AVERAGE MAHLER'S MEASURE AND L_p NORMS OF UNIMODULAR POLYNOMIALS

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ABSTRACT. A polynomial $f \in \mathbb{C}[z]$ is unimodular if all its coefficients have unit modulus. Let $\mathfrak{U}_n$ denote the set of unimodular polynomials of degree $n-1$, and let $\mathfrak{U}_n^*$ denote the subset of reciprocal unimodular polynomials, which have the property that $f(z) = \omega z^{n-1}\overline{f(1/\overline{z})}$ for some complex number ω with $|\omega| = 1$. We study the geometric and arithmetic mean values of both the normalized Mahler's measure $M(f)/\sqrt{n}$ and L_p norm $\|f\|_p/\sqrt{n}$ over the sets $\mathfrak{U}_n$ and $\mathfrak{U}_n^*$, and compute asymptotic values in each case. We show for example that both the geometric and arithmetic mean of the normalized Mahler's measure approach $e^{-\gamma/2} = 0.749306\ldots$ as $n \to \infty$ for unimodular polynomials, and $e^{-\gamma/2}/\sqrt{2} = 0.529839\ldots$ for reciprocal unimodular polynomials. We also show that for large n almost all polynomials in these sets have normalized Mahler's measure or L_p norm very close to the respective limiting mean value.

1. Introduction

Mahler's measure of a polynomial $f(z) \in \mathbb{C}[z]$, denoted $M(f)$, is defined by

$$\log M(f) = \int_0^1 \log |f(e(t))|\, dt,$$

where $e(t)$ denotes the function $e^{2\pi i t}$. If $f(z) = a_{n-1}\prod_{k=1}^{n-1}(z - \alpha_k)$, then it follows from Jensen's formula that

$$M(f) = |a_{n-1}| \prod_{k=1}^{n-1} \max\{1, |\alpha_k|\}.$$

Mahler's measure is also equal to the limit as p approaches zero of the integral defining the L_p norm of f over the unit circle,

$$M(f) = \lim_{p \to 0^+} \|f\|_p\,,$$

where $\|f\|_p$ is defined for $p > 0$ by

$$\|f\|_p = \left(\int_0^1 |f(e(t))|^p\, dt\right)^{1/p}.$$

Of course, this is only a norm for $p \geq 1$, but the restriction $p > 0$ is more natural in this article.

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We say a polynomial $f(z) = \sum_{k=0}^{n-1} a_k z^k \in \mathbb{C}[z]$ is *unimodular* if $|a_k| = 1$ for $0 \le k < n$, and we say f is a *Littlewood polynomial* if in fact $a_k = \pm 1$ for each k. Let $\mathfrak{U}_n$ denote the set of unimodular polynomials of degree $n-1$, and let $\mathfrak{L}_n$ denote the set of Littlewood polynomials of degree $n-1$. Note that by Parseval's formula, if $f \in \mathfrak{U}_n$ then $\|f\|_2 = \sqrt{n}$.

Littlewood posed a number of problems regarding the behavior of unimodular and Littlewood polynomials on the unit disk, see for instance [22, problem 19]. He asked for example if there exist positive constants c_1 and c_2 and a sequence of polynomials $\{f_n\}$ with $f_n \in \mathfrak{U}_n$ (or, more strictly, $f_n \in \mathfrak{L}_n$) such that $c_1\sqrt{n} \le |f_n(z)| \le c_2\sqrt{n}$ for every z with modulus 1. Kahane [20] proved this for unimodular polynomials in 1980, showing in fact that one may choose c_1 and c_2 arbitrarily close to 1 for sufficiently large n. Further, Beck [1] showed in 1991 that constants c_1 and c_2 exist for the set of polynomials whose coefficients are 1200th roots of unity. The problem for Littlewood polynomials however remains open—the Rudin-Shapiro polynomials satisfy the upper bound, and no sequence is known that satisfies the lower bound. We remark also that Erdős [17] conjectured that the constant c_2 must remain bounded away from 1 for large n for the case of the Littlewood polynomials.

Littlewood's problem has been investigated by using L_p norms and Mahler's measure. Since $\|f\|_2 < \|f\|_p < \|f\|_\infty$ when $2 < p < \infty$ and f is not a monomial, Erdős' conjecture regarding c_2 would follow if one could establish for some such p that $\|f\|_p / \|f\|_2$ is bounded away from 1, for every Littlewood polynomial f of positive degree. The special case $p = 4$ is equivalent to the well-known *merit factor problem* of Golay [8]. In the same way, one may ask for unimodular or Littlewood polynomials f whose normalized Mahler's measure $M(f)/\|f\|_2$ or L_p norm $\|f\|_p / \|f\|_2$ with $p < 2$ is especially large. For the case of the measure, in 1963, Mahler [23] proved that the maximum value of the measure of a polynomial with fixed degree having all its coefficients in the closed unit disk is attained by a unimodular polynomial. In 1970, Fielding [18] reported that

$$\lim_{n\to\infty} \sup_{f\in\mathfrak{U}_n} \frac{M(f)}{\sqrt{n}} = 1, \tag{1.1}$$

attributing the proof to an anonymous referee. Beller and Newman [2] also obtained a stronger version of this result in 1973.

For Littlewood polynomials, the problem of determining whether there exists a positive number ϵ such that $M(f)/\|f\|_2 < 1 - \epsilon$ for every $f \in \mathfrak{L}$ with $\deg(f) > 0$ is commonly known as *Mahler's problem* [4], and this problem remains open. The largest known value of $M(f)/\|f\|_2$ for a Littlewood polynomial of positive degree is $0.98636\ldots$, achieved by $f(x) = x^{12} + x^{11} + x^{10} + x^9 + x^8 - x^7 - x^6 + x^5 + x^4 - x^3 + x^2 - x + 1$. The best known asymptotic result in Mahler's problem is due to Erdélyi and Lubinsky [16], who proved that the Fekete polynomials, defined by $f_q(x) = \sum_{k=1}^{q-1} \left(\frac{k}{q}\right) x^{k-1}$, where q is a prime number and $\left(\frac{\cdot}{q}\right)$ denotes the Legendre symbol, satisfy

$$\frac{M(f_q)}{\sqrt{q-1}} > \frac{1}{2} - \epsilon \tag{1.2}$$

for arbitrarily small $\epsilon > 0$ when q is sufficiently large.

Statistical properties of L_p norms and Mahler's measure are also of interest for the unimodular and Littlewood polynomials. Newman and Byrnes [25] determined the mean value of the L_4 norm of a Littlewood polynomial, proving that

$$\operatorname*{mean}_{f\in\mathfrak{L}_n} \|f\|_4^4 = 2n^2 - n.$$

Borwein and Choi [5] found similar formulas for the average value of the L_6 and L_8 norms, showing that

$$\operatorname*{mean}_{f\in\mathfrak{L}_n} \|f\|_6^6 = 6n^3 - 9n^2 + 4n$$

and

$$\operatorname*{mean}_{f\in\mathfrak{L}_n} \|f\|_8^8 = 24n^4 - 66n^3 + 58n^2 - 9n + 3\left((-1)^n - 1\right).$$

Borwein and Lockhart [10] investigated the asymptotic behavior of the mean value of normalized L_p norms of Littlewood polynomials for arbitrary $p > 0$. Using the Lindeberg Central Limit Theorem and dominated convergence, they proved that

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{\|f\|_p^p}{n^{p/2}} = \Gamma(1 + p/2).$$

Their result in fact holds for any class of polynomials whose coefficients are independent, identically distributed random variables with mean 0, variance 1, and finite mth moment for each $m > 2$, so this holds for the set of unimodular polynomials as well. By identifying a polynomial $f \in \mathfrak{U}_n$ by its vector of coefficients, we may consider $\mathfrak{U}_n$ as an n-dimensional torus, and we let μ_n denote Haar measure on this space, normalized so that $\int_{\mathfrak{U}_n} d\mu_n = 1$. Using this notation, Borwein and Lockhart thus established that

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \frac{\|f\|_p^p}{n^{p/2}}\, d\mu_n = \Gamma(1 + p/2). \tag{1.3}$$

for $p > 0$.

Fielding [18] established a lower bound on the expected value of the normalized Mahler's measure for unimodular polynomials, proving that

$$\int_{\mathfrak{U}_n} \frac{M(f)}{\sqrt{n}}\, d\mu_n \geq e^{-\gamma/2}\left(1 + O\left(n^{-1/4+\epsilon}\right)\right) \tag{1.4}$$

for arbitrary $\epsilon > 0$ and sufficiently large n, where γ denotes Euler's constant. Fielding's proof in fact established the precise limiting value for the geometric mean of the normalized Mahler's measure:

$$\int_{\mathfrak{U}_n} \log\left(\frac{M(f)}{\sqrt{n}}\right) d\mu_n = -\frac{\gamma}{2} + O\left(n^{-1/4+\epsilon}\right). \tag{1.5}$$

In this paper, we strengthen (1.4) by proving that the limiting value of the arithmetic mean of the normalized measure of unimodular polynomials of fixed degree is precisely $e^{-\gamma/2} = 0.749306\ldots$. We also recover (1.3) and (1.5) by using a different method, and we establish the asymptotic value of the geometric mean of certain L_p norms of the unimodular polynomials as well. We prove the following theorem in section 2.

Theorem 1. *Let $\mathfrak{U}_n$ denote the set of unimodular polynomials of degree $n-1$. Then*

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \log\left(\frac{M(f)}{\sqrt{n}}\right) d\mu_n = -\frac{\gamma}{2},$$

and for any even positive integer p,

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \log\left(\frac{\|f\|_p}{\sqrt{n}}\right) d\mu_n = \frac{1}{p}\log\Gamma(1+p/2).$$

Furthermore,

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \frac{M(f)}{\sqrt{n}}\, d\mu_n = e^{-\gamma/2},$$

and for any positive real number p,

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \frac{\|f\|_p^p}{n^{p/2}}\, d\mu_n = \Gamma(1+p/2).$$

A polynomial $f(z) \in \mathbb{C}[z]$ of degree d is said to be *reciprocal* if $f(z) = \omega z^d \overline{f(1/\overline{z})}$, for some complex number ω of modulus 1. Reciprocal polynomials often arise in a natural way in problems involving Mahler's measure. For example, Smyth [28] proved that if $f(z) \in \mathbb{Z}[z]$ is nonreciprocal and $f(0) \neq 0$, then $M(f) \geq M(z^3 - z - 1) = 1.32471\ldots$. Let $\mathfrak{U}_n^*$ denote the set of reciprocal unimodular polynomials of degree $n-1$, and let $\mathfrak{L}_n^*$ denote the set of reciprocal Littlewood polynomials of degree $n-1$. In 2007, Borwein and Choi [5] determined the expected value of the L_4 norm of a polynomial $f \in \mathfrak{L}_n^*$, showing that

$$\operatorname*{mean}_{f\in\mathfrak{L}_n^*} \|f\|_4^4 = 3n^2 - 3n + (1-(-1)^n)/2.$$

In addition, recently Choi [13] determined a lower bound on $\|f\|_4$ when f is a reciprocal unimodular polynomial:

$$\|f\|_4^4 \geq \frac{5}{3}n^2 + O(n^{3/2}).$$

In this article, we prove that both the geometric mean and the arithmetic mean of the normalized Mahler's measure of the reciprocal unimodular polynomials approach $e^{-\gamma/2}/\sqrt{2} = 0.529839\ldots$, and we determine limiting values for the mean normalized L_p norms of these polynomials as well. Since $\mathfrak{U}_n^*$ can be identified with a $(1+\lfloor n/2\rfloor)$-dimensional real torus, we let μ_n^* denote Haar measure on the set $\mathfrak{U}_n^*$, normalized so that $\int_{\mathfrak{U}_n^*} d\mu_n^* = 1$. We prove the following theorem in section 3.

Theorem 2. *Let $\mathfrak{U}_n^*$ denote the set of reciprocal unimodular polynomials of degree $n-1$. Then*

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n^*} \log\left(\frac{M(f)}{\sqrt{n}}\right) d\mu_n^* = -\frac{\gamma+\log 2}{2},$$

and for any even positive integer p,

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n^*} \log\left(\frac{\|f\|_p}{\sqrt{n}}\right) d\mu_n^* = \frac{1}{2}\log(2\pi^{1/p}) + \frac{1}{p}\log\Gamma\left(\frac{p+1}{2}\right).$$

Furthermore,

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n^*} \frac{M(f)}{\sqrt{n}}\, d\mu_n^* = \frac{1}{\sqrt{2e^{\gamma}}},$$

and for any positive number p,

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n^*} \frac{\|f\|_p^p}{n^{p/2}}\, d\mu_n^* = \frac{2^{p/2}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right).$$

We may obtain more information on the distribution of values of L_p norms and the Mahler measure by studying their moments, and to this end we establish the following result in section 4.

Theorem 3. *Let $p \geq 2$ be an even integer and m a positive integer. Then*

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \left(\frac{\|f\|_p^p}{n^{p/2}}\right)^m d\mu_n = \left(\lim_{n\to\infty} \int_{\mathfrak{U}_n} \frac{\|f\|_p^p}{n^{p/2}}\, d\mu_n\right)^m \tag{1.6}$$

and

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \left(\frac{M(f)}{\sqrt{n}}\right)^m d\mu_n = \left(\lim_{n\to\infty} \int_{\mathfrak{U}_n} \frac{M(f)}{\sqrt{n}}\, d\mu_n\right)^m. \tag{1.7}$$

Furthermore, equation (1.6) *holds for all positive real numbers p when $m = 2$. The same statements hold for the set $\mathfrak{U}_n^*$.*

Theorem 3 establishes that the limiting value of the variance of each distribution approaches 0 as the degree grows large. It follows that the normalized Mahler's measure or L_p norm (with $p \geq 2$ an even integer) of a polynomial in one of these sets lies arbitrarily close to the corresponding mean value, with probability approaching 1 as the degree grows large. We thus obtain the following corollary regarding convergence in probability of the measure and L_p norms. (The result on L_p norms for $\mathfrak{U}_n$ was also obtained by Borwein and Lockhart [10] for all $p > 0$.)

Corollary 4. *For any $\epsilon > 0$ and $\delta > 0$, there exists a positive integer $N = N(\epsilon, \delta)$ so that if $n \geq N$ then*

$$\Pr\left(f \in \mathfrak{U}_n : \left|\frac{M(f)}{\sqrt{n}} - e^{-\gamma/2}\right| > \delta\right) < \epsilon$$

and, if $p \geq 2$ is an even integer,

$$\Pr\left(f \in \mathfrak{U}_n : \left|\frac{\|f\|_p^p}{n^{p/2}} - \Gamma(1 + p/2)\right| > \delta\right) < \epsilon.$$

Analogous statements hold for the set $\mathfrak{U}_n^*$.

Section 5 adds some remarks on related problems.

2. Unimodular polynomials

We require the following notation. Let $Q_{n,k}$ denote the set of all sequences of length k selected without repetition from the set $\{0, 1, \ldots, n-1\}$, so $|Q_{n,k}| = n!/(n-k)!$ for $0 \leq k \leq n$. Let $R_{n,k}$ denote the set of such sequences if repetition is allowed, so $|R_{n,k}| = n^k$. For two sequences I and J in $R_{n,k}$, we write $I \sim J$ if I is a permutation of J. Let P_k denote the set of permutations of $\{1, 2, \ldots, k\}$, so $|P_k| = k!$. Finally, for a vector $\mathbf{t} \in \mathbb{R}^k$, let $N_{\mathbf{t}}$ denote the number of permutations of P_k that fix $\mathbf{t}$, so

$$N_{\mathbf{t}} = \left|\{\pi \in P_k : t_\ell = t_{\pi(\ell)} \text{ for } 1 \leq \ell \leq k\}\right|.$$

We first establish the following lemma.

Lemma 1. *Let* $\mathbf{t} = (t_1, \ldots, t_k)$ *denote a point in* $[0,1)^k$*. If* $Q_{n,k}$ *and* $N_{\mathbf{t}}$ *are defined as above, then*

$$\sum_{\substack{I,J\in Q_{n,k}\\ I\sim J}} \prod_{\ell=1}^{k} e\left(t_\ell(i_\ell - j_\ell)\right) = n^k N_{\mathbf{t}} + O\left(n^{k-1}\right), \tag{2.1}$$

where the implicit constant may depend on k *and* $\mathbf{t}$ *but is independent of* n*.*

Proof. We compute

$$\begin{aligned}
\sum_{\substack{I,J\in Q_{n,k}\\ I\sim J}} \prod_{\ell=1}^{k} e\left(t_\ell(i_\ell - j_\ell)\right) &= \sum_{\pi\in P_k}\sum_{I\in Q_{n,k}} \prod_{\ell=1}^{k} e\left(t_\ell(i_\ell - i_{\pi(\ell)})\right)\\
&= \sum_{\pi\in P_k}\sum_{I\in R_{n,k}} \prod_{\ell=1}^{k} e\left(t_\ell(i_\ell - i_{\pi(\ell)})\right) + O\left(n^{k-1}\right)\\
&= \sum_{\pi\in P_k}\sum_{I\in R_{n,k}} \prod_{\ell=1}^{k} e\left((t_\ell - t_{\pi^{-1}(\ell)})i_\ell\right) + O\left(n^{k-1}\right)\\
&= \sum_{\pi\in P_k}\prod_{\ell=1}^{k}\sum_{0\le i_\ell<n} e\left((t_\ell - t_{\pi^{-1}(\ell)})i_\ell\right) + O\left(n^{k-1}\right).
\end{aligned}$$

If α is an integer, then certainly

$$\sum_{0\le j<n} e(\alpha j) = n,$$

and if α is not an integer, then

$$\sum_{0\le j<n} e(\alpha j) \ll \frac{1}{\|\alpha\|},$$

where $\|\alpha\|$ denotes the distance from α to the nearest integer, $\|\alpha\| = \min_{n\in\mathbb{Z}} |\alpha - n|$. Thus

$$\sum_{\substack{I,J\in Q_{n,k}\\ I\sim J}} \prod_{\ell=1}^{k} e\left(t_\ell(i_\ell - j_\ell)\right) = n^k N_{\mathbf{t}} + O\left(n^{k-1}\right). \qquad \square$$

In particular, suppose $m_1, \ldots, m_r$ are positive integers, and let $M_j = \sum_{i=1}^{j} m_i$ for each j with $0 \le j \le r$. Suppose also that $t_{M_1}, \ldots, t_{M_r}$ are distinct real numbers in $[0,1)$, and that for each j with $1 \le j \le r$, we have

$$t_{M_{j-1}+i} = t_{M_j}$$

for $1 \le i < m_j$. Let $M = M_r$. Then by Lemma 1 we have

$$\sum_{\substack{I,J\in Q_{n,M}\\ I\sim J}} \prod_{\ell=1}^{M} e\left(t_\ell(i_\ell - j_\ell)\right) = n^M \prod_{\ell=1}^{r} m_\ell! + O\left(n^{M-1}\right). \tag{2.2}$$

We use (2.2) to establish the following theorem regarding the mean value of a particular expression over the set of unimodular polynomials.

Lemma 2. *Suppose $t_1, \ldots, t_r$ are distinct real numbers in $[0,1)$ and $m_1, \ldots, m_r$ are positive integers. Then*

$$\int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}}\, d\mu_n = \prod_{\ell=1}^{r} m_\ell! + O\left(\frac{1}{n}\right),$$

where the implicit constant may depend on the m_ℓ and the t_ℓ but is independent of n. Thus,

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}}\, d\mu_n = \prod_{\ell=1}^{r} m_\ell!.$$

Proof. For clarity, consider first the case $r = 1$. Write m in place of m_1, let $f(z) = \sum_{k=0}^{n} a_k z^k$, and let $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_m)$. Then

$$\begin{aligned}
\int_{\mathfrak{U}_n} \frac{|f(e(t))|^{2m}}{n^m}\, d\mu_n &= \frac{1}{n^m} \int_{\mathfrak{U}_n} \left|\sum_{k=1}^{n} a_k e(kt)\right|^{2m} d\mu_n \\
&= \frac{1}{n^m} \sum_{I,J\in R_{n,m}} e\left(t\sum_{k=1}^{m}(i_k - j_k)\right) \int_{\mathfrak{U}_n} \prod_{k=1}^{m} a_{i_k}\overline{a_{j_k}}\, d\mu_n \\
&= \frac{1}{n^m} \sum_{\substack{I,J\in R_{n,m}\\ I\sim J}} 1 \\
&= \frac{1}{n^m} \sum_{\substack{I,J\in Q_{n,m}\\ I\sim J}} 1 + O\left(\frac{1}{n}\right) \\
&= m! + O\left(\frac{1}{n}\right).
\end{aligned}$$

The general case is similar. Let $M = \sum_{\ell=1}^{r} m_\ell$. We find

$$\int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}}\, d\mu_n = \frac{1}{n^M} \sum_{\substack{I_\ell, J_\ell \in R_{n,m_\ell}\\ (I_1,\ldots,I_r)\sim(J_1,\ldots,J_r)}} \prod_{\ell=1}^{r} e\left(t_\ell \sum_{k=1}^{m_\ell}(i_{\ell,k} - j_{\ell,k})\right),$$

where $I_\ell = (i_{\ell,1}, \ldots, i_{\ell,m_\ell})$ and $J_\ell = (j_{\ell,1}, \ldots, j_{\ell,m_\ell})$. Next, restricting from R_{n,m_ℓ} to Q_{n,m_ℓ} changes the value here by $O(1/n)$, since the terms in the sum over I_ℓ and J_ℓ having repeated elements possess fewer degrees of freedom and contribute only $O(n^{m_\ell - 1})$. Thus

$$\int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}}\, d\mu_n = \frac{1}{n^M} \sum_{\substack{I_\ell, J_\ell \in Q_{n,m_\ell}\\ (I_1,\ldots,I_r)\sim(J_1,\ldots,J_r)}} \prod_{\ell=1}^{r} e\left(t_\ell \sum_{k=1}^{m_\ell}(i_{\ell,k} - j_{\ell,k})\right) + O\left(\frac{1}{n}\right),$$

where the implicit constant depends on the t_ℓ and the m_ℓ. In the same way, we may restrict further to the case where the sequences $I_1, \ldots, I_r$ are disjoint, so

$$\begin{aligned}
\int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}}\, d\mu_n &= \frac{1}{n^M} \sum_{\substack{I,J\in Q_{n,M}\\ I\sim J}} \prod_{\ell=1}^{r}\prod_{k=1}^{m_\ell} e\left(t_\ell(i_k - j_k)\right) + O\left(\frac{1}{n}\right) \\
&= \prod_{\ell=1}^{r} m_\ell! + O\left(\frac{1}{n}\right),
\end{aligned}$$

by (2.2). ☐

The following lemma establishes another mean value over the unimodular polynomials.

Lemma 3. *Let $t_1, \ldots, t_r$ be distinct real numbers in $[0,1)$, and let $s_1, \ldots, s_r$ be complex numbers satisfying $\Re(s_\ell) > 0$ for $1 \le \ell \le r$. Then*

$$\lim_{n\to\infty} \int_{\mathfrak{U}_n} \exp\left(-\sum_{\ell=1}^{r} s_\ell \frac{|f(e(t_\ell))|^2}{n}\right) d\mu_n = \prod_{\ell=1}^{r} \frac{1}{1+s_\ell}.$$

Proof. Let $F(s_1, \ldots, s_r)$ denote the function

$$F(s_1, \ldots, s_r) = \lim_{n\to\infty} \int_{\mathfrak{U}_n} \exp\left(-\sum_{\ell=1}^{r} s_\ell \frac{|f(e(t_\ell))|^2}{n}\right) d\mu_n,$$

so this defines an analytic function on the region $\{(s_1, \ldots, s_r) \in \mathbb{C}^r : \Re(s_\ell) > 0 \text{ for } 1 \le \ell \le r\}$. For real numbers $x_1, \ldots, x_r$ in $[0,1)$, we use Lemma 2 to compute

$$\begin{aligned}
F(x_1, \ldots, x_r) &= \lim_{n\to\infty} \int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \sum_{m_\ell \ge 0} \frac{(-x_\ell)^{m_\ell} |f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell} m_\ell!} \, d\mu_n \\
&= \sum_{\substack{m_\ell \ge 0 \\ 1 \le \ell \le r}} \left(\prod_{\ell=1}^{r} \frac{(-x_\ell)^{m_\ell}}{m_\ell!}\right) \lim_{n\to\infty} \int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}} \, d\mu_n \\
&= \prod_{\ell=1}^{r} \sum_{m_\ell \ge 0} (-x_\ell)^{m_\ell} \\
&= \prod_{\ell=1}^{r} \frac{1}{1+x_\ell}.
\end{aligned}$$

The statement then follows by analytic continuation. ☐

Next, recall that a sequence of random variables $\{X_n\}$ is *uniformly integrable* if

$$\lim_{c\to\infty} \sup_{n} \int_{|X_n|>c} |X_n| \, d\mu_n = 0. \tag{2.3}$$

For a fixed real number $t \in [0,1)$, let $U_{t,n}$ denote the random variable whose value is $|f(e(t))|^2 / n$, where each coefficient of f is selected uniformly over the unit circle. For convenience, let U_n denote the random variable $U_{0,n}$. We show that the sequence of random variables $X_n = \log U_{t,n}$ has the property (2.3).

Lemma 4. *Let t be a fixed real number with $0 \le t < 1$. The sequence of random variables $\{\log U_{t,n}\}$, with $n \ge 1$, is uniformly integrable.*

Proof. By symmetry we may assume that $t = 0$. Let c be a positive real number, let $F_n(x)$ denote the distribution function of U_n, and let $G_n(x) = 1 - F_n(x)$. Certainly

U_n is supported on $[0, n]$, so $F_n(x) = 1$ for $x \geq n$. Thus

$$\begin{aligned}\int_{U_n > e^c} \log U_n \, d\mu_n &= \int_{e^c}^{n} \log x \, dF_n(x) \\ &= \log n - cF_n(c) - \int_{e^c}^{n} \frac{F_n(x)}{x} \, dx \\ &= c\, G_n(e^c) + \int_{e^c}^{n} \frac{G_n(x)}{x} \, dx.\end{aligned}$$

By Markov's inequality, we have $G_n(x) \leq E(U_n)/x = 1/x$, so

$$\int_{U_n > e^c} \log U_n \, d\mu_n \leq (c+1)e^{-c}.$$

For $U_n < e^{-c}$, we require some additional information. Using a result of Kluyver [21] from 1906 on the distribution of the modulus of sums of complex numbers of unit modulus, Fielding [18] determined an asymptotic formula for the density function of the random variable Y_n that measures the value of $|f(1)|$, for a randomly selected polynomial $f \in \mathfrak{U}_n$. This random variable is well-approximated by a Weibull distribution with shape parameter 2 and scale parameter $\sqrt{n}$; its density function is

$$h_n(y) = \begin{cases} \frac{2y}{n} e^{-y^2/n} + O\left(\frac{\sqrt{y}}{n^{7/4-\epsilon}}\right), & \text{if } 0 \leq y \leq n, \\ 0, & \text{if } y > n. \end{cases}$$

Since $U_n = Y_n^2/n$, it follows that U_n has approximately an exponential distribution, with density function

$$u_n(x) = \begin{cases} e^{-x} + O\left(x^{-1/4} n^{-1+\epsilon}\right), & \text{if } 0 \leq x \leq n, \\ 0, & \text{if } x > n. \end{cases} \tag{2.4}$$

Thus

$$\begin{aligned}\int_{U_n < e^{-c}} |\log U_n| \, d\mu_n &= \int_0^{e^{-c}} \left(e^{-x} + O\left(x^{-1/4} n^{-1+\epsilon}\right)\right) |\log x| \, dx \\ &\leq \int_0^{e^{-c}} |\log x| \, dx + O\left(\frac{1}{n^{1-\epsilon}} \int_0^{e^{-c}} \frac{|\log x|}{x^{1/4}} \, dx\right) \\ &= (c+1)e^{-c} + O\left(\frac{ce^{-3c/4}}{n^{1-\epsilon}}\right),\end{aligned}$$

and so

$$\sup_n \int_{U_n < e^{-c}} |\log U_n| \, d\mu_n \ll ce^{-3c/4}.$$

The statement follows. □

The final three lemmata record results about certain integrals.

Lemma 5. *If $t_1, \ldots, t_r$ are real numbers in $[0, 1)$ then*

$$\int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \left| \log\left(\frac{|f(e(t_\ell))|^2}{n}\right) \right| d\mu_n < \infty.$$

Proof. Let $\|\cdot\|_r$ denote the L_r-norm with respect to the measure μ_n, so $\|f\|_r = \left(\int_{\mathfrak{U}_n} |f|^r \, d\mu_n\right)^{1/r}$. Using the arithmetic-geometric mean inequality, followed by the triangle inequality for $\|\cdot\|_r$, we obtain

$$\begin{aligned}
\int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \left| \log\left(\frac{|f(e(t_\ell))|^2}{n}\right)\right| \, d\mu_n &\leq \int_{\mathfrak{U}_n} \left(\frac{1}{r}\sum_{\ell=1}^{r} \left|\log\left(\frac{|f(e(t_\ell))|^2}{n}\right)\right|\right)^r \, d\mu_n \\
&= \frac{1}{r^r}\left\| \sum_{\ell=1}^{r} \left|\log\left(\frac{|f(e(t_\ell))|^2}{n}\right)\right|\right\|_r^r \\
&\leq \frac{1}{r^r}\left(\sum_{\ell=1}^{r} \left\|\left|\log\left(\frac{|f(e(t_\ell))|^2}{n}\right)\right|\right\|_r\right)^r \\
&= E\left(|\log U_n|^r\right).
\end{aligned}$$

From (2.4), we have

$$E(|\log U_n|^r) = \int_0^n |\log x|^r \left(e^{-x} + O\left(x^{-1/4} n^{-1+\epsilon}\right)\right) \, dx.$$

We compute

$$\begin{aligned}
\int_0^1 |\log x|^r \, e^{-x} \, dx &\leq \int_0^1 |\log x|^r \, dx = r!\,, \\
\int_1^\infty |\log x|^r \, e^{-x} \, dx &\leq \int_1^\infty (x-1)^r e^{-x} \, dx = \frac{r!}{e},
\end{aligned}$$

and

$$\frac{1}{n^{1-\epsilon}} \int_0^n \frac{|\log x|^r}{x^{1/4}} \, dx \ll_r n^{-1/4+2\epsilon}.$$

Here, the symbol $\ll_r$ indicates that the implicit constant depends on r. The statement follows. □

Lemma 6. *If $y > 0$ then* $\log y = \displaystyle\int_0^\infty \frac{1}{x}\left(e^{-x} - e^{-xy}\right) \, dx.$

Proof. The left side is $\int_1^y \int_0^\infty e^{-xt} \, dx \, dt$; the right side is $\int_0^\infty \int_1^y e^{-xt} \, dt \, dx$. □

Lemma 7. *If $t_1, \ldots, t_r$ are real numbers in $[0,1]$, then*

$$\int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \int_0^\infty \frac{1}{x_\ell} \left| e^{-x_\ell} - e^{-x_\ell |f(e(t_\ell))|^2/n} \right| \, dx_\ell \, d\mu_n < \infty.$$

Proof. Using a strategy similar to that of the proof of Lemma 5, and adopting the notation used there, we find

$$\begin{aligned}
\int_{\mathfrak{U}_n} \prod_{\ell=1}^{r} \int_0^\infty \frac{1}{x_\ell} \left| e^{-x_\ell} - e^{-x_\ell |f(e(t_\ell))|^2/n} \right| \, dx_\ell \, d\mu_n \\
\leq \left\| \frac{1}{r} \sum_{\ell=1}^{r} \int_0^\infty \frac{1}{x_\ell} \left| e^{-x_\ell} - e^{-x_\ell |f(e(t_\ell))|^2/n} \right| \, dx_\ell \right\|_r^r \\
\leq \left(\frac{1}{r} \sum_{\ell=1}^{r} \left\| \int_0^\infty \frac{1}{x_\ell} \left| e^{-x_\ell} - e^{-x_\ell |f(e(t_\ell))|^2/n} \right| \, dx_\ell \right\|_r \right)^r \\
= \left\| \int_0^\infty \frac{1}{x} \left| e^{-x} - e^{-x |f(e(1))|^2/n} \right| \, dx \right\|_r^r \\
= E\left(|\log U_n|^r \right)
\end{aligned}$$

by Lemma 6, and $E\left(|\log U_n|^r\right) < \infty$ from the proof of Lemma 5. □

We now prove our first main result on unimodular polynomials.

Proof of Theorem 1. For distinct real numbers $t_1, \ldots, t_r$ in $[0,1)$, let $W_{\mathbf{t},n}$ denote the random variable whose value is $\frac{1}{n^r} \prod_{\ell=1}^{r} |f(e(t_\ell))|^2$, where $f \in \mathfrak{U}_n$ is selected at random. Selecting $m_1 = \cdots = m_r = m$ in Lemma 2, we see that the limiting value of the mth moment of $W_{\mathbf{t},n}$ as n grows large is

$$\lim_{n \to \infty} E(W_{\mathbf{t},n}^m) = (m!)^r. \tag{2.5}$$

Since $\sum_{m \geq 1} m!^{-r/2m} = \infty$ for $r = 1$ or 2, by Carleman's test [26, p. 296] there is a unique distribution function F_r having these moments, and further it follows from [14, Theorem 4.5.5] that $W_{\mathbf{t},n}$ converges in distribution to a random variable V_r with distribution function F_r as $n \to \infty$. It is straightforward to verify that the density function of V_r is

$$v_r(x) = \begin{cases} e^{-x} & \text{if } r = 1, \\ \displaystyle\int_0^\infty \frac{1}{t} e^{-t - x/t} \, dt & \text{if } r = 2, \end{cases} \tag{2.6}$$

up to a set of measure zero. Note that $v_2(x) = 2K_0(2\sqrt{x})$, where $K_\nu(x)$ is the modified Bessel function of the second kind.

In particular, for $r = 1$ (and writing t for t_1) it follows from [14, Theorem 4.5.2] that $E(U_{t,n}^p) \to E(V_1^p)$ for all $p > 0$, so

$$\lim_{n \to \infty} \int_{\mathfrak{U}_n} \left(\frac{|f(e(t))|}{\sqrt{n}} \right)^p d\mu_n = \int_0^\infty x^{p/2} e^{-x} \, dx = \Gamma(1 + p/2)$$

for each t. The density function for $U_{t,n}$ is independent of t, so this convergence is uniform in t. Therefore

$$\lim_{n \to \infty} \int_{\mathfrak{U}_n} \frac{\|f\|_p^p}{n^{p/2}} \, d\mu_n = \Gamma(1 + p/2). \tag{2.7}$$

This establishes the asymptotic value for the arithmetic mean of the normalized L_p norm of the unimodular polynomials.

For the geometric mean of the normalized measure, the sequence of random variables $\{\log U_{t,n}\}_{n\geq 1}$ is uniformly integrable by Lemma 4, so $E(\log U_{t,n}) \to E(\log V_1)$ as $n \to \infty$ for each t. Therefore

$$\lim_{n\to\infty} E\left(\log\sqrt{U_{t,n}}\right) = \int_0^\infty e^{-x}\log\sqrt{x}\,dx = -\gamma/2,$$

and so

$$\lim_{n\to\infty}\int_{\mathfrak{U}_n}\log\left(\frac{M(f)}{\sqrt{n}}\right)d\mu_n = -\gamma/2.$$

To calculate the arithmetic mean for the measure, we first observe that

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathfrak{U}_n}\frac{M(f)}{\sqrt{n}}\,d\mu_n &= \lim_{n\to\infty}\int_{\mathfrak{U}_n}\exp\left(\int_0^1\log\left(\frac{|f(e(t))|}{\sqrt{n}}\right)dt\right)d\mu_n\\
&= \lim_{n\to\infty}\int_{\mathfrak{U}_n}\sum_{r\geq 0}\frac{1}{r!}\left(\int_0^1\log\left(\frac{|f(e(t))|}{\sqrt{n}}\right)dt\right)^r d\mu_n\\
&= \lim_{n\to\infty}\sum_{r\geq 0}\frac{1}{r!}\left(\int_0^1\cdots\int_0^1\int_{\mathfrak{U}_n}\prod_{\ell=1}^r\log\left(\frac{|f(e(t_\ell))|}{\sqrt{n}}\right)d\mu_n\,dt_1\,\cdots\,dt_r\right)\\
&= \sum_{r\geq 0}\frac{1}{r!}\left(\int_0^1\cdots\int_0^1\lim_{n\to\infty}\int_{\mathfrak{U}_n}\prod_{\ell=1}^r\log\left(\frac{|f(e(t_\ell))|}{\sqrt{n}}\right)d\mu_n\,dt_1\,\cdots\,dt_r\right).
\end{aligned} \tag{2.8}$$

The validity of switching the order of integration follows from Lemma 5 and Fubini's theorem. Next, using Lemma 3, Lemma 6, Lemma 7, and uniformity in t, and noting that the subset of $[0,1)^n$ where the $t_1, \ldots, t_r$ are not distinct forms a set of measure 0 in $\mathfrak{U}_n$ and so does not affect the integral, we find

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathfrak{U}_n}\prod_{\ell=1}^r\log\left(\frac{|f(e(t_\ell))|}{\sqrt{n}}\right)d\mu_n\\
&\quad= \frac{1}{2^r}\lim_{n\to\infty}\int_{\mathfrak{U}_n}\prod_{\ell=1}^r\left(\int_0^\infty\frac{1}{x_\ell}\left(e^{-x_\ell}-e^{-x_\ell|f(e(t_\ell))|^2/n}\right)dx_\ell\right)d\mu_n\\
&\quad= \frac{1}{2^r}\int_0^\infty\cdots\int_0^\infty\lim_{n\to\infty}\int_{\mathfrak{U}_n}\prod_{\ell=1}^r\frac{1}{x_\ell}\left(e^{-x_\ell}-e^{-x_\ell|f(e(t_\ell))|^2/n}\right)d\mu_n\,dx_1\cdots dx_r\\
&\quad= \frac{1}{2^r}\int_0^\infty\cdots\int_0^\infty\prod_{\ell=1}^r\frac{1}{x_\ell}\left(e^{-x_\ell}-\frac{1}{1+x_\ell}\right)dx_1\cdots dx_r\\
&\quad= \frac{1}{2^r}\left(\int_0^\infty\frac{1}{x}\left(e^{-x}-\frac{1}{1+x}\right)dx\right)^r\\
&\quad= \left(-\frac{\gamma}{2}\right)^r.
\end{aligned}$$

Therefore, from (2.8) we conclude that

$$\lim_{n\to\infty}\int_{\mathfrak{U}_n}\frac{M(f)}{\sqrt{n}}\,d\mu_n = \sum_{r=0}^\infty\frac{1}{r!}\int_0^1\cdots\int_0^1\left(-\frac{\gamma}{2}\right)^r dt_1\cdots dt_r = e^{-\gamma/2}.$$

For the geometric mean of the L_p norm when p is an even integer, we use a similar strategy and (2.5) to obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathfrak{U}_n}\log\left(\int_0^1\frac{|f(e(t))|^p}{n^{p/2}}\,dt\right)d\mu_n\\
&\quad=\lim_{n\to\infty}\int_{\mathfrak{U}_n}\int_0^\infty\frac{1}{x}\left(\exp(-x)-\exp\left(-x\int_0^1\frac{|f(e(t))|^p}{n^{p/2}}dt\right)\right)dx\,d\mu_n\\
&\quad=\lim_{n\to\infty}\int_{\mathfrak{U}_n}\int_0^\infty\frac{1}{x}\left(\exp(-x)-\sum_{r\geq 0}\frac{(-x)^r}{r!}\prod_{\ell=1}^r\left(\int_0^1\frac{|f(e(t_\ell))|^p}{n^{p/2}}\,dt_\ell\right)\right)dx\,d\mu_n\\
&\quad=\int_0^\infty\frac{1}{x}\left(\exp(-x)-\sum_{r\geq 0}\frac{(-x)^r}{r!}\int_0^1\cdots\int_0^1\lim_{n\to\infty}E\left(W_{\mathbf{t},n}^{p/2}\right)dt_1\cdots dt_r\right)dx\\
&\quad=\int_0^\infty\frac{1}{x}\left(\exp(-x)-\sum_{r\geq 0}\frac{(-x)^r}{r!}\Gamma\left(1+\frac{p}{2}\right)^r\right)dx\\
&\quad=\int_0^\infty\frac{1}{x}\left(e^{-x}-e^{-x\Gamma(1+p/2)}\right)dx\\
&\quad=\log\Gamma(1+p/2).
\end{aligned}$$

□

We remark that the limiting distribution of $W_{\mathbf{t},n}$ for $r\geq 3$ does not follow from the computation of the moments in (2.5), since in this case the sum $\sum_{m\geq 1} m!^{-r/2m}$ converges, and so Carleman's criterion is not satisfied. In fact, Berg [3] has shown that for any fixed $r\geq 3$ the sequence of the moments $\{(m!)^r\}_{m\geq 1}$ does not determine a unique distribution function. We suspect that the density function $v_r(x)$ in question is given by

$$v_r(x)=\int_0^\infty\cdots\int_0^\infty\exp\left(-x\prod_{\ell=1}^{r-1}x_\ell^{-1}\right)\prod_{\ell=1}^{r-1}\frac{e^{-x_\ell}}{x_\ell}\,dx_1\cdots dx_{r-1}, \tag{2.9}$$

although we are unable to prove this.

3. Reciprocal polynomials

We use a similar strategy to prove Theorem 2. For a fixed positive integer n, for convenience we say that two nonnegative integers i and j are *complementary* if $i+j=n-1$. Thus, if $f(x)=\sum_k a_k x^k\in\mathfrak{U}_n^*$ and i and j are complementary, then the coefficients a_i and a_j of f satisfy $a_i=\omega\overline{a_j}$, where ω is a fixed complex number of modulus 1. Also, for an odd integer $2k-1$, let $(2k-1)!!$ denote the product $1\cdot 3\cdot 5\cdots(2k-1)=(2k)!/2^k k!$.

Lemma 8. *Suppose $t_1,\ldots,t_r$ are distinct real numbers in $(0,1)$, and $m_1,\ldots,m_r$ are positive integers. Let $M=\sum_{\ell=1}^r m_\ell$. Then*

$$\int_{\mathfrak{U}_n^*}\prod_{\ell=1}^r\frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}}\,d\mu_n^*=\prod_{\ell=1}^r(2m_\ell-1)!!+O\left(\frac{1}{n}\right),$$

where the implicit constant may depend on the m_ℓ and the t_ℓ but is independent of n. Thus,

$$\lim_{n\to\infty}\int_{\mathfrak{U}_n^*}\prod_{\ell=1}^{r}\frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}}\,d\mu_n^* = \prod_{\ell=1}^{r}(2m_\ell-1)!!\,.$$

Proof. We compute

$$\begin{aligned}\int_{\mathfrak{U}_n^*}\prod_{\ell=1}^{r}\frac{|f(e(t_\ell))|^{2m_\ell}}{n^{m_\ell}}\,d\mu_n^* &= \frac{1}{n^M}\int_{\mathfrak{U}_n^*}\prod_{\ell=1}^{r}\left|\sum_{k_\ell=0}^{n-1}a_{k_\ell}e(k_\ell t_\ell)\right|^{2m_\ell}\,d\mu_n^*\\ &= \frac{1}{n^M}\sum_{\substack{I_\ell,J_\ell\in R_{n,m_\ell}\\ 1\le\ell\le r}}\prod_{\ell=1}^{r}e\Bigg(t_\ell\sum_{k=1}^{m_\ell}(i_{\ell,k}-j_{\ell,k})\Bigg)\int_{\mathfrak{U}_n^*}\prod_{\ell=1}^{r}\prod_{k=1}^{m_\ell}a_{i_{\ell,k}}\overline{a_{j_{\ell,k}}}\,d\mu_n^*\\ &= \frac{1}{n^M}\sideset{}{^*}\sum_{\substack{I_\ell,J_\ell\in R_{n,m_\ell}\\ 1\le\ell\le r}}\prod_{\ell=1}^{r}e\Bigg(t_\ell\sum_{k=1}^{m_\ell}(i_{\ell,k}-j_{\ell,k})\Bigg).\end{aligned}$$

Here, $\sum^*$ denotes the sum over only those $2M$-tuples $(I_1,\dots,I_r,J_1,\dots,J_r)$ where the value of each index $i_{\ell,k}$ or $j_{\ell,k}$ either matches that of some other index $i_{\ell,k'}$ or $j_{\ell,k'}$, or complements the value of some other such index. We again restrict our consideration to those terms where $(I_1,\dots,I_r,J_1,\dots,J_r)$ can be grouped into M distinct pairs such that each grouping consists of some number $s\equiv M \bmod 2$ of pairs $(i_{\ell,k},j_{\ell,k'})$ of matching integers, together with $(M-s)/2$ pairs $(i_{\ell,k},i_{\ell,k'})$ and $(M-s)/2$ pairs $(j_{\ell,k},j_{\ell,k'})$ of complementary integers. This way, we have that $\prod_{k=1}^{m_\ell}a_{i_{\ell,k}}\overline{a_{j_{\ell,k}}}=1$ for each ℓ. The number of possible pairings is $\prod_{\ell=1}^{r}(2m_\ell-1)!!$, and the statement follows in the same way as Lemma 2. □

We now prove our main result on reciprocal unimodular polynomials.

Proof of Theorem 2. It is well known that the $2m$th moment of the standard normal distribution is $(2m-1)!!$, so

$$\sqrt{\frac{2}{\pi}}\int_0^\infty x^{2m}e^{-x^2/2}\,dx = (2m-1)!!\,.$$

Therefore,

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty y^{m-1/2}e^{-y/2}\,dy = (2m-1)!!\,,$$

so by Carleman's test the limiting density function as $n\to\infty$ for the random variables $|f(e(t))|^2/n$ on $\mathfrak{U}_n^*$ for $t\neq 0$ is

$$\rho_t(x) = \frac{e^{-x/2}}{\sqrt{2\pi x}}$$

for almost every x in $[0,\infty)$. Thus, for $t\neq 0$, we have

$$\lim_{n\to\infty}\int_{\mathfrak{U}_n^*}\log\left(\frac{|f(e(t))|}{\sqrt{n}}\right)\,d\mu_n^* = \int_0^\infty\frac{e^{-x/2}}{\sqrt{2\pi x}}\log\sqrt{x}\,dx = -\frac{\gamma+\log 2}{2}$$

and

$$\lim_{n\to\infty}\int_{\mathfrak{U}_n^*}\left(\frac{|f(e(t))|}{\sqrt{n}}\right)^p\,d\mu_n^* = \int_0^\infty\frac{x^{p/2}e^{-x/2}}{\sqrt{2\pi x}}\,dx = \frac{2^{p/2}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right),$$

and the results on the geometric mean of the normalized measure and the arithmetic mean of the normalized L_p norm follow as in the proof of Theorem 1. The remaining statements may be proved in the same way as Theorem 1. □

4. Moments of the distributions

We now establish the result on the moments of the distributions of the L_p norms and Mahler's measure of unimodular polynomials.

Proof of Theorem 3. For any even integer $p \geq 2$, we compute

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathfrak{U}_n} \left(\frac{\|f\|_p^p}{n^{p/2}} \right)^m d\mu_n &= \lim_{n\to\infty} \int_{\mathfrak{U}_n} \left(\prod_{\ell=1}^m \int_0^1 \frac{|f(e(t))|^p}{n^{p/2}}\, dt \right) d\mu_n \\
&= \int_0^1 \cdots \int_0^1 \lim_{n\to\infty} \int_{\mathfrak{U}_n} \prod_{\ell=1}^m \frac{|f(e(t_\ell))|^p}{n^{p/2}}\, d\mu_n\, dt_1 \cdots dt_m \\
&= \int_0^1 \cdots \int_0^1 \Gamma(1+p/2)^m\, dt_1 \cdots dt_m \\
&= \left(\lim_{n\to\infty} \int_{\mathfrak{U}_n} \frac{\|f\|_p^p}{n^{p/2}}\, d\mu_n \right)^m
\end{aligned}$$

by Lemma 2. The computation for Mahler's measure is similar. When $m = 2$, we may use the density function (2.6) to determine that

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathfrak{U}_n} \left(\frac{\|f\|_p^p}{n^{p/2}} \right)^2 d\mu_n &= \lim_{n\to\infty} \int_{\mathfrak{U}_n} \int_0^1 \int_0^1 \frac{|f(e(t_1))|^p}{n^{p/2}} \frac{|f(e(t_2))|^p}{n^{p/2}}\, dt_1\, dt_2\, d\mu_n \\
&= \int_0^1 \int_0^1 \lim_{n\to\infty} \int_{\mathfrak{U}_n} \frac{|f(e(t_1))|^p}{n^{p/2}} \frac{|f(e(t_2))|^p}{n^{p/2}}\, d\mu_n\, dt_1\, dt_2 \\
&= \int_0^1 \int_0^1 \int_0^\infty x^{p/2} \int_0^\infty \frac{1}{t} e^{-t-x/t}\, dt\, dx\, dt_1\, dt_2 \\
&= \Gamma(1+p/2)^2 \\
&= \left(\lim_{n\to\infty} \int_{\mathfrak{U}_n} \frac{\|f\|_p^p}{n^{p/2}}\, d\mu_n \right)^2
\end{aligned}$$

for $p > 0$. The computations for $\mathfrak{U}_n^*$ are similar. (The density function for the case $m = 2$ in this case is $K_0(\sqrt{x})/\pi\sqrt{x}$.) □

5. Concluding remarks

It would be of interest to apply this method to determine the expected value in the limit for the normalized Mahler's measure of the Littlewood polynomials, and to study the reciprocal Littlewood polynomials in the same way. We conjecture that the limiting mean values for the Littlewood polynomials match the ones stated in Theorems 1 and 2 for the unimodular polynomials. (This was established already in [10] for the arithmetic mean of the L_p norms with $p > 0$ in $\mathfrak{L}_n$.) Verifying this for Mahler's measure in $\mathfrak{L}_n$ would thus improve the best known result (1.2) in Mahler's problem. Indeed, by calculating sequences of moments one may establish analogues of Lemmata 2 and 8 for the Littlewood polynomials. However, the moment sequences are no longer independent of t in this case, as $t = 0$ and $t = 1/2$ behave differently from other $t \in (0, 1)$.

Finally, we remark that lower bounds on Mahler's measure of noncyclotomic Littlewood polynomials are found in [7, 9, 15, 19], and a list of the maximum values of Mahler's measure for polynomials in $\mathfrak{L}_n$ for $n \leq 25$ can be found in [11]. Also, the distribution of the values of Mahler's measure for polynomials with arbitrary real or complex coefficients is studied in [12, 27], and average L_p norms of other families of polynomials are considered in [5, 6, 24].

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