.7) is

$$
\begin{aligned}
&= \sum_{\substack{n=1 \\ p^M | n}}^{\infty} f\left(\frac{n}{p^M}\right) f\left(\frac{n}{p^m}\right) n^{-s} \\
&= \frac{1}{p^{Ms}} \sum_{n=1}^{\infty} f(n) f(np^{M-m}) n^{-s} \\
&= \frac{1}{p^{Ms}} \sum_{l=0}^{\infty} \sum_{\substack{n=1 \\ (p,n)=1}}^{\infty} f(np^l) f(np^{M-m+l}) p^{-ls} n^{-s} \\
&= \frac{1}{p^{Ms}} \sum_{l=0}^{\infty} f(p^l) f(p^{M-m+l}) p^{-ls} \sum_{\substack{n=1 \\ (p,n)=1}}^{\infty} \frac{f^2(n)}{n^s} \qquad (3.8)
\end{aligned}
$$

since $f(n)$ is multiplicative. By writing

$$
\sum_{n=1}^{\infty} \frac{f^2(n)}{n^s} = \sum_{l=0}^{\infty} \sum_{\substack{n=1 \\ (p,n)=1}}^{\infty} \frac{f^2(np^l)}{(np^l)^s} = \sum_{l=0}^{\infty} \frac{f^2(p^l)}{p^{ls}} \sum_{\substack{n=1 \\ (p,n)=1}}^{\infty} \frac{f^2(n)}{n^s},
$$

we deduce that

$$
\sum_{\substack{n=1 \\ (p,n)=1}}^{\infty} \frac{f^2(n)}{n^s} = L_{f^2}(s) \left(\sum_{l=0}^{\infty} f^2(p^l) p^{-ls} \right)^{-1}. \tag{3.9}
$$

Using (3.7), (3.8) and (3.9), we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} A^2(n) n^{-s} = L_{f^2}(s) &\left(\sum_{l=0}^{\infty} f^2(p^l) p^{-ls} \right)^{-1} \times \\
&\times \sum_{m_1, m_2=0}^{\infty} \frac{a_{m_1} a_{m_2}}{p^{\max(m_1, m_2)s}} \sum_{l=0}^{\infty} \frac{f(p^l) f(p^{l+|m_1-m_2|})}{p^{ls}}. \qquad (3.10)
\end{aligned}
$$

The contribution corresponding to $m_1 = m_2$ in the above double summation is

$$
\sum_{m=0}^{\infty} \frac{{a_m}^2}{p^{ms}} \sum_{l=0}^{\infty} \frac{f^2(p^l)}{p^{ls}} \tag{3.11}
$$

and the contribution corresponding to $m_1 \neq m_2$ is

$$
\begin{aligned}
&= 2 \sum_{m_2 < m_1}^{\infty} \frac{a_{m_1} a_{m_2}}{p^{m_1 s}} \sum_{l=0}^{\infty} \frac{f(p^l) f(p^{l+m_1-m_2})}{p^{ls}} \\
&= 2 \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{a_{m+k} a_m}{p^{(m+k)s}} \sum_{l=0}^{\infty} \frac{f(p^l) f(p^{l+k})}{p^{ls}} \\
&= 2 \sum_{k=1}^{\infty} \left\{ \sum_{m=0}^{\infty} \frac{a_{m+k} a_m}{p^{ms}} \right\} \left\{ \sum_{l=0}^{\infty} \frac{f(p^l) f(p^{l+k})}{p^{ls}} \right\} \frac{1}{p^{ks}}. \qquad (3.12)
\end{aligned}
$$

Now (3.6) follows from (3.10), (3.11) and (3.12). □

On applying Lemma 3.1 to (3.2) and (3.4) and using (2.2), we have

$$\mathcal{R}_4(s) = 64\frac{(8\cdot 2^{3-3s} - 10\cdot 2^{2-2s} + 2^{1-s} + 1)\zeta(s-2)\zeta^2(s-1)\zeta(s)}{(1+2^{1-s})\zeta(2s-2)};$$

and

$$\mathcal{R}_8(s) = 256\frac{(32\cdot 2^{6-2s} - 3\cdot 2^{3-s} + 1)\zeta(s-6)\zeta^2(s-3)\zeta(s)}{(1+2^{3-s})\zeta(2s-6)}.$$

Therefore, we have completed the proof of the following Theorem.

Theorem 3.2. *We may write*

$$\mathcal{R}_2(s) = \frac{(4\zeta(s)L_{-4}(s))^2}{(1+2^{-s})\zeta(2s)}, \qquad \Re(s) > 1;$$

$$\mathcal{R}_4(s) = 64\frac{(8\cdot 2^{3-3s} - 10\cdot 2^{2-2s} + 2^{1-s} + 1)\zeta(s-2)\zeta^2(s-1)\zeta(s)}{(1+2^{1-s})\zeta(2s-2)}, \qquad \Re(s) > 3;$$

$$\mathcal{R}_6(s) = 16\frac{(17-32\cdot 2^{-s})}{(1-16\cdot 2^{-2s})}\frac{\zeta(s-4)L_{-4}^2(s-2)\zeta(s)}{\zeta(2s-4)} - \frac{128}{(1+4\cdot 2^{-s})}\frac{L_{-4}(s-4)\zeta^2(s-2)L_{-4}(s)}{\zeta(2s-4)}, \qquad \Re(s) > 5;$$

and

$$\mathcal{R}_8(s) = 256\frac{(32\cdot 2^{6-2s} - 3\cdot 2^{3-s} + 1)\zeta(s-6)\zeta^2(s-3)\zeta(s)}{(1+2^{3-s})\zeta(2s-6)} \qquad \Re(s) > 7.$$

Since $\epsilon\zeta(1+\epsilon) \to 1$ as $\epsilon \to 0$, the value of the $\lim_{\epsilon\to 0}\epsilon\mathcal{R}_N(N-1+\epsilon)$ at its largest pole is, respectively:

$$\lim_{\epsilon\to 0}\epsilon\mathcal{R}_4(3+\epsilon) = 96\zeta(3) = 3W_4$$

$$\lim_{\epsilon\to 0}\epsilon\mathcal{R}_6(5+\epsilon) = 240\zeta(5) = 5W_6$$

and

$$\lim_{\epsilon\to 0}\epsilon\mathcal{R}_8(7+\epsilon) = \frac{4480}{17}\zeta(7) = 7W_8.$$

The formulae for $\mathcal{R}_N(s)$ in Theorem 3.2 enable us to estimate the average order of $r_N^2(n)$ for $N = 2, 4, 6, 8$. Following from Sierpinski's result on the circle problem (cf. Satz 509 of [17])

$$\sum_{n\le x} r_2(n) = \pi x + O(x^{1/3}), \tag{3.13}$$

we have

$$\sum_{n\le x} r_2^2(n) = 4x\log x + 4\alpha x + O(x^{2/3}) \tag{3.14}$$

where $\alpha := 2\gamma + \frac{8}{\pi}L'_{-4}(1) - \frac{12}{\pi^2}\zeta'(2) + \frac{1}{3}\log 2 - 1 = 2.0166216\cdots$. Indeed, one can prove (3.14) as follows. Let

$$\sum_{n=1}^{\infty} h_n n^{-s} := \{4\zeta(s)L_{-4}(s)\}^2 = \left(\sum_{n=1}^{\infty} r_2(n)n^{-s}\right)^2. \tag{3.15}$$

By the hyperbola method and (3.13), one has

$$\begin{aligned} H(x) &:= \sum_{n\le x} h_n = \sum_{\substack{m,d\ge 1\\ md\le x}} r_2(m)r_2(d) \\ &= 2\sum_{m\le\sqrt{x}} r_2(m)\sum_{n\le x/m} r_2(n) - \left(\sum_{n\le\sqrt{x}} r_2(n)\right)^2 \\ &= 2\sum_{m\le\sqrt{x}} r_2(m)\left\{\pi\frac{x}{m} + O\left(\frac{x^{1/3}}{m^{1/3}}\right)\right\} - \{\pi x^{1/2} + O(x^{1/6})\}^2 \\ &= \pi^2 x\log x + C_1 x + O(x^{2/3}), \end{aligned}$$

for some constant C_1. Now by (2.5) we have

$$\mathcal{R}_2(s) = \sum_{n=1}^{\infty} r_2^2(n)n^{-s} = \sum_{m=1}^{\infty} h_m m^{-s}\sum_{n=1}^{\infty} l_n n^{-s}$$

where h_n is given (3.15) and

$$\sum_{n=1}^{\infty} l_n n^{-s} = (1+2^{-s})^{-1}\zeta^{-1}(2s) = \sum_{j=0}^{\infty}(-1)^j 2^{-js}\sum_{k=1}^{\infty}\mu(k)k^{-2s}$$

has abscissa of absolute convergence 1/2 and

$$\sum_{n\le x}|l_n| = O(x^{1/2}\log x).$$

Here $\mu(n)$ is the Möbius function. Now by an elementary convolution argument

$$\begin{aligned} \sum_{n\le x} r_2^2(n) &= \sum_{n\le x} l_n H(x/n) \\ &= \sum_{n\le x} l_n\{\pi^2\frac{x}{n}\log\frac{x}{n} + C_1\frac{x}{n} + O(\frac{x^{2/3}}{n^{2/3}})\} \\ &= 4x\log x + C_2 x + O(x^{2/3}) \qquad (3.16) \end{aligned}$$

for some constant C_2. To evaluate the value of C_2, we first note that for any $\sigma > 1$, we have

$$\sum_{n\le x}\frac{r_2^2(n)}{n^\sigma} = \int_{1^-}^{x} u^{-\sigma}d\sum_{n\le u} r_2^2(n)$$

and hence from (3.16) and letting $x \longrightarrow +\infty$, we get

$$\mathcal{R}_2(\sigma) = \sigma\int_1^\infty\left(\frac{\sum_{n\le u} r_2^2(n) - 4u\log u - C_2 u}{u^{\sigma+1}}\right)du + \frac{4}{(\sigma-1)^2} + \frac{4+\sigma C_2}{\sigma-1}.$$

The above integral converges when $\sigma \longrightarrow 1^+$ and hence

$$\lim_{\sigma\longrightarrow 1^+}\left\{\mathcal{R}_2(\sigma) - \frac{4}{(\sigma-1)^2}\right\}(\sigma-1) = 4 + C_2. \qquad (3.17)$$

Now in view of (2.5), $\mathcal{R}_2(s)$ has a pole at $s = 1$ of order 2. So the limit in (3.17) in fact is the residue of $\mathcal{R}_2(s)$ at $s = 1$ which can be evaluated by the method in §5 below and it is equal to

$$4\left(2\gamma+\frac{8}{\pi}L'_{-4}(1)-\frac{12}{\pi^2}\zeta'(2)+\frac{1}{3}\log 2\right).$$

This completes the proof of (3.14)

It is also worth to note that Sierpinski's result has been slightly improved and so the error term in (3.14) could be improved accordingly. For example, the term $O(x^{2/3})$ can be replaced by $O(x^{284/429})$ if we employ Nowak's result in [19] which replaces the term $O(x^{1/3})$ in (3.13) by $O(x^{139/429})$.

We now consider the case $N=4$. In view of Theorem 3.2, $\mathcal{R}_4(s)/\zeta(s-2)$ is equal to the product of a finite Dirichlet series and the five Dirichlet series $\zeta(s-1),\zeta(s-1),\zeta(s),\zeta^{-1}(2s-2)$ and $(1+2^{1-s})^{-1}$, each of which has the property that the coefficient of n^{-s} is $O(n)$. Hence from the formula for $\mathcal{R}_4(s)$ in Theorem 3.2,

$$\mathcal{R}_4(s)=\zeta(s-2)\sum_{n=1}^{\infty}g_n n^{-s},$$

where $|g_n|=O(nd_5(n))$ and $d_k(n)$ is the number of ways of expressing n in the form $n=n_1n_2\cdots n_k$ with $n_1,n_2,\cdots,n_k$ positive integers. It follows that

$$\begin{aligned}\sum_{n\le x}r_4^2(n) &= \sum_{n\le x}g_n\sum_{m\le x/n}m^2\\ &= \sum_{n\le x}g_n\left(\frac{1}{3}\left(\frac{x}{n}\right)^3+O\left(\frac{x^2}{n^2}\right)\right)\\ &= \frac{x^3}{3}\sum_{n=1}^{\infty}\frac{g_n}{n^3}+O\left(x^3\left|\sum_{n>x}\frac{g_n}{n^3}\right|\right)+O\left(x^2\sum_{n\le x}\frac{|g_n|}{n^2}\right)\\ &= \frac{x^3}{3}\sum_{n=1}^{\infty}\frac{g_n}{n^3}+O\left(x^3\sum_{n>x}\frac{d_5(n)}{n^2}\right)+O\left(x^2\sum_{n\le x}\frac{d_5(n)}{n}\right)\\ &= \frac{x^3}{3}\sum_{n=1}^{\infty}\frac{g_n}{n^3}+O(x^2\log^5 x)\end{aligned}$$

because $\sum_{n\le x}d_k(n)\sim xP_k(\log x)$ for some polynomial $P_k(X)$ of degree $k-1$ (see Chapter XII in [26]). Now since

$$\sum_{n=1}^{\infty}\frac{g_n}{n^3}=\lim_{s\longrightarrow 3^+}\mathcal{R}_4(s)/\zeta(s-2)=\lim_{\epsilon\longrightarrow 0}\epsilon\mathcal{R}_4(3+\epsilon)=3W_4$$

so we have

$$\sum_{n\le x}r_4^2(n)=W_4x^3+O(x^2\log^5 x).$$

The cases for $N=6$ and $N=8$ can be treated in the same manner as

$$\mathcal{R}_6(s)=\zeta(s-4)\sum_{n=1}^{\infty}\frac{b_n}{n^s}+L_{-4}(s-4)\sum_{n=1}^{\infty}\frac{c_n}{n^s}$$

and

$$\mathcal{R}_8(s)=\zeta(s-6)\sum_{n=1}^{\infty}\frac{d_n}{n^s}$$

where b_n and c_n are $\ll n^2 d_5(n)$ and d_n is $\ll n^3 d_5(n)$. Therefore, we have

$$\sum_{n\le x} r_N^2(n) = W_N x^{N-1} + O(x^{N-2}) \tag{3.18}$$

for $N = 6, 8$ with W_N given by (1.3).

For $N \ne 2, 4, 6, 8$, lacking the closed forms for $\mathcal{R}_N(s)$, we can't follow the argument above to estimate the average order for $r_N^2(n)$. However, as suggested by the referee, the asymptotic value for $\sum_{n\le x} r_N^2(n)$, at least for $N \ge 5$, can be obtained from the singular series formula for $r_N(n)$ given by Hardy (see p.342 of [12] or p.155 of [11]), which may be written as

$$r_N(n)\frac{\Gamma(N/2)}{\pi^{N/2}}n^{1-N/2} = \sum_{k=1}^{\infty}\sum_{\substack{1\le h\le k\\(h,k)=1}}\left(\frac{G(h,k)}{k}\right)^N e^{-2\pi i h n/k} + O(n^{1-N/4}) \tag{3.19}$$

where $G(h,k) = \sum_{j=1}^{k} e^{2\pi i h j^2/k}$ is the standard quadratic Gauss sum. In fact, using a well-known result on quadratic Gauss sum (e.g. p.138 of [11])

$$|G(h,k)| = \begin{cases}\sqrt{k} & \text{if } k\equiv 1 \pmod 2;\\ 0 & \text{if } k\equiv 2 \pmod 4;\\ \sqrt{2k} & \text{if } k\equiv 0 \pmod 4;\end{cases} \tag{3.20}$$

for $(h,k) = 1$, we have

$$\begin{aligned} r_N(n)\frac{\Gamma(N/2)}{\pi^{N/2}}n^{1-N/2} &= \sum_{k\le x^{1/2}}\sum_{\substack{1\le h\le k\\(h,k)=1}}\left(\frac{G(h,k)}{k}\right)^N e^{-2\pi i h n/k} + O(x^{1-N/4})\\ &:= P(n) + O(x^{1-N/4})\end{aligned}$$

for $N \ge 5$ and $n \le x$. By (3.20), we have $|P(n)| \ll 1$ and hence

$$r_N(n)^2 = \frac{\pi^N}{\Gamma(N/2)^2} n^{N-2}|P(n)|^2 + O(x^{3N/4-1}).$$

It follows that

$$\sum_{n\le x} r_N(n)^2 = \frac{\pi^N}{\Gamma(N/2)^2}\sum_{n\le x} n^{N-2}|P(n)|^2 + O(x^{3N/4}). \tag{3.21}$$

It remains to estimate the sum $\sum_{n\le x} n^{N-2}|P(n)|^2$ which is equal to

$$\sum_{1\le k_1,k_2\le x^{1/2}}\ \sum_{\substack{1\le h_i\le k_i\\(h_i,k_i)=1, i=1,2}}\left(\frac{G(h_1,k_1)}{k_1}\right)^N\left(\frac{\overline{G(h_2,k_2)}}{k_2}\right)^N\sum_{n\le x} n^{N-2}e^{-2\pi i n(\frac{h_1}{k_1}-\frac{h_2}{k_2})}. \tag{3.22}$$

We now note that when $\frac{h_1}{k_1} \ne \frac{h_2}{k_2}$, we have

$$\left|\sum_{n\le x} e^{-2\pi i n(\frac{h_1}{k_1}-\frac{h_2}{k_2})}\right| \le k_1 k_2$$

and hence the contribution for those terms $\frac{h_1}{k_1} \neq \frac{h_2}{k_2}$ to (3.22) is

$$\ll x^{N-2} \left(\sum_{k \leq x^{1/2}} k^{2-N/2} \right)^2 .$$

Using this, (3.22) and (3.21), we have

$$\begin{aligned} \sum_{n \leq x} r_N(n)^2 &= \frac{\pi^N}{(N-1)\Gamma(N/2)^2} \left(\sum_{k \leq x^{1/2}} B(k) \right) x^{N-1} + O(x^{N-2} + x^{3N/4}) \\ &= \frac{\pi^N}{(N-1)\Gamma(N/2)^2} \left(\sum_{k=1}^{\infty} B(k) \right) x^{N-1} + O(x^{N-2} + x^{3N/4}) \end{aligned}$$

where

$$B(k) := \sum_{\substack{1 \leq h \leq k \\ (h,k)=1}} \left| \frac{G(h,k)}{k} \right|^{2N} .$$

Note that when $N = 6$, we have a better error term in (3.18). The function $k \to B(k)$ is multiplicative in k (see p.156 of [11]) and from (3.20), $B(1) = 1, B(2) = 0, B(2^l) = 2^{-N(l-1)}\phi(2^l)$ for any $l \geq 2$ and $B(p^j) = p^{-Nj}\phi(p^j)$ for any $j \geq 1$ and odd prime p. It then follows from the Euler product formula that

$$\sum_{k=1}^{\infty} B(k) = (1 - 2^{-(N-1)})^{-1} \prod_{p>3} \frac{1 - p^{-N}}{1 - p^{-(N-1)}} = \frac{1}{(1 - 2^{-N})} \frac{\zeta(N-1)}{\zeta(N)}.$$

We finally conclude that

Theorem 3.3. *We have*

$$\sum_{n \leq x} r_2^2(n) = 4x \log x + 4\alpha x + O(x^{2/3})$$

$$\sum_{n \leq x} r_4^2(n) = W_4 x^3 + O(x^2 \log^5 x)$$

and

$$\sum_{n \leq x} r_6^2(n) = W_6 x^5 + O(x^4)$$

For $N \geq 5, N \neq 6$ and $x \geq 1$, we have

$$\sum_{n \leq x} r_N^2(n) = W_N x^{N-1} + O(x^{N-2} + x^{3N/4}).$$

Here $\alpha = 2\gamma + \frac{8}{\pi} L'_{-4}(1) - \frac{12}{\pi^2}\zeta'(2) + \frac{1}{3}\log 2 - 1 = 2.0166216\cdots$.

This proves Wagon's conjecture for $N \geq 4$. Theorem 3.3 can also be found in [8] and it contains the same basic arguments for getting the error bounds on $r_N^2(n)$ summatory for $N \geq 5$. The estimate $O(x^{N-2})$ in fact is the best possible as will be discussed elsewhere.

4. Closed Forms for Dirichlet Series of Quadratic Forms

There is a rich parallel theory of L-functions over imaginary quadratic fields. In this vein, let $r_{2,P}(n)$ be the number of solutions to $x^2 + Py^2 = n$ (again counting sign and order). Denote

$$\mathcal{L}_{2,P}(s) := \sum_{n=1}^{\infty} r_{2,P}(n) n^{-s}, \qquad \mathcal{R}_{2,P}(s) := \sum_{n=1}^{\infty} r_{2,P}(n)^2 n^{-s}.$$

It is known that when the quadratic form $x^2 + Py^2$ has disjoint discriminants (that is, it has exactly one form per genus), then one has the following formula (see (9.2.8) in [3])

$$\begin{aligned} \mathcal{L}_{2,P} &= 2^{1-t} \sum_{\mu|P} L_{\epsilon_\mu \mu}(s) L_{-4P\epsilon_\mu/\mu}(s) \\ &= \sum_{n=1}^{\infty} \left\{ 2^{1-t} \sum_{\mu|P} \left(\frac{\epsilon_\mu \mu}{n}\right) * \left(\frac{-4P\epsilon_\mu/\mu}{n}\right) \right\} n^{-s} \qquad (4.1) \end{aligned}$$

where P is an odd square-free number, t is the number of distinct factors of P and $\epsilon_\mu := \left(\frac{-1}{\mu}\right)$.

Explicitly, (4.1) holds for all *type one* numbers. These include and may comprise:

$$P = 5, 13, 21, 33, 37, 57, 85, 93, 105, 133, 165, 177, 253, 273, 345, 357, 385, 1365.$$

It is known that there are only finitely many such disjoint discriminants. We call such P **solvable**. Using (4.1), we have

$$\begin{aligned} \mathcal{R}_{2,P}(s) &= \sum_{n=1}^{\infty} 2^{2-2t} \sum_{\mu_1\mu_2|P} \left[\left(\frac{\epsilon_{\mu_1}\mu_1}{n}\right) * \left(\frac{-4P\epsilon_{\mu_1}/\mu_1}{n}\right)\right] \cdot \left[\left(\frac{\epsilon_{\mu_2}\mu_2}{n}\right) * \left(\frac{-4P\epsilon_{\mu_2}/\mu_2}{n}\right)\right] n^{-s} \\ &= 2^{2-2t} \sum_{\mu_1\mu_2|P} \sum_{n=1}^{\infty} \left[\left(\frac{\epsilon_{\mu_1}\mu_1}{n}\right) * \left(\frac{-4P\epsilon_{\mu_1}/\mu_1}{n}\right)\right] \cdot \left[\left(\frac{\epsilon_{\mu_2}\mu_2}{n}\right) * \left(\frac{-4P\epsilon_{\mu_2}/\mu_2}{n}\right)\right] n^{-s}. \end{aligned}$$

We now notice that $\mathcal{R}_{2,P}(s)$ is a sum of Dirichlet series in the form of Theorem 2.1. We may apply Theorem 2.1 on letting

$$f_i(n) := \left(\frac{\epsilon_{\mu_i}\mu_i}{n}\right), \qquad g_i(n) := \left(\frac{-4P\epsilon_{\mu_i}/\mu_i}{n}\right),$$

for $i = 1, 2$. Then

$$\begin{aligned} L_{f_1 f_2}(s) &= \sum_{n=1}^{\infty} \left(\frac{\epsilon_{\mu_1}\mu_1}{n}\right)\left(\frac{\epsilon_{\mu_2}\mu_2}{n}\right) n^{-s} \\ &= \sum_{\substack{n=1 \\ (n,(\mu_1,\mu_2))=1}}^{\infty} \left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{n}\right) n^{-s} \\ &= L_{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}(s) \prod_{p|(\mu_1,\mu_2)} \left(1 - \left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{p}\right) p^{-s}\right) \end{aligned}$$

where $\mu_i^* := \mu_i/(\mu_1, \mu_2)$ and $\prod_{p|n}$ denotes the product over all prime factors of n. Similarly, we have

$$L_{g_1 g_2}(s) = L_{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}(s) \prod_{p|\frac{2P}{[\mu_1,\mu_2]}} \left(1 - \left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{p}\right) p^{-s}\right);$$

$$L_{f_1 g_2}(s) = L_{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}(s) \prod_{p|\mu_1^*} \left(1 - \left(\frac{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}{p}\right) p^{-s}\right);$$

$$L_{f_2 g_1}(s) = L_{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}(s) \prod_{p|\mu_2^*} \left(1 - \left(\frac{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}{p}\right) p^{-s}\right)$$

and

$$L_{f_1 f_2 g_1 g_2}(s) = \zeta(s) \prod_{p|2P} (1 - p^{-s}).$$

Our basic Theorem 2.1 gives

$$\begin{aligned}\mathcal{R}_{2,P}(s) = 2^{2(1-t)} \sum_{\mu_1,\mu_2|P} & L^2_{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}(s) L^2_{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}(s)\zeta(2s)^{-1} \\ & \times \prod_{p|2P} \left\{1 + \left[\left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{p}\right) + \left(\frac{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}{p}\right)\right] p^{-s}\right\}^{-1}.\end{aligned}$$

We have similar closed forms of L-functions for the quadratic form $x^2 + 2Py^2$ with discriminant $-8P$ (see (9.2.9) in [3]):

$$\mathcal{L}_{2,2P} = 2^{1-t} \sum_{\mu|P} L_{\epsilon_\mu \mu}(s) L_{-8P\epsilon_\mu/\mu}(s).$$

We deduce from Theorem 2.1, in the same way, that

$$\begin{aligned}\mathcal{R}_{2,2P}(s) = 2^{2(1-t)} \sum_{\mu_1,\mu_2|P} & L^2_{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}(s) L^2_{-8P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}(s)\zeta(2s)^{-1} \\ & \times \prod_{p|2P} \left\{1 + \left[\left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{p}\right) + \left(\frac{-8P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}{p}\right)\right] p^{-s}\right\}^{-1}\end{aligned}$$

for the *type two* integers

$$P = 1, 3, 5, 11, 15, 21, 29, 35, 39, 51, 65, 95, 105, 165, 231.$$

We note that $210 = 2 \times 105$ yields the invariant which Ramanujan sent to Hardy in his famous letter.

We may reprise with the following theorem:

Theorem 4.1. *Let P be a solvable square-free integer and let t be the number of distinct factors of P. We have for P respectively of type one and type two:*

$$\begin{aligned}\mathcal{R}_{2,P}(s) = 2^{2(1-t)} \sum_{\mu_1,\mu_2|P} & L^2_{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}(s) L^2_{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}(s)\zeta(2s)^{-1} \\ & \times \prod_{p|2P} \left\{1 + \left[\left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{p}\right) + \left(\frac{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}{p}\right)\right] p^{-s}\right\}^{-1},\end{aligned} \tag{4.2}$$

and

$$\mathcal{R}_{2,2P}(s) = 2^{2(1-t)} \sum_{\mu_1,\mu_2|P} L^2_{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}(s) L^2_{-8P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}(s)\zeta(2s)^{-1}$$
$$\times \prod_{p|2P}\left\{1+\left[\left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{p}\right)+\left(\frac{-8P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}{p}\right)\right]p^{-s}\right\}^{-1}$$

where $\epsilon_\mu = \left(\frac{-1}{\mu}\right)$ *and* $\mu_i^* = \mu_i/(\mu_1,\mu_2)$.

In particular, the prime cases provide:

Corollary 4.2. *We have*

$$\mathcal{R}_{2,p}(s) = \frac{2\zeta^2(s)L^2_{-4p}(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L^2_p(s)L^2_{-4}(s)}{(1-2^{-s})(1+p^{-s})\zeta(2s)}$$

for $p = 5, 13, 37$, *while*

$$\mathcal{R}_{2,2}(s) = \frac{4\zeta^2(s)L^2_{-8}(s)}{(1+2^{-s})\zeta(2s)}.$$

Similarly,

$$\mathcal{R}_{2,2p}(s) = \frac{2\zeta^2(s)L^2_{-8p}(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L^2_{-p}(s)L^2_8(s)}{(1-2^{-s})(1-p^{-s})\zeta(2s)},$$

for $p = 3, 11$ *while*

$$\mathcal{R}_{2,2p}(s) = \frac{2\zeta^2(s)L^2_{-8p}(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L^2_p(s)L^2_{-8}(s)}{(1-2^{-s})(1-p^{-s})\zeta(2s)}$$

for $p = 5, 29$.

Closed forms for $\mathcal{L}_{2,P}(s)$ are also accessible for some P other than those of *type one* or *type two*. For example, (see Table VI of [10]) one has

$$\mathcal{L}_{2,3}(s) = \left(2+4^{1-s}\right)\zeta(s)L_{-3}(s). \tag{4.3}$$

and hence by Theorem 2.1 and Lemma 3.1, we obtain

$$\mathcal{R}_{2,3}(s) = 4\frac{1+2^{3-2s}}{1+3^{-s}}\frac{\left(\zeta(s)\,L_{-3}(s)\right)^2}{\zeta(2s)}. \tag{4.4}$$

We may also derive many formulae for non-square free integers via modular transformations [3]. We contain ourselves with the simplest example which is

$$\mathcal{R}_{2,4}(s) = \frac{4-2^{2-s}+2^{4-2s}}{1+2^{-s}}\frac{\left(\zeta(s)\,L_{-4}(s)\right)^2}{\zeta(2s)}$$

as a consequence of a quadratic transformation leading to

$$\mathcal{L}_{2,4}(s) = (2^{-1}-2^{-1-s}+4^{-s})\mathcal{L}_2(s).$$

There are some simple closed forms of the generating functions for more general binary quadratic forms found in [10]. Let

$$\mathcal{L}_{(a,b,c)}(s) := \sum_{(n,m)\neq(0,0)} \frac{1}{(am^2+bmn+cn^2)^s} = \sum_{n=1}^{\infty}\frac{r_{(a,b,c)}(n)}{n^s}$$

and $\mathcal{R}_{(a,b,c)}(s) := \sum_{n=1}^{\infty} \frac{r_{(a,b,c)}(n)^2}{n^s}$ where $r_{(a,b,c)}(n)$ is the number of representations of n by the quadratic form $ax^2 + bxy + cy^2$. Then, we have (e.g. (26) of [25])

$$\sum_{h(D)} \mathcal{L}_{(a,b,c)}(s) = \omega(D)\zeta(s)L_D(s)$$

where the sum is taken over the $h(D)$ inequivalent reduced quadratic forms of discriminant $D := b^2 - 4ac$ and $\omega(-3) = 6, \omega(-4) = 4$ and $\omega(D) = 2$ for $D < -4$. In particular, for $c = 2, 3, 5, 11, 17, 41$, $h(D) = 1$ and the result is especially simple:

$$\mathcal{L}_{(1,1,c)}(s) = 2\zeta(s)L_D(s).$$

Hence from Theorem 2.1, we have

$$\mathcal{R}_{(1,1,c)}(s) = \frac{4(\zeta(s)L_D(s))^2}{(1 + |D|^{-s})\zeta(2s)},$$

with similar formulae for $(a, b, c) = (1, 1, 1)$ and $(1, 0, 1)$.

Thanks to the *On-Line Encyclopedia of Integer Sequences*

`http://www.research.att.com/~njas/sequences/`

we discover that the sequence $2, 3, 5, 11, 17, 41$ is exactly the so-called Euler "lucky" numbers which are the numbers n such that $m \to m^2 - m + n$ has prime values for $m = 0, \cdots, n-1$.

5. The Average Order of $r_{2,P}(n)$

We start with the average order of $r_{2,P}$. The results in this section, in fact, can be obtained by a convolution argument such as we used to prove (3.18) in §3. This, however, does not seem to yield better error estimates, especially in the power of N, in Theorem 5.1 and 5.3 below. So we instead apply Perron's formula. Both methods would seem to add an unnecessary if unobtrusive 'ε'.

Theorem 5.1. *Let P be a solvable square-free integer, $x > 1$ and $\epsilon > 0$. We have for either $N = P$ of type one or $N = 2P$ of type two:*

$$\sum_{n \le x} r_{2,N}(n) = \frac{\pi}{\sqrt{N}} x + O((xN)^{\frac{1}{2}+\epsilon}).$$

where the implicit constants are independent of x and P.

Proof. In view of (4.1), we have for $n \ge 1$

$$r_{2,P}(n) = 2^{1-t} \sum_{\mu|P} \left(\frac{\epsilon_\mu \mu}{n}\right) * \left(\frac{-4P\epsilon_\mu/\mu}{n}\right) \le 2^{1-t} \sum_{\mu|P} \sigma_0(n) \le 2\sigma_0(n). \quad (5.1)$$

It follows from (1.1) that

$$\mathcal{L}_{2,P}(\sigma) \ll \sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n^\sigma} = \zeta(\sigma)^2 \ll \frac{1}{(\sigma-1)^2}$$

as $\sigma \to 1^+$. Now in view of Perron's formula (see Theorem 1 in §1 of Chapter V in [16]), for any $c > 1$, $\epsilon > 0$ and $x, T \ge 1$ we have

$$\sum_{n \le x} r_{2,P}(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \mathcal{L}_{2,P}(s) \frac{x^s}{s} ds + O(x^c T^{-1}(c-1)^{-2} + x^{1+\epsilon} T^{-1}). \quad (5.2)$$

In order to evaluate the above integral, we need the following well-known estimates for $\zeta(s)$ and L-functions.

Lemma 5.2. *We have*

$$\zeta(\sigma+i\xi) \ll \begin{cases} \frac{1}{\sigma-1} & \text{if } 1<\sigma\le 2 \text{ and } \xi=0 \\ \log|\xi| & \text{if } 1\le\sigma \text{ and } |\xi|\ge e \\ |\xi|^{\frac{1-\sigma}{2}}\log|\xi| & \text{if } 0\le\sigma\le 1 \text{ and } |\xi|\ge e \end{cases}$$

and

$$\frac{1}{\zeta(\sigma+i\xi)} \ll \log^7|\xi|$$

if $\sigma\ge 1$ *and* $|\xi|\ge e$. *If* χ *is a non-principal character modulo* q, *we have*

$$L(\sigma+i\xi,\chi) \ll \log q(|\xi|+2)$$

for $\sigma\ge 1$ *while if* χ *is a primitive character modulo* $q\ge 3$ *and* $0\le\sigma\le 1$, *then*

$$L(\sigma+i\xi,\chi) \ll (q(|\xi|+2))^{\frac{1-\sigma}{2}}\log q(|\xi|+2).$$

As usual, we estimate the integral in (5.2) by replacing the integral over the rectangle R with vertices $b\pm iT$ and $c\pm iT$ with $b=\frac{1}{\log x}$ and then calculate the residues of the poles of the integrand inside R. In view of (4.2), the only pole of $\mathcal{R}_{2,P}(s)\frac{x^s}{s}$ inside R is $s=1$, which comes from $\zeta(s)$, and its residue at $s=1$ is $2^{1-t}L_{-4P}(1)x$ because $\lim_{s\to 1}(s-1)\zeta(s)=1$.

For solvable P, i.e, x^2+Py^2 having one form per genus, the class number equals the number of genera — which is 2^t (see p. 198 of [24]). Hence $L_{-4P}(1)=\frac{2^{t-1}\pi}{\sqrt{P}}$ for type one P and $L_{-8P}(1)=\frac{2^{t-1}\pi}{\sqrt{2P}}$ for type two P by (4.11) in [11]. Thus, the residue of $\mathcal{R}_{2,P}(s)\frac{x^s}{s}$ at $s=1$ is $\frac{\pi}{\sqrt{P}}x$.

Next, using the estimates in Lemma 5.2 and (4.2), we may prove that for $|\xi|\le T$,

$$\mathcal{L}_{2,P}(\sigma+i\xi) \ll \begin{cases} (P(|\xi|+2))^{(1-\sigma)}\log^2(PT) & \text{if } b\le\sigma\le 1, \\ \log^2(PT) & \text{if } 1\le\sigma\le c. \end{cases}$$

It then follows that

$$\begin{aligned} \frac{1}{2\pi i}\int_{b-iT}^{b+iT}\mathcal{L}_{2,P}(s)\frac{x^s}{s}ds &\ll \int_{-T}^{T}|\mathcal{L}_{2,P}(b+i\xi)|\frac{x^b}{|b+i\xi|}d\xi \\ &\ll PT\log^2(PT) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} &\frac{1}{2\pi i}\int_{b\pm iT}^{c\pm iT}\mathcal{L}_{2,P}(s)\frac{x^s}{s}ds \\ \ll\ &\left\{\int_b^1+\int_1^c\right\}|\mathcal{L}_{2,P}(\sigma\pm iT)|\frac{x^\sigma}{T}d\sigma \\ \ll\ &P(\log PT)^2\int_b^1\left(\frac{x}{PT}\right)^\sigma d\sigma + T^{-1}(\log PT)^2\int_1^c x^\sigma d\sigma \\ \ll\ &x^cT^{-1}\log^2(PT)\log x. \end{aligned} \tag{5.4}$$

Now by choosing $c = 1 + \frac{1}{\log x}$ and $T = (x/P)^{\frac{1}{2}}$, we get from (5.2)–(5.4) that

$$\sum_{n\leq x} r_{2,P}(n) = \frac{\pi}{\sqrt{P}}x + O((xP)^{\frac{1}{2}+\epsilon}).$$

The case for type two P can be proved in the same way. This completes the proof of Theorem 5.1. □

For any square-free integer N, we define a constant α by:

$$\alpha(N) := 2\gamma + \sum_{p|2N} \frac{\log p}{p+1} + 2\frac{L'_{-4N}(1)}{L_{-4N}(1)} - \frac{12}{\pi^2}\zeta'(2) - 1 \tag{5.5}$$

where γ is Euler's constant and $\sum_{p|n}$ is the summation over all prime factors of n.

Theorem 5.3. *Let P be a solvable square-free integer. Let $x > 1$ and $\epsilon > 0$. We have for either $N = P$ of type one or $N = 2P$ of type two:*

$$\sum_{n\leq x} r_{2,N}(n)^2 = \frac{3}{N}\left(\prod_{p|2N}\frac{2p}{p+1}\right)(x\log x + \alpha(N)x) + O(N^{\frac{1}{4}+\epsilon}x^{\frac{3}{4}+\epsilon})$$

where the implicit constants are independent of both x and P.

Proof. It follows from (1.2) and (5.1) that

$$\mathcal{R}_{2,P}(\sigma) \ll \sum_{n=1}^{\infty}\frac{\sigma_0(n)^2}{n^\sigma} = \frac{\zeta^4(\sigma)}{\zeta(2\sigma)} \ll \frac{1}{(\sigma-1)^4}$$

as $\sigma \to 1^+$. Similar to (5.2), for any $c > 1$, $\epsilon > 0$ and $x, T \geq 1$, we have

$$\sum_{n\leq x} r_{2,P}(n)^2 = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\mathcal{R}_{2,P}(s)\frac{x^s}{s}ds + O(x^cT^{-1}(c-1)^{-4} + x^{1+\epsilon}T^{-1}). \tag{5.6}$$

We estimate the integral in (5.3) by replacing the integral over the rectangle R with vertices $\frac{1}{2} \pm iT$ and $c \pm iT$ and then calculate the residues of the poles of the integrand inside R. In view of (4.2), the only pole of $\mathcal{R}_{2,P}(s)\frac{x^s}{s}$ inside R is $s = 1$ of order 2 which comes from $\zeta(s)^2$ and corresponds to the terms when $\mu_1 = \mu_2$ in the double summation of (4.2):

$$2^{2(1-t)}\sigma_0(P)\zeta(s)^2L_{-4P}(s)^2\zeta(2s)^{-1}\prod_{p|2P}(1+p^{-s})^{-1}\frac{x^s}{s} := F(s) \tag{5.7}$$

and its residue at $s = 1$ is

$$\begin{aligned} &= \lim_{s\to 1}\frac{d}{ds}\left\{(s-1)^2F(s)\right\} \\ &= \lim_{s\to 1}(s-1)^2F(s)\lim_{s\to 1}\frac{d}{ds}\log\left\{(s-1)^2F(s)\right\}. \end{aligned}$$

Since P is solvable, so

$$\begin{aligned} \lim_{s\to 1}(s-1)^2F(s) &= 2^{2(1-t)}\sigma_0(P)L^2_{-4P}(1)\zeta(2)^{-1}\prod_{p|2P}(1+p^{-1})^{-1}x \\ &= \frac{3}{P}\left(\prod_{p|2P}\frac{2p}{p+1}\right)x. \end{aligned}$$

In view of (5.5) and (5.7), we have

$$\begin{aligned}
&\lim_{s\to 1}\frac{d}{ds}\log\left\{(s-1)^2F(s)\right\}\\
= &\quad 2\gamma+\sum_{p|2P}\frac{\log p}{p+1}+2\frac{L'_{-4P}(1)}{L_{-4P}(1)}-\frac{12}{\pi^2}\zeta'(2)-1+\log x\\
= &\quad \alpha(P)+\log x
\end{aligned}$$

because $\lim_{s\to 1}\left(\frac{1}{s-1}+\frac{\zeta'(s)}{\zeta(s)}\right)=\gamma$. Therefore the residue of $\mathcal{R}_{2,P}(s)\frac{x^s}{s}$ at $s=1$ is

$$\frac{3}{P}\left(\prod_{p|2P}\frac{2p}{p+1}\right)(x\log x+\alpha(P)x)\,. \tag{5.8}$$

Next using the estimates in Lemma 5.2 and (4.2), one can prove that for $|\xi|\le T$,

$$\mathcal{R}_{2,P}(\sigma+i\xi)\ll\begin{cases}P^{(1-\sigma)+\epsilon}(|\xi|+2)^{2(1-\sigma)}\log^A T & \text{if } \frac{1}{2}\le\sigma\le 1,\\ P^{\epsilon}\log^A T & \text{if } 1\le\sigma\le c.\end{cases}$$

It then follows that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\mathcal{R}_{2,P}(s)\frac{x^s}{s}ds &\ll \int_{-T}^{T}|\mathcal{R}_{2,P}(\frac{1}{2}+i\xi)|\frac{x^{\frac{1}{2}}}{|\frac{1}{2}+i\xi|}d\xi\\
&\ll P^{\frac{1}{2}+\epsilon}x^{\frac{1}{2}}T\log^A T
\end{aligned}\tag{5.9}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\frac{1}{2}\pm iT}^{c\pm iT}\mathcal{R}_{2,P}(s)\frac{x^s}{s}ds\\
\ll &\quad \left\{\int_{\frac{1}{2}}^{1}+\int_{1}^{c}\right\}|\mathcal{R}_{2,P}(\sigma\pm iT)|\frac{x^\sigma}{T}d\sigma\\
\ll &\quad P^{1+\epsilon}T(\log T)^A\int_{\frac{1}{2}}^{1}\left(\frac{x}{PT^2}\right)^\sigma d\sigma+P^\epsilon T^{-1}(\log T)^A\int_1^c x^\sigma d\sigma\\
\ll &\quad P^\epsilon x^c T^{-1}\log^A T.
\end{aligned}\tag{5.10}$$

Now by choosing $c=1+\frac{1}{\log x}$ and $T=(x/P)^{\frac{1}{4}}$, we get from (5.6) and (5.8)-(5.10) that

$$\sum_{n\le x}r_{2,P}(n)^2=\frac{3}{P}\left(\prod_{p|2P}\frac{2p}{p+1}\right)(x\log x+\alpha(P)x)+O(P^{\frac{1}{4}+\epsilon}x^{\frac{3}{4}+\epsilon}).$$

The case for type two P can be proved in the same way. This completes the proof of Theorem 5.3. □

In particular, we have established:

Theorem 5.4. *For any $x\ge 1$, we have*

$$\sum_{n\le x}r_{2,p}(n)^2=\frac{8}{p+1}(x\log x+\alpha(p)x)+O(x^{\frac{3}{4}+\epsilon})$$

for $p = 5, 13, 37$ *and*

$$\sum_{n\le x} r_{2,2p}(n)^2 = \frac{4}{p+1}(x\log x + \alpha(2p)x) + O(x^{\frac{3}{4}+\epsilon})$$

for $p = 1, 3, 5, 11, 29$. *Here the implicit constants are again independent of* x.

Similarly, in view of (4.3) and (4.4), we have for $x > 1$,

$$\sum_{n\le x} r_{2,3}(n) = \frac{\pi}{\sqrt{3}}x + O(x^{\frac{1}{2}+\epsilon})$$

and

$$\sum_{n\le x} r_{2,3}(n)^2 = 2(x\log x + \alpha_3 x) + O(x^{\frac{3}{4}+\epsilon}) \tag{5.11}$$

where $\alpha_3 := 2\gamma - \frac{4}{3}\log 2 + \frac{1}{4}\log 3 + \frac{6\sqrt{3}}{\pi}L'_{-3}(1) - \frac{12}{\pi^2}\zeta'(2) - 1$.

Also

$$\sum_{n\le x} r_{2,4}(n) = \frac{\pi}{2}x + O(x^{\frac{1}{2}+\epsilon})$$

and

$$\sum_{n\le x} r_{2,4}(n)^2 = \frac{3}{2}(x\log x + \alpha_4 x) + O(x^{\frac{3}{4}+\epsilon})$$

where $\alpha_4 := 2\gamma - \frac{2}{3}\log 2 + \frac{8}{\pi}L'_{-4}(1) - \frac{12}{\pi^2}\zeta'(2) - 1$.

Akin to Wagon's conjecture, we make the following conjecture.

Quadratic Conjecture. *For any square-free* P,

$$\sum_{n\le x} r_{2,P}(n) \sim \frac{\pi}{\sqrt{P}}x$$

and

$$\sum_{n\le x} r_{2,P}(n)^2 \sim \frac{3}{P}\left(\prod_{p|2P}\frac{2p}{p+1}\right)x\log x$$

as $x \to \infty$.

In view of Theorem 5.3, (3.14) and (5.11), our conjecture is true for solvable P and for $P = 1, 3$. We have also confirmed it for $P = 7$ and 15 from the representations of

$$\mathcal{L}_{2,7}(s) = 2\,(1 - 2^{1-s} + 2^{1-2s})\zeta(s)L_{-7}(s)$$

and

$$\mathcal{L}_{2,15}(s) = (1 - 2^{1-s} + 2^{1-2s})\zeta(s)L_{15}(s) + (1 + 2^{1-s} + 2^{1-2s})L_{-3}(s)L_5(s)$$

again given in [10], which leads to

$$\mathcal{R}_{2,7}(s) = 4\frac{(1 - 3\cdot 2^{-s} + 2^{2-2s})}{(1+2^{-s})(1+7^{-s})}\frac{(\zeta(s)L_{-7}(s))^2}{\zeta(2s)}$$

and

$$\mathcal{R}_{2,15}(s) = \frac{2(1 - 3\cdot 2^{-s} + 2^{2-2s})}{(1+2^{-s})(1+3^{-s})(1+5^{-s})}\frac{(\zeta(s)L_{-15}(s))^2}{\zeta(2s)} + \frac{2(1 + 3\cdot 2^{-s} + 2^{2-2s})}{(1-2^{-s})(1-3^{-s})(1-5^{-s})}\frac{(L_{-3}(s)L_5(s))^2}{\zeta(2s)},$$

and may be analyzed by the methods above.

6. Sums of Three Squares and Other Powers

6.1. **Three Squares.** Odd squares are notoriously less amenable to closed forms. In this subsection, we primarily record some results for $r_3(n)$, the number of representations of n as a sum of three squares. Following Hardy, Bateman in [2] gives the following formula for $r_3(n)$. Let

$$\chi_2(n) := \begin{cases} 0 & \text{if } 4^{-a}n \equiv 7 \pmod 8; \\ 2^{-a} & \text{if } 4^{-a}n \equiv 3 \pmod 8; \\ 3\cdot 2^{-1-a} & \text{if } 4^{-a}n \equiv 1,2,5,6 \pmod 8) \end{cases}$$

where a is the highest power of 4 dividing n.

Then

$$r_3(n) = \frac{16\sqrt{n}}{\pi} L_{-4n}(1)\,\chi_2(n) \times \prod_{p^2|n} \left(\frac{p^{-\tau}-1}{p^{-1}-1} + p^{-\tau}\left(1 - \frac{1}{p}\left(\frac{-p^{-2\tau}n}{p}\right)\right)^{-1}\right) \tag{6.1}$$

where $\tau = \tau_p$ is the highest power of p^2 dividing n.

The Dirichlet series for $r_3(n)$ deriving from (6.1) is not as malleable as those of (3.1)-(3.4), but we are able to derive a nice expression in terms of Bessel functions.

Let K_s be the *modified Bessel function of the second kind.* Then we have (see [27], p. 183)

$$K_s(x) = \frac{1}{2}\left(\frac{x}{2}\right)^s \int_0^\infty e^{-t-\frac{x^2}{4t}} \frac{dt}{t^{s+1}}. \tag{6.2}$$

By the substitution $t = \frac{1}{u}$ in (6.2), we get

$$K_s(x) = \frac{1}{2}\left(\frac{x}{2}\right)^s \int_0^\infty e^{-\frac{x^2u}{4}-\frac{1}{u}} u^{s-1} du. \tag{6.3}$$

Let

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

be the classical Jacobean theta function. In view of the Poisson summation formula, we have, for $t > 0$

$$\theta_3(e^{-\pi t}) = t^{-\frac{1}{2}}\theta_3(e^{-\pi/t}).$$

Since the Mellin transform of $e^{-\alpha t}$ for $\alpha \neq 0$ is $M_s(e^{-\alpha t}) = \Gamma(s)\alpha^{-s}$, so we have (letting $q = e^{-\pi t}$)

$$\begin{aligned}
\mathcal{L}_3(s) &= 3 \sum_{n,m,p\in\mathbb{Z}} \frac{n^2}{(n^2+m^2+p^2)^{s+1}} \\
&= \frac{3\pi^{s+1}}{\Gamma(s+1)} \sum_{n,m,p\in\mathbb{Z}} n^2 M_{s+1}(q^{n^2+m^2+p^2}) \\
&= \frac{3\pi^{s+1}}{\Gamma(s+1)} M_{s+1}\left(\sum_{n\in\mathbb{Z}} n^2 q^{n^2} \theta_3^2(q)\right) \\
&= \frac{3\pi^{s+1}}{\Gamma(s+1)} \sum_{n\in\mathbb{Z}} n^2 \int_0^\infty e^{-n^2\pi t}\theta_3^2(e^{-\pi/t}) t^{s-1} dt \\
&= \frac{3\pi^{s+1}}{\Gamma(s+1)} \sum_{n\in\mathbb{Z}} n^2 \sum_{m=1}^\infty r_2(m) \int_0^\infty e^{-n^2\pi t - \frac{\pi m}{t}} t^{s-1} dt \\
&\quad + \frac{3\pi^{s+1}}{\Gamma(s+1)} \sum_{n\in\mathbb{Z}} n^2 \int_0^\infty e^{-n^2\pi t} t^{s-1} dt. \qquad (6.4)
\end{aligned}$$

The first term of (6.4) is

$$\begin{aligned}
&= \frac{6\pi^{s+1}}{\Gamma(s+1)} \sum_{n=1}^\infty n^2 \sum_{m=1}^\infty r_2(m) \int_0^\infty e^{-n^2\pi t - \frac{\pi m}{t}} t^{s-1} dt \\
&= \frac{6\pi^{s+1}}{\Gamma(s+1)} \sum_{m=1}^\infty r_2(m)(\pi m)^s \sum_{n=1}^\infty n^2 \int_0^\infty e^{-n^2\pi^2 m x - 1/x} x^{x-1} dx, \qquad (x = \frac{t}{\pi m}) \\
&= \frac{12\pi^{s+1}}{\Gamma(s+1)} \sum_{m=1}^\infty r_2(m) m^{s/2} \sum_{n=1}^\infty \frac{1}{n^{s-2}} K_s(2\pi n\sqrt{m})
\end{aligned}$$

by (6.3) and the second term is

$$\begin{aligned}
&= \frac{6\pi^{s+1}}{\Gamma(s+1)} \sum_{n=1}^\infty \frac{1}{n^{2s-2}\pi^s} \int_0^\infty e^{-x} x^{s-1} ds \\
&= \frac{6\pi}{s}\zeta(2s-2).
\end{aligned}$$

This proves the following result:

$$\mathcal{L}_3(s) = \frac{6\pi}{s}\zeta(2s-2) + \frac{12\pi^{s+1}}{\Gamma(s+1)} \sum_{m=1}^\infty r_2(m) m^{s/2} \sum_{n=1}^\infty \frac{1}{n^{s-2}} K_s(2\pi n\sqrt{m}). \qquad (6.5)$$

There is a corresponding formula for $\sum(-1)^n r_3(n)/n^s$ which corresponds to Madelung's constant (see p. 301 in [3]). The second term of (6.5) can be rewritten as

$$\frac{12\pi^{s+1}}{\Gamma(s+1)} \sum_{k>0} k^{\frac{s}{2}} K_s(2\pi\sqrt{k}) \sum_{n^2|k} \frac{r_2(k/n^2)}{n^{2s-2}}.$$

Moreover, these Bessel functions are elementary when s is a half-integer. Most nicely, for 'jellium', which is the Wigner sum analogue of Madelung's constant, we have

$$\mathcal{L}_3(1/2) = -\pi + 3\pi \sum_{m>0} \frac{r_2(m)}{\sinh^2(\pi\sqrt{m})},$$

and the exponential convergence is entirely apparent.

For a survey of other rapidly convergent lattice sums of this type see [3] and [6].

There is a corresponding formula for $\mathcal{L}_N(s)$, for all $N \geq 2$, in which we obtain a Bessel-series in $r_{N-1}(m)$:

$$\begin{aligned} \mathcal{L}_N(s) = \sum_{n>0} \frac{r_N(n)}{n^s} &= \frac{2N\,\Gamma(s-\frac{N-3}{2})}{\Gamma(s+1)} \pi^{\frac{N-1}{2}} \zeta(2s-N+1) \\ &+ \frac{4N\,\pi^{s+1}}{\Gamma(s+1)} \sum_{m>0} \frac{m^{\frac{1}{2}s}\, r_{N-1}(m)}{m^{\frac{N-3}{4}}} \sum_{n>0} \frac{n^{\frac{N+1}{2}}}{n^s} K_{s-\frac{N-3}{2}}(2n\pi\sqrt{m}). \end{aligned} \tag{6.6}$$

There is an equally attractive integral representation (see [27] p. 172) for:

$$K_s(x) = \left(\frac{2}{x}\right)^s \frac{\Gamma(s+1/2)}{\Gamma(1/2)} \int_0^\infty \frac{\cos(xt)}{(1+t^2)^{s+1/2}}\, dt$$

at least when $x > 1/2$. This leads to

$$\sum_{n>0} \frac{r_3(n)}{n^s} = 2L_{-4}(s+\frac{1}{2},\frac{1}{2}) \sum_{m>0} r_2(m) \int_0^\infty \frac{C_{s-2}(\sqrt{m}t)}{(1+t^2)^{s+1/2}}\, dt$$

where

$$C_s(x) = \sum_{n>0} \frac{\cos(2\pi\, nx)}{n^s}$$

is a *Clausen-type* function. For $s = 2k$, even integer, this evaluates to

$$C_{2k}(x) = \frac{(2\pi)^{2k}}{(-1)^{k-1}2(2k)!} B_{2k}(x)$$

where B_k is a Bernoulli polynomial.

Obviously this also extends to reworkings of (6.6). For example, the $N = 2$ case yields

$$4\,L_{-4}(s+\frac{1}{2},\frac{1}{2})\zeta(2\,s-1) + \frac{16\,\pi^{1+s}}{\Gamma(s+1)} \sum_{n=1}^\infty \frac{\sigma_{2\,s-1}(n)}{n^{s-\frac{3}{2}}} K_{s+\frac{1}{2}}(2\,n\pi) = 4\,\zeta(s)L_{-4}(s).$$

This in turn, with $s = 2$, becomes

$$4\,\pi^3 \sum_{n=1}^\infty \sigma_3(n) e^{-2\,n\pi} \left(1 + \frac{3}{2}\frac{1}{n\pi} + \frac{3}{4}\frac{1}{n^2\pi^2}\right) \frac{1}{n} = \frac{2}{3}\pi^2 \mathrm{G} - \frac{3}{2}\zeta(3),$$

where $\mathrm{G} := \sum_{\mathrm{n}\geq 0} (-1)^{\mathrm{n}}\,(2\mathrm{n}+1)^{-2}$ is *Catalan's constant.*

There is a puissant formula for θ_2^3 due to Andrews [1] (given with a typographical error in [3] p. 286). It is

$$\theta_2^3(q) = 8 \sum_{n=0}^\infty \sum_{j=0}^{2\,n} \left(\frac{1+q^{4\,n+2}}{1-q^{4\,n+2}}\right) q^{(2\,n+1)^2-(j+1/2)^2}. \tag{6.7}$$

Lamentably we have not been able to use it to study $\mathcal{R}_3$, or even $\mathcal{L}_3$ any further than was achieved in [6].

6.2. **Twelve and Twenty-four Squares.** Explicit 'divisor' formulae for $r_{12}(n)$ and $r_{24}(n)$ are also known (e.g. p. 200 of [20] and §9 of Chapter 9 in [15]): they are

$$r_{12}(n) = 8(-1)^{n-1} \sum_{d|n} (-1)^{d+n/d} d^5 + 16\omega(n)$$

and

$$r_{24}(n) = \frac{16}{691}\sigma_{11}^*(n) + \frac{128}{691}\left((-1)^{n-1} 259\tau(n) - 512\tau(\frac{1}{2}n)\right)$$

where $\sigma_{11}^*(n) = \sum_{d|n} d^{11}$ if n is odd and $\sigma_{11}^*(n) = \sum_{d|n}(-1)^d d^{11}$ if n is even,

$$q((1-q^2)(1-q^4)(1-q^6)\cdots)^{12} = \sum_{n=1}^{\infty} \omega(n) q^n$$

and

$$q((1-q)(1-q^2)(1-q^3)\cdots)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Here $\tau(n)$ is the famous Ramanujan's τ-function.

We have recorded these representations because, while $N = 12$ and $N = 24$ (due to Ramanujan, see Chapter IX of [13]) are the next most accessible even cases, neither directly lead to an appropriate closed form for $\mathcal{L}_N$ let alone for $\mathcal{R}_N$. This is thanks to the impediment offered by ω and τ respectively: which encode knowledge, via the Jacobi triple-product, of all the representations of n as a sum of 4 or 8 squares. The divisor functions do produce appropriate L-function representations. Thus, using Ramanujan's ζ-function

$$g_{24}(s) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1},$$

which is discussed in detail in Chapter X of [13], it transpires that τ is multiplicative, with the preceding lovely Euler product. Additionally,

$$\begin{aligned} \mathcal{L}_{24}(s) = \sum_{n=1}^{\infty} \frac{r_{24}(n)}{n^s} &= \frac{16}{691}(2^{12-2s} - 2^{1-s} + 1)\zeta(s)\zeta(s-11) \\ &+ \frac{128}{691}(259 + 745 \cdot 2^{4-s} + 259 \cdot 2^{12-2s}) g_{24}(s). \end{aligned}$$

Similarly with $g_{12}(s) := \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s}$ one has

$$\mathcal{L}_{12}(s) = \sum_{n=1}^{\infty} \frac{r_{12}(n)}{n^s} = 8(1 - 2^{6-2s})\zeta(s)\zeta(s-5) + 16 g_{12}(s).$$

We also note that the analysis in [13], due to Rankin (see [22]), provides an 'almost closed form' for

$$f(s) := \sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^s} = \prod_p \left(1 + \tau^2(p)p^{-s} - p^{22-2s} - \frac{2\tau^2(p)p^{-s}}{1 + p^{11-s}}\right)^{-1}.$$

Rankin studied the above function $f(s)$ in [22] and showed that $f(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ with the only poles at $s = 12$ and at the complex zeros of $\zeta(2s - 22)$, all lying to the left of $\Re(s) = 12$. In [22], Rankin proved his famous result that $\tau(n) = O(n^{29/5})$. His proof depends on a functional equation of $f(s)$, namely,

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s-11)\zeta(2s-22)f(s) =$$
$$(2\pi)^{2s-46}\Gamma(23-s)\Gamma(12-s)\zeta(24-2s)f(23-s).$$

is invariant as $s \to 23 - s$. Finally, we note that a recent paper by Ewell [9] has a new divisor like recursion for τ.

Acknowledgments

The second author wishes to thank Professor P. Borwein for his support concerning this paper. The authors also wish to thank Greg Fee for some useful computational assistance and Stan Wagon and Richard Crandall for many stimulating exchanges. Finally, the authors wish to express their gratitude to Professor Paul Bateman for his thoughtful and gracious comments and suggestions, especially for improving and simplifying the error estimates to the average order of $r_N^2(n)$ in §3.

References

[1] G. E. Andrews, "The Fifth and Seventh Order Mock Theta Functions," *Transactions of the AMS*, **293** (1986), 113-134.
[2] P. Bateman, "On the Representation of a Number as the Sum of Three Squares", *Transactions of the AMS*, **71** (1951), 70-101.
[3] J.M. Borwein and P.B. Borwein, *Pi and the AGM. A study in analytic number theory and computational complexity*, CMS, Monographs and Advanced Texts, 4. John Wiley & Sons, New York, 1987. Paperback, 1998.
[4] L. Carlitz, "A note on the multiplication formulas for the Bernoulli and Euler polynomials," *Proceedings of the AMS*, **4** (1953), 184-188.
[5] R.D. Connors and J.P. Keating, "Degeneracy moments for the square billiard," *J. Phys. G: Nucl. Part. Phys.* **25** (1999), 555-562.
[6] R. E. Crandall, "New representations for the Madelung constant," *Experimental Mathematics*, **8:4** (1999), 367-379.
[7] R. E. Crandall, "Signal processing applications in additive number theory," (2001) preprint.
[8] R. Crandall and S. Wagon, "Sums of squares: Computational aspects," (2001) preprint.
[9] J.A. Ewell, "New representations of Ramanujan's tau function," *Proc. Amer. Math. Soc.* **128** (1999), 723-726.
[10] M. Glasser and I. Zucker. "Lattice Sums," in *Theoretical Chemistry : Advances and Perspectives,* **5** (1980), 67-139.
[11] E. Grosswald, *Representations of Integers as Sums of Squares*, Springer-Verlag, 1985.
[12] G.H. Hardy, *Collected Papers*, Vol I, Oxford University Press, 1969.
[13] G.H. Hardy, *Ramanujan*, Cambridge University Press, 1940. Revised Amer. Math. Soc. , 1999.
[14] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 5th Ed., Oxford, 1979.
[15] L.K. Hua, *Introduction to Number Theory*, Springer-Verlag, 1982.
[16] A.A. Karatsuba, *Basic Analytic Number Theory*, Springer-Verlag, 1991.
[17] E. Landau, *Vorlesungen über Zahlentheorie*, Leipzig, Hirzel, 1927.
[18] E. Landau, *Collected works*, Vol. 4. (German) Edited and with a preface in English by P. T. Bateman, L. Mirsky, H. L. Montgomery, W. Schaal, I. J. Schoenberg, W. Schwarz and H. Wefelscheid. Thales-Verlag, Essen, 1986.
[19] W. Nowak, "Zum Kreisproblem", österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **194** (1985), no. 4-10, 265-271.
[20] H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag, 1973.
[21] S. Ramanujan, "Some formulae in the analytic theory of numbers" , *Messenger of Math.*, **45** (1916), 81-84.

[22] R. Rankin, "Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions (I), (II), (III)", *Proc. Cambridge Philos. Soc.*, **35, 36** (1939), (1940), 351-356, 357-372, 150-151.
[23] M.M. Robertson and I.J. Zucker, "Exact Values for Some Two-dimensional Lattice Sums," *J. Phys. A: Math. Gen.* **8** (1975), 874-881.
[24] H.E. Rose, *A Course in Number Theory*, Oxford Science Publications, 2nd Ed, 1994.
[25] D. Shanks, "Calculation and Applications of Epstein Zeta Functions", *Math. Comp.*, **29** (1975), 271-287.
[26] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford Science Publications, 2nd Ed, 1986.
[27] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1966.

CECM, Department of Mathematics, Simon Fraser University, Burnaby B.C., Canada, V5A 1S6. Email: jborwein@cecm.sfu.ca, choi@cecm.sfu.ca