DIRICHLET SERIES FOR SQUARES OF SUMS OF SQUARES: A SUMMARY

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This note is a summary version of the results that the authors presented at CNTA VII in Montreal and detailed proofs may be found in [2].

Let σ_k denote the sum of kth powers of the divisors of n. There is a beautiful formula for the generating functions of $\sigma_k(n)$ (see Theorem 291 in Chapter XVII of [8])

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k), \qquad \Re(s) > \max\{1, k+1\} \tag{1}$$

which is in terms of only the Riemann Zeta function $\zeta(s)$. Following Hardy and Wright, by standard techniques, one can prove the following remarkable identity due to Ramanujan (also see Theorem 305 in Chapter XVII of [8] or [13])

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} \tag{2}$$

for $\Re(s) > \max\{1, a+1, b+1, a+b+1\}$. In this paper, we identify other arithmetical functions enjoying similarly explicit representations. We begin by generalizing the above result and proving that

Theorem 1. *If f_i and g_i are completely multiplicative arithmetical functions, then we have*

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)}{n^s} = \frac{L_{f_1 f_2}(s) L_{g_1 g_2}(s) L_{f_1 g_2}(s) L_{g_1 f_2}(s)}{L_{f_1 f_2 g_1 g_2}(2s)}$$

where $L_f(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$ is the Dirichlet series corresponding to f and the convolution $f * g$

$$f * g(n) := \sum_{d|n} f(d) g(n/d),$$

provided that $\Re(s)$ is greater than all the abscissa of absolute convergence of $L_{f_i}(s)$ and $L_{g_i}(s)$.

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This result recovers Hardy and Wright's formulae (1) and (2) immediately. In fact, Rankin and Selberg discovered the formula of Theorem 1 when they considered the convolution L-functions attached to automorphic forms (e.g. see Chapter 13 in [9] or [11]), as was pointed out by Ram Murty to the authors.

More generally, for certain classes of Dirichlet series, $\sum_{n=1}^{\infty} A(n)n^{-s}$, our Theorem 1 can be applied to obtain closed forms for the series $\sum_{n=1}^{\infty} A^2(n)n^{-s}$. In particular, if the generating function $L_f(s)$ of an arithmetic function f is expressible as a sum of products of two L-functions:

$$L_f(s) = \sum_{\chi_1,\chi_2} a(\chi_1,\chi_2) L_{\chi_1}(s) L_{\chi_2}(s)$$

for appropriate coefficients $a(\chi_1,\chi_2)$ and Dirichlet characters χ_i, then we are able to find a simple closed form (in term of L-functions) for the generating function $L_f^2(s) := \sum_{n=1}^{\infty} f^2(n)n^{-s}$.

One of our central applications is to the study of the number of representations as a sum of squares. Let $r_N(n)$ be the number of solutions to $x_1^2 + x_2^2 + \cdots + x_N^2 = n$ (counting permutations and signs). Hardy and Wright record a classical closed form, due to Lorenz, of the generating function for $r_2(n)$ in the terms of $\zeta(s)$ and a Dirichlet L-function, namely,

$$\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = 4\zeta(s)L_{-4}(s)$$

where $L_\mu(s) = \sum_{n=1}^{\infty} \left(\frac{\mu}{n}\right) n^{-s}$ is the *primitive L-function* corresponding to the *Kronecker symbol* $\left(\frac{\mu}{n}\right)$. Define

$$\mathcal{L}_N(s) := \sum_{n=1}^{\infty} \frac{r_N(n)}{n^s} \qquad \text{and} \qquad \mathcal{R}_N(s) := \sum_{n=1}^{\infty} \frac{r_N^2(n)}{n^s}.$$

Simple closed forms for $\mathcal{L}_N(s)$ are known for $N = 2, 4, 6$ and 8, due to the known explicit formula of $r_N(n)$ for these N (e.g. §91 in [12]); indeed the corresponding $q-$series were known to Jacobi,

$$\begin{aligned}
\mathcal{L}_2(s) &= 4\zeta(s)L_{-4}(s), \\
\mathcal{L}_4(s) &= 8(1-4^{1-s})\zeta(s)\zeta(s-1), \\
\mathcal{L}_6(s) &= 16\zeta(s-2)L_{-4}(s) - 4\zeta(s)L_{-4}(s-2), \\
\mathcal{L}_8(s) &= 16(1-2^{1-s}+4^{2-s})\zeta(s)\zeta(s-3).
\end{aligned}$$

Now applying an extension of Theorem 1 to these $\mathcal{L}_N$, we obtain the following closed forms for $\mathcal{R}_N(s)$:

Theorem 2. *We have*

$$\mathcal{R}_2(s) = \frac{(4\zeta(s)L_{-4}(s))^2}{(1+2^{-s})\zeta(2s)}, \qquad \Re(s) > 1,$$

$$\mathcal{R}_4(s) = 64\frac{(8\cdot 2^{3-3s} - 10\cdot 2^{2-2s} + 2^{1-s} + 1)\zeta(s-2)\zeta^2(s-1)\zeta(s)}{(1+2^{1-s})\zeta(2s-2)}, \qquad \Re(s) > 3,$$

$$\mathcal{R}_6(s) = 16\frac{(17-32\cdot 2^{-s})}{(1-16\cdot 2^{-2s})}\frac{\zeta(s-4)L_{-4}^2(s-2)\zeta(s)}{\zeta(2s-4)} - \frac{128}{(1+4\cdot 2^{-s})}\frac{L_{-4}(s-4)\zeta^2(s-2)L_{-4}(s)}{\zeta(2s-4)}, \qquad \Re(s) > 5,$$

and

$$\mathcal{R}_8(s) = 256\frac{(32\cdot 2^{6-2s} - 3\cdot 2^{3-s}+1)\zeta(s-6)\zeta^2(s-3)\zeta(s)}{(1+2^{3-s})\zeta(2s-6)} \qquad \Re(s) > 7.$$

An immediate application of Theorem 2 is to the estimation of the average order of $r_N^2(n)$. Using the usual hyperbola method of Dirichlet and a convolution argument, we can prove that

Theorem 3. *We have*

(3) $$\sum_{n\le x} r_2^2(n) = 4x\log x + 4\alpha x + O(x^{2/3}),$$

$$\sum_{n\le x} r_4^2(n) = W_4 x^3 + O(x^2\log^5 x),$$

$$\sum_{n\le x} r_6^2(n) = W_6 x^5 + O(x^4)$$

and

$$\sum_{n\le x} r_8^2(n) = W_8 x^7 + O(x^6)$$

where $\alpha := 2\gamma + \frac{8}{\pi}L'_{-4}(1) - \frac{12}{\pi^2}\zeta'(2) + \frac{1}{3}\log 2 - 1 = 2.0166216\cdots$ *and*

$$W_N := \frac{1}{(N-1)(1-2^{-N})}\frac{\pi^N}{\Gamma^2(\frac{1}{2}N)}\frac{\zeta(N-1)}{\zeta(N)}, \qquad (N\ge 3).$$

Here γ is the Euler's constant.

This technique can be adjusted to handle all integers $N \ge 2$ except $N = 3$ as below and so we establish all but the most difficult case of the following general conjecture due to Wagon (see [4]):

Wagon's Conjecture. *For integer $N \ge 3$, $\sum_{n\le x} r_N^2(n) \sim W_N x^{N-1}$ as $x \longrightarrow \infty$.*

Despite lacking the closed form for $\mathcal{R}_N(s)$ in general for $N \ge 5$, we can use the singular series formula for $r_N(n)$ given by Hardy (see p.342 of [7] or p.155 of [6]), which may be written as

$$r_N(n)\frac{\Gamma(N/2)}{\pi^{N/2}}n^{1-N/2} = \sum_{k=1}^{\infty}\sum_{\substack{1\le h\le k\\(h,k)=1}}\left(\frac{G(h,k)}{k}\right)^N e^{-2\pi i h n/k} + O(n^{1-N/4}),$$

where $G(h,k) = \sum_{j=1}^{k} e^{2\pi i h j^2/k}$ is the standard quadratic Gauss sum, to obtain

(4) $$\sum_{n\le x} r_N^2(n) = W_N x^{N-1} + O(x^{N-2} + x^{3N/4}) \qquad (N\ge 5).$$

The error term in (3) is not as good as the best known result. Indeed M. Kühleitner proved in [10] that

$$\sum_{n\le x} r_2^2(n) = 4x\log x + 4\alpha x + O(x^{1/2}(\log x)^{11/3}(\log\log x)^{1/3}).$$

However, for $N \ge 5$, the estimate $O(x^{N-2})$ in (4) in fact is the best possible.

We similarly study the number of representations by a binary quadratic forms. Let $r_{2,P}(n)$ be the number of solutions of the binary quadratic form $x^2 + Py^2 = n$. Define

$$\mathcal{L}_{2,P}(s) := \sum_{n=1}^{\infty} \frac{r_{2,P}(n)}{n^s} \qquad \text{and} \qquad \mathcal{R}_{2,P}(s) := \sum_{n=1}^{\infty} \frac{r_{2,P}(n)^2}{n^s}.$$

The closed forms of $\mathcal{L}_{2,P}(s)$ have been studied by a number of people, particular by Glasser, Zucker and Robertson (see [5] and [14]). In finding the exact evaluation of lattice sums, they are interested in expressing a multiple sum, such as the generating functions of $r_{2,P}(n)$, as a product of simple sums. As a result, plenty of closed forms of Dirichlet series $\sum_{(n,m)\neq(0,0)}(am^2+bmn+cn^2)^{-s}$ in terms of L-functions have been found. One of the most interesting cases is when the binary quadratic forms have *disjoint discriminants*, i.e, have only one form per genus. Then there are simple closed forms for the corresponding $\mathcal{L}_{2,P}(s)$ (see (9.2.8) in [1])

$$\mathcal{L}_{2,P}(s) = 2^{1-t}\sum_{\mu|P} L_{\epsilon_\mu \mu}(s) L_{-4P\epsilon_\mu/\mu}(s) \tag{5}$$

where P is an odd square-free number, t is the number of distinct factors of P and $\epsilon_\mu := \left(\frac{-1}{\mu}\right)$. Explicitly, (5) holds for all *type one* numbers. These include and may comprise:

$$P = 5, 13, 21, 33, 37, 57, 85, 93, 105, 133, 165, 177, 253, 273, 345, 357, 385, 1365.$$

It is known that there are only finitely many such disjoint discriminants. We call such P **solvable**. We have similar closed forms of L-functions for the quadratic form $x^2 + 2Py^2$ with discriminant $-8P$ (see (9.2.9) in [1]):

$$\mathcal{L}_{2,2P}(s) = 2^{1-t}\sum_{\mu|P} L_{\epsilon_\mu \mu}(s) L_{-8P\epsilon_\mu/\mu}(s)$$

for the *type two* integers

$$P = 1, 3, 5, 11, 15, 21, 29, 35, 39, 51, 65, 95, 105, 165, 231.$$

Again applying extensions of Theorem 1, we obtain closed forms for $\mathcal{R}_{2,P}(s)$ and $\mathcal{R}_{2,2P}(s)$.

Theorem 4. *Let P be a solvable square-free integer and let t be the number of distinct factors of P. We have for P respectively of type one and type two:*

$$\mathcal{R}_{2,P}(s) = 2^{2(1-t)} \sum_{\mu_1,\mu_2|P} L^2_{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}(s) L^2_{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}(s)\zeta(2s)^{-1}$$
$$\times \prod_{p|2P}\left\{1 + \left[\left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{p}\right) + \left(\frac{-4P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}{p}\right)\right]p^{-s}\right\}^{-1},$$

and

$$\mathcal{R}_{2,2P}(s) = 2^{2(1-t)} \sum_{\mu_1,\mu_2|P} L^2_{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}(s) L^2_{-8P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}(s)\zeta(2s)^{-1}$$
$$\times \prod_{p|2P} \left\{1+\left[\left(\frac{\epsilon_{\mu_1^*\mu_2^*}\mu_1^*\mu_2^*}{p}\right)+\left(\frac{-8P\epsilon_{\mu_1^*\mu_2^*}/\mu_1^*\mu_2^*}{p}\right)\right]p^{-s}\right\}^{-1}$$

where $\epsilon_\mu = \left(\frac{-1}{\mu}\right)$ *and* $\mu_i^* = \mu_i/(\mu_1,\mu_2)$.

In particular, the prime cases provide:

Corollary 5. *We have*

$$\mathcal{R}_{2,p}(s) = \frac{2\zeta^2(s)L^2_{-4p}(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L^2_p(s)L^2_{-4}(s)}{(1-2^{-s})(1+p^{-s})\zeta(2s)}$$

for $p = 5, 13, 37$, *while*

$$\mathcal{R}_{2,2}(s) = \frac{4\zeta^2(s)L^2_{-8}(s)}{(1+2^{-s})\zeta(2s)}.$$

Similarly,

$$\mathcal{R}_{2,2p}(s) = \frac{2\zeta^2(s)L^2_{-8p}(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L^2_{-p}(s)L^2_8(s)}{(1-2^{-s})(1-p^{-s})\zeta(2s)},$$

for $p = 3, 11$ *while*

$$\mathcal{R}_{2,2p}(s) = \frac{2\zeta^2(s)L^2_{-8p}(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L^2_p(s)L^2_{-8}(s)}{(1-2^{-s})(1-p^{-s})\zeta(2s)}$$

for $p = 5, 29$.

Closed forms for $\mathcal{L}_{2,P}(s)$ are also accessible for some P other than those of *type one* or *type two*. For example, (see Table VI of [5]) one has

$$\mathcal{L}_{2,3}(s) = (2+4^{1-s})\,\zeta(s)L_{-3}(s)$$

and hence we obtain

$$\mathcal{R}_{2,3}(s) = 4\frac{1+2^{3-2s}}{1+3^{-s}}\frac{(\zeta(s)\,L_{-3}(s))^2}{\zeta(2s)}. \tag{6}$$

We may also derive many formulae for non-square free integers via modular transformations [1]. We contain ourselves with the simplest example which is

$$\mathcal{R}_{2,4}(s) = \frac{4-2^{2-s}+2^{4-2s}}{1+2^{-s}}\frac{(\zeta(s)\,L_{-4}(s))^2}{\zeta(2s)}$$

as a consequence of a quadratic transformation leading to

$$\mathcal{L}_{2,4}(s) = (2^{-1}-2^{-1-s}+4^{-s})\mathcal{L}_2(s).$$

There are some simple closed forms of the generating functions for more general binary quadratic forms found in [5]. Let

$$\mathcal{L}_{(a,b,c)}(s) := \sum_{(n,m)\neq(0,0)} \frac{1}{(am^2+bmn+cn^2)^s} = \sum_{n=1}^\infty \frac{r_{(a,b,c)}(n)}{n^s}$$

and $\mathcal{R}_{(a,b,c)}(s) := \sum_{n=1}^{\infty} \frac{r_{(a,b,c)}(n)^2}{n^s}$ where $r_{(a,b,c)}(n)$ is the number of representations of n by the quadratic form $ax^2 + bxy + cy^2$. Then, we have (e.g. (26) of [15])

$$\sum_{h(D)} \mathcal{L}_{(a,b,c)}(s) = \omega(D)\zeta(s)L_D(s)$$

where the sum is taken over the $h(D)$ inequivalent reduced quadratic forms of discriminant $D := b^2 - 4ac$ and $\omega(-3) = 6, \omega(-4) = 4$ and $\omega(D) = 2$ for $D < -4$. In particular, for $c = 2, 3, 5, 11, 17, 41$, $h(D) = 1$ and the result is especially simple:

$$\mathcal{L}_{(1,1,c)}(s) = 2\zeta(s)L_D(s).$$

Hence from Theorem 1, we have

$$\mathcal{R}_{(1,1,c)}(s) = \frac{4(\zeta(s)L_D(s))^2}{(1+|D|^{-s})\zeta(2s)},$$

with similar formulae for $(a, b, c) = (1, 1, 1)$ and $(1, 0, 1)$. Thanks to the *On-Line Encyclopedia of Integer Sequences*

`http://www.research.att.com/~njas/sequences/`

we discover that the sequence $2, 3, 5, 11, 17, 41$ is exactly the so-called Euler "lucky" numbers which are the numbers n such that $m \to m^2 - m + n$ has prime values for $m = 0, \cdots, n-1$.

Applying standard complex integration methods to $\mathcal{R}_{2,P}(s)$ and $\mathcal{R}_{2,2P}(s)$, we derive the asymptotic formula

Theorem 6. *Let P be a solvable square-free integer. Let $x > 1$ and $\epsilon > 0$. We have for either $N = P$ of type one or $N = 2P$ of type two:*

$$\sum_{n\le x} r_{2,N}(n)^2 = \frac{3}{N}\left(\prod_{p|2N}\frac{2p}{p+1}\right)(x\log x + \alpha(N)x) + O(N^{\frac{1}{4}+\epsilon}x^{\frac{3}{4}+\epsilon})$$

where the implicit constants are independent of both x and P and

$$\alpha(N) := 2\gamma + \sum_{p|2N}\frac{\log p}{p+1} + 2\frac{L'_{-4N}(1)}{L_{-4N}(1)} - \frac{12}{\pi^2}\zeta'(2) - 1$$

and $\sum_{p|n}$ is the summation over all prime factors of n.

Akin to Wagon's conjecture, we make the following conjecture.

Quadratic Conjecture. *For any square-free P,*

$$\sum_{n\le x} r_{2,P}(n)^2 \sim \frac{3}{P}\left(\prod_{p|2P}\frac{2p}{p+1}\right)x\log x$$

as $x \to \infty$.

In view of Theorem 6, (3) and (6), our conjecture is true for solvable P and for $P = 1, 3$. We have also confirmed it for $P = 7$ and 15 from the representations of

$$\mathcal{L}_{2,7}(s) = 2\,(1 - 2^{1-s} + 2^{1-2s})\zeta(s)L_{-7}(s)$$

and

$$\mathcal{L}_{2,15}(s) = (1 - 2^{1-s} + 2^{1-2s})\zeta(s)L_{15}(s) + (1 + 2^{1-s} + 2^{1-2s})L_{-3}(s)L_5(s)$$

again given in [5], which leads to

$$\mathcal{R}_{2,7}(s) = 4\frac{(1-3\cdot 2^{-s}+2^{2-2s})}{(1+2^{-s})(1+7^{-s})}\frac{(\zeta(s)L_{-7}(s))^2}{\zeta(2s)}$$

and

$$\mathcal{R}_{2,15}(s) = \frac{2(1-3\cdot 2^{-s}+2^{2-2s})}{(1+2^{-s})(1+3^{-s})(1+5^{-s})}\frac{(\zeta(s)L_{-15}(s))^2}{\zeta(2s)} + \frac{2(1+3\cdot 2^{-s}+2^{2-2s})}{(1-2^{-s})(1-3^{-s})(1-5^{-s})}\frac{(L_{-3}(s)L_5(s))^2}{\zeta(2s)},$$

and may be analyzed by the methods above.

For the negative P, we have studied only the case $P=-1$. By the elementary formula due to Sierpinski [16]

$$r_{2,-1}(n) = 2d(n) - 4d(\frac{n}{2}) + 4d(\frac{n}{4})$$

we obtain

$$\mathcal{L}_{2,-1}(s) = 2(1-2^{1-s}+2^{1-2s})\zeta(s)^2$$

and

$$\mathcal{R}_{2,-1}(s) = 4\frac{(1-3\cdot 2^{-s}+4\cdot 2^{-2s})\zeta(s)^4}{(1+2^{-s})\zeta(2s)}$$

where $d(n)$ is the divisor function and $d(x)=0$ if x is not an integer.

We also studied $\mathcal{L}_N(s)$, for all $N \geq 2$, and obtained a Bessel-series in $r_{N-1}(m)$:

$$\mathcal{L}_N(s) = \sum_{n>0}\frac{r_N(n)}{n^s} = \frac{2N\,\Gamma(s-\frac{N-3}{2})}{\Gamma(s+1)}\pi^{\frac{N-1}{2}}\zeta(2s-N+1) + \frac{4N\,\pi^{s+1}}{\Gamma(s+1)}\sum_{m>0}\frac{m^{\frac{1}{2}s}\,r_{N-1}(m)}{m^{\frac{N-3}{4}}}\sum_{n>0}\frac{n^{\frac{N+1}{2}}}{n^s}K_{s-\frac{N-3}{2}}(2n\pi\sqrt{m}). \tag{7}$$

Here K_s is the *modified Bessel function of the second kind.* This is especially attractive for $N=3,5,7,9$ since in these cases $r_{N-1}(n)$ is an explicit divisor function.

Most pleasantly, for 'jellium', which is the Wigner sum analogue of Madelung's constant, we have

$$\mathcal{L}_3(1/2) = -\pi + 3\pi\sum_{m>0}\frac{r_2(m)}{\sinh^2(\pi\sqrt{m})},$$

in which the exponential convergence is entirely apparent.

Some similar identities are

$$\mathcal{L}_5(3/2) = -\frac{10}{9}\pi^2\left(1+3\sum_{m>0}\frac{r_4(m)}{\sinh^2(\pi\sqrt{m})}\right),$$

$$\mathcal{L}_7(5/2) = -\frac{28}{45}\pi^3\left(1+3\sum_{m>0}\frac{r_6(m)}{\sinh^2(\pi\sqrt{m})}\right),$$

$$\mathcal{L}_9(7/2) = -\frac{8}{35}\pi^4\left(1+3\sum_{m>0}\frac{r_8(m)}{\sinh^2(\pi\sqrt{m})}\right),$$

and

$$R_{2,4}(2) = 8\mathrm{G}^2$$

where $\mathrm{G} := \sum_{\mathrm{n}\geq 0}(-1)^{\mathrm{n}}\,(2\mathrm{n}+1)^{-2}$ is *Catalan's constant.*

For a survey of other rapidly convergent lattice sums of this type see [1] and [3]. Unfortunately we have not been able to extend this analysis to $R_N(s)$.

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