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A NUMERICAL BOUND FOR BAKER'S CONSTANT - SOME EXPLICIT ESTIMATES FOR SMALL PRIME SOLUTIONS OF LINEAR EQUATIONS

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In 1989, M.C. Liu and K.M. Tsang proved that there is an absolute constant $V > 0$ such that the linear equation $a_1p_1 + a_2p_2 + a_3p_3 = b$ has prime solutions p_j's if $b \gg (\max_j a_j)^V$ and $a_j > 0$. Apart from the numerical value of V, the bound is sharp. In this manuscript, we obtain a numerical bound for V. We also obtain a numerical bound for the small prime solutions of the above equation if the a_j's are not all of the same sign.

1 INTRODUCTION

Let a_1, a_2, a_3 be any nonzero integers such that

$$\gcd(a_1, a_2, a_3) = 1. \tag{1.1}$$

Let b be any integer satisfying

$$b \equiv a_1 + a_2 + a_3 (\mathrm{mod} 2) \quad \text{and} \quad \gcd(b, a_i, a_j) = 1 \text{ for } 1 \le i < j \le 3. \tag{1.2}$$

Write $A := \max\{3, |a_1|, |a_2|, |a_3|\}$.

In [11], M.C. Liu and K. M. Tsang studied a problem of A. Baker [1], namely, the solubility and the size of small prime solutions p_1, p_2, p_3 of the linear equation

$$a_1p_1 + a_2p_2 + a_3p_3 = b. \tag{1.3}$$

They proved that

THEOREM I *Suppose a_1, a_2, a_3 are all positive and satisfy (1.1). Then there exists an effective absolute constant $V > 0$ such that if b satisfies (1.2) and $b \ge A^V$, then the equation (1.3) has a solution in primes p_1, p_2, p_3.*

THEOREM II *Suppose a_1, a_2, a_3 are not all of the same sign and satisfy (1.1). Then there exists an effective absolute constant $B > 0$ such that whenever b satisfies (1.2), the equation (1.3) has a solution in primes p_1, p_2, p_3 satisfying*

$$\max\{p_1, p_2, p_3\} \le 3|b| + A^B.$$

They also mentioned in [11] that apart from the numerical values of V, B, the bounds A^V, A^B are sharp, i.e., of the right order of infinity. Furthermore, by putting $a_1 = a_2 = a_3 = 1$, we see that Vinogradov's famous three primes theorem is a special case of Theorem I. In [12] they define the Baker Constant $\mathcal{B}$ and the Vinogradov Constant $\mathcal{V}$ to be the infima for all possible values of the constant B and the constant V respectively. It can be shown that (see, for example, §2 in [12]) the Linnik Theorem on the smallest prime p in arithmetical progressions (i.e., for $1 \le l \le q$ with $(l, q) = 1$, there is an absolute constant $L > 0$ such that $p \ll q^L$) can be derived easily and directly from Theorem I or Theorem II, $\mathcal{B} \ge \mathcal{L}$ and $\mathcal{V} \ge \mathcal{L}+1$, where $\mathcal{L}$ is the Linnik Constant defined as the infimum for all possible values of the constant L. They also made in [12] the conjecture that the Baker constant $\mathcal{B}$ is 1 and the Vinogradov constant $\mathcal{V}$ is 2.

In view of the relation with Linnik's constant, it is now worthwhile to obtain explicit values of V and B. Our main objective in this manuscript is to prove that $V \le 4191$ and $B \le 4190$. In particular, we have the following results:

Theorem 1 *Suppose a_1, a_2, a_3 are all positive and satisfy (1.1). If b satisfies (1.2) and $b \gg (a_1a_2a_3)^{1397}$, then the equation (1.3) has a solution in primes p_1, p_2, p_3.*

Theorem 2 *Suppose a_1, a_2, a_3 are not all of the same sign and satisfy (1.1). Whenever b satisfies (1.2), the equation (1.3) has a solution in primes p_1, p_2, p_3 satisfying*

$$\max_{1 \le j \le 3} \{|a_j|p_j\} \ll |b| + |a_1a_2a_3|^{1397}.$$

A conditional result of Theorems 1 and 2 under the Generalized Riemann Hypothesis (GRH) was studied in [2], [3] and [4]. Roughly speaking, in [4] we proved that under GRH, $\mathcal{V} \le 5$ and $\mathcal{B} \le 4$.

In proving Theorems 1 and 2, we require quantitative estimates for zero-free regions and the Deuring-Heilbronn phenomenon of Dirichlet's L-functions. These are stated in Lemmas 2.1 and 2.2. We also require a form of Linnik's density theorem.

2 ZERO-FREE REGION FOR L-FUNCTION

Let χ be a Dirichlet character modulo q and $s = \sigma + it$.

Lemma 2.1 *For $Q \ge 15$ and $T \ge 1$, there is at most one primitive character χ to a modulus $\le Q$ for which the corresponding $L(s, \chi)$ has a zero in the region*

$$\sigma \ge 1 - \frac{1}{\lambda_1 \log(QT)}, \quad |t| \le T, \tag{2.1}$$

where $\lambda_1 = 9.645908801$ and if there is such an exception, the exceptional zero is real, simple and unique.

Proof. This follows directly from Theorems 1 and 2 in [13]. □

If the exceptional zero in Lemma 2.1 exists, we denote the exceptional zero by $\tilde{\beta}$ and the corresponding exceptional character by $\tilde{\chi}(\mathrm{mod}\tilde{r})$.

Lemma 2.2 *Let $\chi_1(\mathrm{mod}q)$ be a real non-principal character and $\chi(\mathrm{mod}q)$ be any character. Let $\beta_1 = 1-\delta_1$ be a real zero of $L(s,\chi_1)$ and $\rho = \beta + i\tau = 1-\delta+i\tau$ be a zero of $L(s,\chi)$ with $\delta < 1/6, \beta < \beta_1$. Let ϵ be arbitrary small positive real number. Suppose that $D = q(|\tau|+1)$ is sufficiently large that is, $D \geq D_0(\epsilon)$. Then*

$$\delta_1 \geq \frac{2}{3}(1-6\delta)D^{-(3/2+\epsilon)\delta/(1-6\delta)}/\log D.$$

Proof. This is Theorem 2 in [16]. □

The next lemma is the Deuring-Heilbronn phenomenon which is a direct consequence of Lemma 2.2.

Lemma 2.3 *Let ϵ be arbitrary small positive real number. Suppose the exceptional zero $\tilde{\beta}$ in Lemma 2.1 exists. If Q and T are sufficiently large and $0 < c < 1$, then the zero-free region in (2.1) can be extended to*

$$\sigma \geq 1 - \min\left\{c/6, \frac{(1-c)(2/3-\epsilon)}{\log(Q^2T)}\log\left(\frac{(1-c)(2/3-\epsilon)}{(1-\tilde{\beta})\log(Q^2T)}\right)\right\}, |t| \leq T.$$

Proof. Lemma 2.3 follows easily from Lemmas 2.1 and 2.2. □

3 DENSITY THEOREM OF LINNIK

In proving his famous theorem on the least prime in the arithmetic progression [9], [10], Linnik proved two theorems concerning the distribution of the zeros of Dirichlet's L-functions: Linnik's density theorem and Linnik's theorem on the Siegel zeros (the Deuring-Heilbronn phenomenon).

Let $N_\chi(\alpha,T)$ be the number of zeros of Dirichlet's L-function $L(s,\chi)$ in the rectangle : $\alpha \leq Re(s) < 1, |Im(s)| \leq T$. Linnik's density theorem states that for $1/2 \leq \alpha \leq 1$,

$$Z(Q,\alpha,T) := \sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi(\bmod q)} N_\chi(\alpha,T) \ll (Q^2T)^{c(1-\alpha)}.$$

Here $\sum^*_{\chi(\bmod q)}$ denotes the summation over all primitive characters $\chi(\mathrm{mod}q)$ and c is effective absolute constant. Similar result was also proved by Gallagher in [6]. Linnik's density theorem is very delicate near $\sigma = 1$ and is essential in the proofs of Theorems I and II in [11]. In [8], M. Jutila proved that for $4/5 \leq \alpha \leq 1, T \geq 1$, we have

$$Z(Q,\alpha,T) \leq K(\epsilon)(Q^2T)^{(2+\epsilon)(1-\alpha)}.$$

However, to obtain explicit values for V, B in Theorems I and II in [11], the numerical value of $K(\epsilon)$ is needed. So, in this section we shall prove the following theorem.

Theorem 3 *Let QT be sufficiently large. For $4/5 \le \alpha < 1$,*

$$Z(Q,\alpha,T) \le c_1(Q^2T)^{c_2(1-\alpha)}, \tag{3.1}$$

where $c_1 = 474.6438$ and $c_2 = 17.25$.

To prove Theorem 3, we follow closely the arguments in the proof of Theorem 1 in [8]. First, we prove the following lemma.

Lemma 3.1 *For any non-principal primitive character χ modulo $q \le Q$, $1/2 \le \alpha < 1$ and $\epsilon > 0$, the number of zeros of $L(s,\chi)$ in the rectangle : $\alpha \le Re(s) \le 1$, $|Im(s) - t| \le r(1-\alpha)$ is*

$$\le \left(\frac{(1+r)^2 + r^2}{1+r}\right)\left(\frac{1}{2}(1+\epsilon)(1-\alpha)\log Q(|t|+1) + \frac{1}{r}\right), \tag{3.2}$$

provided that Q exceeds a certain bound, depending on r and ϵ only.

Proof. The proof is similar to Lemma 8 in [8]. □

Now, we come to prove Theorem 3. Let ϵ and ϵ' be arbitrary small positive numbers. In view of [7, p297-298] and Lemma 2.1, we can assume that $1-\epsilon \le \alpha \le 1 - \frac{1}{\lambda_1 \log QT}$. Also, in view of the zero-free region for $\zeta(s)$ [14, p.87, (11.7)] and the estimate for the number of zeros of $\zeta(s)$ [15, p.235, (9.18.3)], we have

$$N_{\chi_0}(\alpha,T) \le \epsilon(Q^2T)^{c_2(1-\alpha)}, \tag{3.3}$$

for sufficiently large T. From now on, we only consider the zeros of $L(s,\chi)$ with $\chi \ne \chi_0$.

Let $D = Q^2T$ and $\Delta = (\log D)^{-1}$. We split up the rectangle $R(\alpha,T)$ into smaller rectangles $R_j(\alpha,T), j = 0, \pm1, \cdots$, where

$$R_j(\alpha,T) := \{\sigma + it : \alpha \le \sigma \le 1, \max(-T, j\Delta) \le t \le \min(T,(j+1)\Delta)\}. \tag{3.4}$$

For any primitive character $\chi(\mathrm{mod} q)$ with $q \le Q$, if the function $L(s,\chi)$ has zeros in $R_j(\alpha,T)$, we choose arbitrary one zero as a representative. Then we group all such representative zeros and divide them into two sets according as j is even or odd. We denote by J the cardinality of the set which contains more zeros. Let $\rho_1, \cdots, \rho_J$ be the zeros of this set and $\chi_1(\mathrm{mod} q_1), \cdots, \chi_J(\mathrm{mod} q_J)$ be the primitive characters corresponding to $\rho_1, \cdots, \rho_J$ respectively. From the above construction, we know that the distance between any two zeros in $\{\rho_1, \cdots, \rho_J\}$ is greater than Δ.

Let $f = 0.524998, e_1 = 3.1, e_2 = 4.3$ and $c = 5.324999$. We also let

$$R := D^f, z_1 := D^{e_1}, z_2 := D^{e_2}, X := D^c \text{ and } x := D^c \log^2 D. \tag{3.5}$$

Define

$$\lambda_d := \begin{cases} \mu(d) & \text{if } 1 \le d < z_1, \\ \mu(d)\log(z_2/d)/\log(z_2/z_1) & \text{if } z_1 \le d \le z_2, \\ 0 & \text{if } z_2 < d, \end{cases}$$

and

$$a(n) := \sum_{d|n} \lambda_d,$$

where $\mu(n)$ is the Möbius function. Also, let $\psi(n) = \mu(n)\phi(n)$ and $\psi_r(n) = \psi((n,r))$, where $\phi(n)$ is the Euler totient function.

Now using similar proof to Lemma 6 in [8], we have for $j = 1, 2, \cdots J$,

$$\frac{q_j}{\phi(q_j)}|g(\rho_j, \chi_j)| \geq \frac{6f}{(1+\epsilon')\pi^2}\log D \tag{3.6}$$

where

$$g(s,\chi) := \sum_{z_1<n\leq x} a(n)\chi(n)e^{-n/X}n^{-s} \sum_{\substack{r\leq R\\(r,q)=1}} \frac{\mu^2(r)\psi_r(n)}{r}.$$

Let M and N be variables such that $M = e^{\xi}$ and $N = e^{\eta}$ where $\frac{1}{2}\log z_1 \leq \xi \leq \log z_1$ and $\log x \leq \eta \leq \frac{3}{2}\log x$. Let

$$c(n, \chi(\mathrm{mod} q)) := \sum_{\substack{r\leq R\\(r,q)=1}} \mu^2(r)r^{-1}\psi_r(n).$$

By (3.7) in [8] with $|\eta_j| = q_j\phi(q_j)^{-1}, j = 1, 2, \cdots, J$, we have

$$\begin{aligned} &\left(\sum_{j=1}^{J} \frac{q_j}{\phi(q_j)}|g(\rho_j,\chi_j)|\right)^2 \\ &\leq \sum_{z_1<n\leq x} a(n)^2 e^{-2n/X}\alpha_n^{-1} \sum_{j,k=1}^{J} \bar{\eta_j}\eta_k B(\bar{\rho_j}+\rho_k, \chi_j, \chi_k), \end{aligned} \tag{3.7}$$

where $\alpha_n := n^{2\alpha-1}(e^{-n/N} - e^{-n/M})$ and

$$B(\bar{\rho_j}+\rho_k, \chi_j, \chi_k) = \sum_{m=1}^{\infty} c(m,\chi_j)c(m,\chi_k)\alpha_m\bar{\chi_j}\chi_k(m)m^{-\bar{\rho_j}-\rho_k}.$$

Using the elementary inequality $e^{-2n/X}(e^{-n/N} - e^{-n/M})^{-1} \leq (1-e^{-1})^{-1}(1+\epsilon')$ for $z_1 < n \leq x$, Lemma 4 in [8] and (3.5), it follows from (3.6) and (3.7) that

$$J^2\log^2 D \leq k_1 x^{2(1-\alpha)} \sum_{j,k=1}^{J} \bar{\eta_j}\eta_k B(\bar{\rho_j}+\rho_k, \chi_j, \chi_k), \tag{3.8}$$

where $k_1 = \frac{(c+\epsilon'-e_1)(1+\epsilon')^4\pi^4}{36f^2(1-e^{-1})(e_2-e_1)}$. As proved in [8], we have

$$\begin{aligned} B(s,\chi_j,\chi_k) &= E(\bar{\chi_j}\chi_k)\beta(2\alpha - s) \times \\ &\times \sum_{\substack{r\leq R\\(r,q_j)=(r,q_k)=1}} \frac{\mu^2(r)\phi(r)}{r^2} + O((Q^2T)^{1/2}M^{-1+\epsilon'}R^2), \end{aligned} \tag{3.9}$$

where $E(\chi) = \phi(q)/q$ or 0 according as χ is principal or not and $\beta(s) = \Gamma(s)(N^s - M^s)$. Now fix j in $1 \le j \le J$, let $\chi_{k_i}(\text{mod} q_j)(i = 1, \cdots, t)$ be the primitive characters among $\chi_1, \cdots, \chi_J$ such that $\chi_{k_i} = \chi_j$ for $i = 1, \cdots, t$ and ρ_{k_i} be the corresponding selected zeros. For convenience, we let $\rho_{k_1} = \rho_j$ and $|Im(\rho_j - \rho_{k_i})| \ge (i - \frac{3+(-1)^{i+1}}{2})\Delta$ for $i = 2, \cdots, t$. Therefore, from (3.8) and (3.9), we have

$$\begin{aligned} J^2 \log^2 D &\le k_1 x^{2(1-\alpha)} \sum + O(x^{2(1-\alpha)} J^2 R^2 D^{1/2} z_1^{-\frac{1-\epsilon'}{2}} \log^2 D) \\ &\le (1+\epsilon') k_1 x^{2(1-\alpha)} \sum \end{aligned} \tag{3.10}$$

where

$$\sum = \sum_{j=1}^{J} \sum_{i=1}^{t} \bar{\eta}_j \eta_{k_i} \frac{\phi(q_j)}{q_j} \beta(2\alpha - \bar{\rho}_j - \rho_{k_i}) \sum_{\substack{r \le R \\ (r, q_j)=1}} \frac{\mu^2(r)\phi(r)}{r^2}.$$

Let $\sum = \sum_1 + \sum_2$ where $\sum_1$ is the summation over $1 \le j \le J$ with $i = 1$ and $\sum_2$ is the double summation over $1 \le j \le J$ and $2 \le i \le t$. Since $-2\epsilon \le \theta \le 0$, so using the inequality $\beta(\theta) \le (1+\epsilon') \log(N/M)$ and Lemma 5 in [8], we have

$$\sum\nolimits_1 \le (1+\epsilon')^2 f(\frac{3}{2}c + \epsilon' - \frac{1}{2}e_1) J \log^2 D := k_2 J \log^2 D. \tag{3.11}$$

Concerning $\sum_2$, we first note that $-2\epsilon \le Re\theta_i \le 0$ and $|\theta_i| \ge (i - \frac{3+(-1)^{i+1}}{2})\Delta$, for $i = 2, \cdots, t$. Then we integrate with respect to ξ and η and obtain

$$\begin{aligned} &\left| \int_{\frac{1}{2}\log z_1}^{\log z_1} \int_{\log x}^{\frac{3}{2}\log x} \sum\nolimits_2 d\eta d\xi \right| (\frac{1}{2}\log x)^{-1} (\frac{1}{2}\log z_1)^{-1} \\ \le\ & 4(c + e_1 + \epsilon') \sum_{j=1}^{J} \frac{q_j}{\phi(q_j)} \sum_{\substack{r \le R \\ (r, q_j)=1}} \frac{\mu^2(r)\phi(r)}{r^2} \sum_{i=2}^{t} \left| \frac{\Gamma(\theta_i)}{\theta_i} \right| \frac{\log D}{\log z_1 \log x}. \end{aligned} \tag{3.12}$$

Since $-2\epsilon \le Re\theta_i \le 0$, we have $|\Gamma(\theta_i)| \le \frac{(1+\epsilon')}{|Im(\theta_i)|}$. Hence $\sum_{i=2}^{t} \left| \frac{\Gamma(\theta_i)}{\theta_i} \right| \le \frac{\pi^2}{4}(1 + \epsilon') \log^2 D$ and by this, Lemma 5 in [8], (3.5) and (3.12), we have

$$\left| \int \int \sum\nolimits_2 d\eta d\xi \right| (\frac{1}{2}\log x)^{-1} (\frac{1}{2}\log z_1)^{-1} \le k_3 J \log^2 D. \tag{3.13}$$

where $k_3 := \frac{\pi^2 f(c+e_1+\epsilon')(1+\epsilon')^2}{e_1 c}$. In view of (3.10), (3.11) and (3.13), we have

$$J \le k_1(k_2 + k_3)(1+\epsilon') x^{2(1-\alpha)},$$

for sufficiently large D. In view of (3.4), by putting $r = \lambda_1/2$ into (3.2) in Lemma 3.1 and choosing a suitable ϵ' in terms of ϵ, we obtain for any $\lambda > 0$,

$$Z(Q, \alpha, T) \le 2J \left(\frac{(1+\frac{\lambda_1}{2})^2 + (\frac{\lambda_1}{2})^2}{1+\frac{\lambda_1}{2}} \right) \left(\frac{1}{2}(1+\epsilon')(1-\alpha)\log D + \frac{2}{\lambda_1} \right)$$

$$\leq \frac{3132.649}{\lambda} D^{(2c+\lambda+\epsilon)(1-\alpha)}. \tag{3.14}$$

Then (3.1) follows from (3.3) and (3.14) by putting $\lambda = 6.6$ and thus completes the proof of Theorem 3.

4 STRUCTURE OF THE PROOFS OF THEOREMS 1 AND 2

Let

$$\begin{cases} Q := N^{\delta}, T := N^{\delta c_3}, \tau := N^{-1} T^{2/3} Q^{1/6} \quad \text{and} \quad \epsilon_0 = 10^{-10}, \\ c_3 = 2.375001 \quad \text{and} \quad \delta = 0.002864. \end{cases} \tag{4.1}$$

Suppose

$$N \geq |a_1 a_2 a_3|^{4/\delta + \epsilon_0} \quad \text{and} \quad N \geq N_0, \tag{4.2}$$

where N_0 is an effective large positive constant.

In view of Lemmas 2.1, 2.3 and (4.1), there is at most one primitive character χ to a modulus $\leq Q$ for which the corresponding $L(\sigma + it, \chi)$ has a zero in the region : $\sigma \geq 1 - \eta(T), |t| \leq T$ where $\eta(T) = c_4 / \log T$ and $c_4 = 0.072953$ and if the exceptional zero $\tilde{\beta}$ exists, then it is real, simple, unique and the zero-free region can be extended to $\eta(T) := \frac{c_5}{\log T} \log \left\{ \frac{c_5}{(1-\tilde{\beta}) \log T} \right\}$ where $c_5 = 0.361904$ or 0.248339 according as $(1 - \tilde{\beta}) \log T > T^{-\epsilon_0}$ or not. Let $\tilde{\chi}(\text{mod}\tilde{r}), \tilde{r} \leq Q$ be the corresponding exceptional character if the exceptional zero $\tilde{\beta}$ exists. Then by [5, p.96, (12)], we have

$$\frac{c_6}{\sqrt{\tilde{r}} \log^2 \tilde{r}} \leq 1 - \tilde{\beta} \leq \frac{c_4}{\log T}, \tag{4.3}$$

where c_6 is an effective positive constant which explicit value is irrelevant to our calculation.

For any $y > 0$ and any $\chi(\text{mod} q)$ with $q \leq Q$, let

$$S_\chi(y, T) := \sum_{|\gamma| \leq T}' y^{\beta - 1}, \tag{4.4}$$

where $\sum_{|\gamma| \leq T}'$ denotes the summation over all zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ lying inside the region: $|\gamma| \leq T, 1/2 \leq \beta < 1 - \eta(T)$ and $\eta(T)$ is defined as before according as $\tilde{\beta}$ exists or not. Let

$$\Omega := \begin{cases} (1 - \tilde{\beta}) \log T & \text{if } \tilde{\beta} \text{ exists}, \\ 1 & \text{otherwise}. \end{cases} \tag{4.5}$$

Lemma 4.1 *For any fixed constant $c > 0$, if $y \geq cN|a_1 a_2 a_3|^{-1}$, we have*

$$\sum_{q \leq Q} \sum_{\chi(\bmod q)}^{*} S_\chi(y, T) \leq K_1 \Omega^3, \tag{4.6}$$

where K_1 *is* 1.63×10^{-23} *or* 0.136276 *according as* $\tilde\beta$ *exists or not.*

Proof. If $y \ge cN|a_1a_2a_3|^{-1}$, by (4.2) we have $y \ge N^{1-\delta/4}$. Thus by (4.4),

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi(\bmod q)} S_\chi(y,T) \le -\int_{1/2}^{1-\eta(T)} N^{(1-\delta/4)(\alpha-1)} dZ(Q,\alpha,T). \tag{4.7}$$

By Theorem 12.2 in [14] and (4.1), it is easy to show that the integration over $[\frac{1}{2},\frac{4}{5}]$ in (4.7) is $\le N^{-0.487}+N^{-0.193}$. Also, using (3.1) in Theorem 3 and (4.1), the integration over $[\frac{4}{5},1-\eta(T)]$ in (4.7) is $\le \frac{c_1(1-\delta/4)}{c_7}N^{-c_7\eta(T)}+N^{-0.156}$, where $c_7 = 1-\{\frac{1}{4}+(2+c_3)c_2\}\delta$. Hence (4.6) follows from (4.5) and (4.7). □

Let $\Lambda(n)$ denote the von Mangoldt function. For any real y, we write $e(y)$ for $e^{2\pi i y}$ and $e_q(y)$ for $e(y/q)$. Let for $j=1,2,3$,

$$N_j := N|a_j|^{-1}, \quad N_j' := N(4|a_j|)^{-1}$$

and define

$$S_j(y) := \sum_{N_j'<n\le N_j} \Lambda(n)e(a_jny), \quad S_j(\chi,y) := \sum_{N_j'<n\le N_j} \Lambda(n)\chi(n)e(a_jny), \tag{4.8}$$

$$I_j(y) := \int_{N_j'}^{N_j} e(a_jxy)dx, \quad \tilde I_j(y) := \int_{N_j'}^{N_j} x^{\tilde\beta-1}e(a_jxy)dx, \tag{4.9}$$

$$I_j(\chi,y) := \int_{N_j'}^{N_j} e(a_jxy) \sideset{}{'}\sum_{|\gamma|\le T} x^{\rho-1}dx, \tag{4.10}$$

for any $\chi(\bmod q)$ with $q\le Q$.

For any integers h and q satisfying $1\le h\le q\le Q$ and $(h,q)=1$, let $m(h,q)$ be the open interval $((h-\tau)/q,(h+\tau)/q)$. Let $\mathcal{M}$ denote the union of these intervals $m(h,q)$ and $\mathcal{M}'$ denote the complement of $\mathcal{M}$ in $[\tau,1+\tau]$. In view of (4.1), the $m(h,q)$'s are disjoint sub-intervals of $[\tau,1+\tau]$. If we put $I(b) := \sum_{n_j}\Lambda(n_1)\Lambda(n_2)\Lambda(n_3)$, where the summation is over $N_j' < n_j \le N_j, j=1,2,3$ satisfying $\sum_{j=1}^3 a_jn_j = b$ then

$$I(b) = \int_\tau^{1+\tau} e(-bx)\prod_{j=1}^3 S_j(x)dx := I_1(b)+I_2(b), \text{ say.} \tag{4.11}$$

Here $I_1(b)$ and $I_2(b)$ are the integrations over the major arcs $\mathcal{M}$ and minor $\mathcal{M}'$ respectively. We shall follows closely the arguments in [11] and show that $I_1(b)$ constitutes the main term and the contribution of $I_2(b)$ is negligible. Since many lemmas below can be proved in a very similar way in [11], so we will omit the proofs of them.

5 SIMPLIFICATION OF $I_1(B)$

For any character $\chi(\text{mod } q)$, let $C_\chi(m) := \sum_{l=1}^{q} \chi(l) e_q(ml)$ and $C_q(m) := C_{\chi_0}(m)$. If $x \in m(h,q)$, we write $x = hq^{-1} + \eta$ so that $(h,q) = 1$ and $|\eta| < \tau/q$. By (4.8) and the orthogonality relation of characters we see that [5, p.147, (2)]

$$S_j(x) = \phi(q)^{-1} \sum_{\chi(\text{mod } q)} C_{\bar{\chi}}(a_j h) S_j(\chi, \eta) + O(\log^2 N). \tag{5.1}$$

Lemma 5.1 *For any real y and any $\chi(\text{mod } q)$ with $q \le Q$, we have for $j = 1,2,3$,*

$$S_j(\chi, y) = \delta_\chi I_j(y) - \delta'_\chi \widetilde{I}_j(y) - I_j(\chi, y) + O((1 + N|y|) N_j T^{-1} \log^2 N),$$

where $I_j(y), \widetilde{I}_j(y)$ and $I_j(\chi, y)$ are defined in (4.9) and (4.10),

$$\delta_\chi := \begin{cases} 1 & \text{if } \chi = \chi_0 (\text{mod } q), \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \delta'_\chi := \begin{cases} 1 & \text{if } \chi = \widetilde{\chi}\chi_0 (\text{mod } q), \\ 0 & \text{otherwise}, \end{cases}$$

Proof. This is Lemma 3.1 in [11]. □

For $j = 1,2,3$, define

$$\begin{cases} \mathcal{G}_j(h,q,\eta) := \sum_{\chi(\text{mod } q)} C_{\bar{\chi}}(a_j h) I_j(\chi, \eta), \\ H_j(h,q,\eta) := C_q(a_j h) I_j(\eta) - \delta(q) C_{\widetilde{\chi}\chi_0(\text{mod } q)}(a_j h) \widetilde{I}_j(\eta) \\ \qquad -\mathcal{G}_j(h,q,\eta), \end{cases} \tag{5.2}$$

where $\delta(q) := \begin{cases} 1 & \text{if } \tilde{r}|q, \\ 0 & \text{otherwise.} \end{cases}$

Let

$$\Omega_1 := N^2 Q^{5/12} T^{-1/3} |a_1 a_2 a_3|^{-1} d(a_3) \log^4 N,$$

where $d(n)$ is the number of positive divisors of n. In view of Lemma 5.1, (5.1) and (5.2), we have the following lemma.

Lemma 5.2 *We have*

$$I_1(b) = \sum_{q \le Q} \phi(q)^{-3} {\sum_{h=1}^{q}}' e_q(-bh) \int_{-\tau/q}^{\tau/q} e(-b\eta) \prod_{j=1}^{3} H_j(h,q,\eta) d\eta + O(\Omega_1),$$

where ${\sum_{h=1}^{q}}'$ is the summation over all $1 \le h \le q$ satisfying $(h,q) = 1$.

Proof. This can be proved in a similar way as proving (3.17) in [11]. □

6 SINGULAR SERIES AND SINGULAR INTEGRAL

In this section, we come to analyze the singular series and singular integral. We first discuss the singular series and establish some of its arithmetical properties.

For the integers a_1, a_2, a_3 and b satisfying (1.1) and (1.2) we define

$$A(q) := \phi(q)^{-3} \sum_{h=1}^{q}{}' e_q(-bh) \prod_{j=1}^{3} C_q(a_j h)$$

and

$$N(q) := \text{Card}\{(l_1, l_2, l_3) : 1 \le l_j \le q, (l_j, q) = 1, \sum_{j=1}^{3} a_j l_j \equiv b (\text{mod} q)\}. \tag{6.1}$$

In view of Lemma 4.1 in [11], both $A(q)$ and $N(q)$ are multiplicative functions of q. For any prime p, let $s(p) := 1 + A(p)$.

We partition the primes into the following two sets:

$$P_G := \{p : p \nmid a_1 a_2 a_3\} \text{ and } P_B := \{p : p | a_1 a_2 a_3\}.$$

Under the hypothesis (1.1) and (1.2), we have the following lemma.

Lemma 6.1 *We have*

(i) *For any prime p, we have $-(p-1)^{-2} \le A(p) \le (p-1)^{-1}$.*
(ii) *For any $p \in P_G$, we have $|A(p)| \le (p-1)^{-2}$.*

Proof. The proof is similar to Lemma 4.2 in [11]. □

The following lemma proves that $\sum A(q)$ is absolutely convergent.

Lemma 6.2 *We have*

(i) *For any $y \ge 1$, we have*

$$\sum_{y \le q} |A(q)| \ll y^{-1} d(a_1 a_2 a_3)^{\log_2 3} \log(y+2),$$

(ii)

$$\prod_p (1 + |A(p)|) \le K_2 \prod_p (1 + A(p)),$$

where $K_2 = \prod_{p \ge 3} (1 + \frac{2}{p(p-2)}) < 2.140782$.

Proof. Part (i) can be proved in the same way as proving Lemma 4.4(4) in [11] and part (ii) follows from Lemma 6.1 □

Denote the least common multiple of the r_j's by $[r_1, r_2, r_3]$. For $j = 1, 2, 3$, let $\chi_j (\text{mod} r_j)$ be primitive characters. Let

$$Z(q) := \sum_{h=1}^{q}{}' e_q(-bh) \prod_{j=1}^{3} C_{\chi_j \chi_0}(a_j h),$$

where $[r_1, r_2, r_3] | q$ and χ_0 is the principal character modulo q.

Lemma 6.3 *Let* $r = [r_1, r_2, r_3]$. *We have*

$$\sum_{\substack{q\le Q\\ r|q}} \phi(q)^{-3}|Z(q)| \le K_2 \prod_p s(p),$$

where K_2 *is defined in Lemma 6.2.*

Proof. By Lemma 6.2(ii), Lemma 6.3 can be proved in the same way as proving Lemma 4.6 in [11]. □

Lemma 6.4 *For any complex number* ρ_j *satisfying* $0 < Re\rho_j \le 1$, *for* $j = 1, 2, 3$, *we have*

$$\int_{-\infty}^{\infty} \left(\prod_{j=1}^{3} \int_{N_j'}^{N_j} x^{\rho_j - 1} e(a_j x\eta) dx \right) e(-b\eta) d\eta = N^2 |a_3|^{-1} \int_D \prod_{j=1}^{3} (Nx_j)^{\rho_j - 1} dx_1 dx_2,$$

were

$$x_3 := f(x_1, x_2) := a_3^{-1}(bN^{-1} - a_1x_1 - a_2x_2)$$

and

(6.2) $$D := \{(x_1, x_2) : (4|a_j|)^{-1} \le x_j \le |a_j|^{-1}, j = 1, 2, 3\}.$$

Proof. This is essential Lemma 4.7 in [11]. □

7 ESTIMATES ON MAJOR ARCS

In view of (5.2), when we multiply out the product $\prod_{j=1}^{3} H_j(h, q, \eta)$, we get 27 terms (if $\tilde{\beta}$ exists). These are grouped into the following three categories:

- (τ_1) : the term $\prod_{j=1}^{3} C_q(a_j h) I_j(\eta)$,
- (τ_2) : 19 terms (if $\tilde{\beta}$ exists), each has at least one $\mathcal{G}_j(h, q, \eta)$ as factor,
- (τ_3) : the 7 terms remaining (if $\tilde{\beta}$ exists).

For $i = 1, 2, 3$, define

$$M_i := \sum_{q\le Q} \phi(q)^{-3} {\sum_{h=1}^{q}}' e_q(-bh) \int_{-\tau/q}^{\tau/q} \{\text{sum of the terms in } \tau_i\} e(-b\eta) d\eta.$$

In view of Lemma 5.2, we have

(7.1) $$I_1(b) = M_1 + M_2 + M_3 + O(\Omega_1).$$

Let

(7.2) $$M_0 := N^2 |a_3|^{-1} \left(\prod_p s(p) \right) \int_D dx_1 dx_2,$$

where D is defined in (6.2) of Lemma 6.4.

We shall show that M_1 is the main term and M_2, M_3 are the remainders. We now come to discuss M_1.

Lemma 7.1 *We have*

$$M_1 = M_0 + O(\Omega_2), \tag{7.3}$$

where $\Omega_2 := N^2 Q^{-1} |a_1 a_2 a_3|^{-1} d(a_1 a_2 a_3)^{\log_2 3} \log^2 N$.

Proof. The proof is similar to Lemma 5.1 in [11]. □

We abbreviate by $\sum_\sim$ the summation over (l_1, l_2, l_3) satisfying the conditions:

$$1 \le l_j \le \tilde{r}, (l_j, \tilde{r}) = 1 \quad \text{and} \quad \sum_{j=1}^{3} a_j l_j \equiv b(\text{mod} \tilde{r}). \tag{7.4}$$

For $1 \le m_1 < m_2 < \cdots \le 3$, we define

$$G(m_1, m_2, \cdots) := \sum_{\sim} \tilde{\chi}(l_{m_1}) \tilde{\chi}(l_{m_2}) \cdots \tag{7.5}$$

and

$$P(m_1, m_2, \cdots) := \int_D (Nx_{m_1})^{\tilde{\beta}-1} (Nx_{m_2}) \tilde{\beta} - 1 \cdots dx_1 dx_2, \tag{7.6}$$

where D is defined in (6.2).

Similar to Lemma 5.2 in [11], we have the following lemma which takes care of M_3 when $\tilde{r}/Q$ is small.

Lemma 7.2 *If the exceptional zero* $\tilde{\beta}$ *exists, then*

$$\begin{aligned} M_3 &= N^2 |a_3|^{-1} \tilde{r} \phi(\tilde{r})^{-3} \left(\prod_{p \nmid \tilde{r}} s(p) \right) \{ -\sum_{j=1}^{3} G(j) P(j) + \sum_{1 \le i < j \le 3} G(i,j) P(i,j) \\ &\quad - G(1,2,3) P(1,2,3) \} + O(\tilde{r} \Omega_2). \end{aligned}$$

Lemma 7.3 *If the exceptional zero* $\tilde{\beta}$ *exists, then*

$$M_1 + M_3 \ge K_3 \Omega^3 M_0 + O(\tilde{r} \Omega_2),$$

where $K_3 := (1 - \exp\{-(1 - \delta/4) c_4 / (c_3 \delta)\})^3 c_4^{-3} = 2575.3820 \cdots$.

Proof. In view of (4.16) in [11], (6.1) and (7.4), we have

$$\prod_{p | \tilde{r}} s(p) = \tilde{r} \phi(\tilde{r})^{-3} N(\tilde{r}) = \tilde{r} \phi(\tilde{r})^{-3} \sum_{\sim} 1. \tag{7.7}$$

By (7.3) and Lemma 7.2, we have

$$\begin{aligned} M_1 + M_3 &= M_0 + N^2 |a_3|^{-1} \tilde{r} \phi(\tilde{r})^{-3} \prod_{p \nmid \tilde{r}} s(p) \{ -\sum_{j=1}^{3} G(j) P(j) + \\ &\quad + \sum_{1 \le i < j \le 3} G(i,j) P(i,j) - G(1,2,3) P(1,2,3) \} + O(\tilde{r} \Omega_2) \end{aligned}$$

$$(7.8) \qquad = N^2|a_3|^{-1}\tilde{r}\phi(\tilde{r})^{-3}\prod_{p\nmid\tilde{r}} s(p)\sum_{\sim}\int_D\prod_{j=1}^{3}\{1-\tilde{\chi}(l_j)(Nx_j)^{\tilde{\beta}-1}\}dx_1dx_2 + O(\tilde{r}\Omega_2).$$

The last equality follows from (7.2) and (7.5)-(7.7). Now,

$$\begin{aligned}\sum_{\sim}\int_D\prod_{j=1}^{3}\{1-\tilde{\chi}(l_j)(Nx_j)^{\tilde{\beta}-1}\}dx_1dx_2 &\geq \sum_{\sim}\int_D\prod_{j=1}^{3}(1-(Nx_j)^{\tilde{\beta}-1})dx_1dx_2\\ &\geq \left(\sum_{\sim}\int_D dx_1dx_2\right)\prod_{j=1}^{3}(1-(N_j')^{\tilde{\beta}-1}), \qquad (7.9)\end{aligned}$$

since in (6.2), $Nx_j \geq N_j'(j=1,2,3)$. In view of (4.2), we have $N_j' \geq N^{(1-\delta/4)}$. Then by (4.5) and $\Omega \leq c_4$

$$(7.10) \qquad 1 - N_j'^{\tilde{\beta}-1} \geq \left(\frac{1-\exp\{-(1-\delta/4)c_4/(c_3\delta)\}}{c_4}\right)\Omega.$$

Hence by (7.7)-(7.10) and (7.2)

$$\begin{aligned} M_1 + M_3 &\geq K_3\Omega^3N^2|a_3|^{-1}\tilde{r}\phi(\tilde{r})^{-3}\left(\prod_{p\nmid\tilde{r}} s(p)\right)\sum_{\sim}\int_D dx_1dx_2 + O(\tilde{r}\Omega_2)\\ &= K_3\Omega^3M_0 + O(\tilde{r}\Omega_2).\end{aligned}$$

This proves Lemma 7.3. □

If $\tilde{r}/Q$ is large, the error term in Lemma 7.2 will also be large. So the lower bound in this lemma is useful only when $\tilde{r}/Q$ is small. When $\tilde{r}/Q$ is not so small, we have the following lemma.

Lemma 7.4 *If the exceptional zero $\tilde{\beta}$ exists, then*

$$M_1 + M_3 = M_0 + O(\Omega_2 + N^2\tilde{r}^{-1}\log N).$$

Proof. This can be proved in the similar way as proving Lemma 5.5 in [11]. □

Concerning M_2, we have the following lemma.

Lemma 7.5 *We have*

$$|M_2| \leq K_4\Omega^3M_0 + O(\Omega_1),$$

where

$$K_4 := \begin{cases} K_2K_1(12+6K_1c_4^3+K_1^2c_4^6), & \text{if } \tilde{\beta} \text{ exists,}\\ K_2K_1(3+3K_1+K_1^2), & \text{otherwise,}\end{cases}$$

and K_1, K_2 are defined in Lemmas 4.1 and 6.2 respectively. Note that $K_4 < 4.18737\times10^{-22}$ if $\tilde{\beta}$ exists and $K_4 < 0.99999$ otherwise.

Proof. This can be shown in the same way as proving Lemma 6.1 in [11] except using Lemmas 2.1 and 4.6 in [11], we use their explicit forms Lemmas 4.1 and 6.3. □

Lemma 7.6 *We have*

$$I_1(b) \gg \begin{cases} M_0 + O(\Omega_1), & \text{if } \tilde{\beta} \text{ does not exist,} \\ M_0 + O(N^2 Q^{-\frac{1}{4}} \log N), & \text{if } \tilde{\beta} \text{ exists and } Q^{\frac{1}{4}} < \tilde{r}, \\ \Omega^3 M_0 + O(\Omega_1), & \text{if } \tilde{\beta} \text{ exists and } Q^{\frac{1}{4}} \geq \tilde{r}, \end{cases}$$

Proof. If the exceptional zero $\tilde{\beta}$ does not exist, then there is no M_3. Thus by (7.1), Lemmas 7.1 and 7.5, we have

$$I_1(b) \geq (1 - K_4)M_0 + O(\Omega_1) \gg M_0 + O(\Omega_1).$$

If $\tilde{\beta}$ exists then the proof is similar except using Lemma 7.1, we use Lemmas 7.3 and 7.4 according as $Q^{1/4} \geq \tilde{r}$ or $Q^{1/4} < \tilde{r}$. □

8 MINOR ARCS AND COMPLETION OF THE PROOFS OF THE THEOREMS

As usual, we employ Vinogradov's lemma to estimate the contribution of $I_2(b)$.

Lemma 8.1 *We have*

$$I_2(b) \ll N^2 Q^{-\frac{1}{2}} |a_1 a_2 a_3|^{-\frac{1}{2}} \log^5 N.$$

Proof. This is essential Lemma 7.1 in [11]. □

The next lemma provides us a result to obtain a lower bound for $I(b)$.

Lemma 8.2 *If*

(i) all the a_j's are positive and $b = N$, or

(ii) not all the a_j's are of the same sign and $N \geq 3|b|$, then

$$M_0 \gg N^2 |a_1 a_2 a_3|^{-1}.$$

Proof. This is essential Lemma 7.2 in [11]. □

To complete the proofs of the theorems, first, by (4.1) and (4.2) and Lemma 8.1, we can deduce that

$$I_2(b) = O(N^2 Q^{-3/8} |a_1 a_2 a_3|^{-1} (\log N)^{-3}).$$

Also, in view of Lemma 7.6, Lemma 8.2 and $\Omega \gg Q^{-1/8}(\log N)^{-1}$ if $\tilde{\beta}$ exists and $Q^{1/4} \geq \tilde{r}$, we have

$$I_1(b) \gg N^2 Q^{-3/8} |a_1 a_2 a_3|^{-1} (\log N)^{-3}.$$

Then by (4.11), we conclude that

$$I(b) = I_1(b) + I_2(b) \gg N^2 Q^{-3/8} |a_1 a_2 a_3|^{-1} (\log N)^{-3}.$$

This proves Theorems 1 and 2.

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