AVERAGE MAHLER'S MEASURE AND L_p NORMS OF LITTLEWOOD POLYNOMIALS

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Abstract. Littlewood polynomials are polynomials with each of their coefficients in the set $\{-1, 1\}$. We compute asymptotic formulas for the arithmetic mean values of the Mahler's measure and the L_p norms of Littlewood polynomials of degree $n-1$. We show that the arithmetic means of the Mahler's measure and the L_p norms of Littlewood polynomials of degree $n-1$ are asymptotically $e^{-\gamma/2}\sqrt{n}$ and $\Gamma(1+p/2)^{1/p}\sqrt{n}$, respectively, as n grows large. We also compute asymptotic formulas for the power means M_α of the L_p norms of Littlewood polynomials of degree $n-1$ for any $p > 0$ and $\alpha > 0$. We are able to compute asymptotic formulas for the geometric means of the Mahler's measure of the "truncated" Littlewood polynomials $\hat{f}$ defined by $\hat{f}(z) := \min\{|f(z)|, 1/n\}$ associated with Littlewood polynomials f of degree $n-1$. These "truncated" Littlewood polynomials have the same limiting distribution functions as the Littlewood polynomials. Analogous results for the unimodular polynomials, i.e., with complex coefficients of modulus 1, were proved in [4]. Our results for Littlewood polynomials were expected by people for a long time but it looked beyond reach as an analogous result of Fielding was not available for Littlewood polynomials.

1. Introduction and Main Results

The *Mahler's measure* $M(f)$ of a polynomial f with complex coefficients is defined by

$$\log M(f) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(e^{it})\right| dt.$$

It is well known that if the polynomial f is of the form

$$f(z) = a_m \prod_{k=1}^{m} (z - \alpha_k),$$

then

$$M(f) = |a_m| \prod_{k=1}^{m-1} \max\{1, |\alpha_k|\}$$

and

$$M(f) = \lim_{p\to 0^+} \|f\|_p,$$

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where $\|f\|_p$ is defined by

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{it}) \right|^p \, dt \right)^{1/p}, \qquad p > 0\,.$$

A polynomial f is called Littlewood polynomial if each of its coefficients is in the set $\{-1,1\}$. A polynomial f is called unimodular if each of its coefficients is a complex number of modulus 1. Let $\mathfrak{L}_n$ denote the set of all Littlewood polynomials of degree $n-1$. Let $\mathfrak{U}_n$ denote the set of all unimodular polynomials of degree $n-1$. Note that Parseval's formula implies $\|f\|_2 = \sqrt{n}$ for all $f \in \mathfrak{U}_n$. We also introduce the set $\mathfrak{L} := \cup_{n=1}^{\infty} \mathfrak{L}_n$ of all Littlewood polynomials and the set $\mathfrak{U} := \cup_{n=1}^{\infty} \mathfrak{U}_n$ of all unimodular polynomials.

Littlewood posed a number of problems regarding the behavior of Littlewood and unimodular polynomials on the unit circle; see for instance [11, problem 19]. He asked if there exist absolute constants $c_1 > 0$ and $c_2 > 0$ and a sequence (f_n) of polynomials $f_n \in \mathfrak{U}_n$ (or perhaps $f_n \in \mathfrak{L}_n$) such that

$$c_1 \sqrt{n} \leq |f_n(z)| \leq c_2 \sqrt{n}\,, \qquad z \in \mathbb{C}, \quad |z| = 1\,.$$

Kahane [9] proved that there is such a sequence (f_n) of polynomials $f_n \in \mathfrak{U}_n$, showing in fact that for every $\varepsilon > 0$ there is a sequence (f_n) of polynomials $f_n \in \mathfrak{U}_n$ such that

$$(1-\varepsilon)\sqrt{n} \leq |f_n(z)| \leq (1+\varepsilon)\sqrt{n}\,, \qquad z \in \mathbb{C}, \quad |z| = 1\,.$$

for all sufficiently large n. Whether or not there is such a sequence (f_n) of polynomials $f_n \in \mathfrak{L}_n$ is still open. The Rudin-Shapiro polynomials of degree $n = 2^k - 1$ satisfy the upper bound with $c_2 = \sqrt{2}$ and no sequence (f_n) of polynomials $f_n \in \mathfrak{L}_n$ is known that satisfies just a lower bound with an absolute constant $c_1 > 0$. Erdős conjectured that there is an absolute constant $\varepsilon > 0$ such that the maximum modulus of any Littlewood polynomial $f_n \in \mathfrak{L}_n$ on the unit circle is at least $(1+\varepsilon)\sqrt{n}$.

Problems regarding the existence of unimodular or Littlewood polynomials with certain flatness properties on the unit circle also arise in the context of the Mahler's measure. In 1963, Mahler [12] proved that the maximum value of the Mahler's measure of a polynomial of degree at most m having all its coefficients in the closed unit disk is attained by a unimodular polynomial. Mahler posed the problem of determining the mean value of the Mahler's measure of these polynomials to Fielding, who proved in 1970 that

$$\lim_{n\to\infty} \sup_{f \in \mathfrak{U}_n} \frac{M(f)}{\sqrt{n}} = 1, \tag{1.1}$$

and

$$\operatorname*{mean}_{f \in \mathfrak{U}_n} \frac{M(f)}{\sqrt{n}} \geq e^{-\gamma/2} \left(1 + O\left(n^{-1/4+\varepsilon} \right) \right), \tag{1.2}$$

where γ denotes Euler's constant and ε is an arbitrarily small positive constant. See [8]. Here

$$\operatorname*{mean}_{f \in \mathfrak{U}_n} (F(f)) := \frac{1}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} F(f)\, d\theta_0\, d\theta_1 \, \cdots \, d\theta_{n-1}\,.$$

In [4], the authors strengthen (1.2) by proving that the limiting value of both the geometric and the arithmetic means of the normalized Mahler's measure of unimodular polynomials $f \in \mathfrak{U}_n$ is exactly $e^{-\gamma/2} = 0.749306\cdots$.

Theorem 1.1. *Let $\mathfrak{U}_n$ denote the set of unimodular polynomials of degree $n-1$. Then*

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{U}_n} \log\left(\frac{M(f)}{\sqrt{n}}\right) = -\gamma/2$$

and

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{U}_n} \log\left(\frac{\|f\|_p}{\sqrt{n}}\right) = \frac{1}{p}\log\Gamma(1+p/2)\,, \qquad p>0.$$

Also

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{U}_n} \frac{M(f)}{\sqrt{n}} = e^{-\gamma/2}$$

and

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{U}_n} \frac{\|f\|_p^p}{\sqrt{n}} = \Gamma(1+p/2)\,, \qquad p>0.$$

In this paper first we determine the limiting values of the arithmetic means of the normalized Mahler's measure and L_p norms of Littlewood polynomials $f\in\mathfrak{L}_n$. The values we obtain are the same as those found in [4] in the unimodular case. To do so we will employ a result of Konyagin and Schlag (Lemma 2.4 below) in the place of Fielding's result about the unimodular polynomials. We prove the following theorem in the next section. For any $f\in\mathfrak{L}_n$, we define $\hat{f}(z) := \max\{|f(z)|, n^{-1}\}$.

Theorem 1.2. *We have*

$$\lim_{n\to\infty} \exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n} \log\left(\frac{M(\hat{f})}{\sqrt{n}}\right)\right) = e^{-\gamma/2}, \tag{1.3}$$

and

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{M(f)}{\sqrt{n}} = e^{-\gamma/2}, \tag{1.4}$$

where $\operatorname*{mean}_{f\in\mathfrak{L}_n}(\cdot) := \frac{1}{2^n}\sum_{f\in\mathfrak{L}_n}(\cdot)$*. We also have*

$$\lim_{n\to\infty} \exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n} \log\left(\frac{\|\hat{f}\|_p}{\sqrt{n}}\right)\right) = \Gamma(1+p/2)^{1/p}, \qquad p>0\,, \tag{1.5}$$

and

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{\|f\|_p}{\sqrt{n}} = \Gamma(1+p/2)^{1/p}\,, \qquad p>0\,. \tag{1.6}$$

Result (1.6) is proved in Theorem 1 of [2], while (1.3), (1.4) and (1.5) are new results. Hopefully, the results (1.3) and (1.5) will shed some light on how to compute the limiting value of the geometric means of the normalized Mahler's measure of Littlewood polynomials $f\in\mathfrak{L}_n$ as well in the future.

Although we cannot offer an asymptotic formula for the geometric means of the Mahler's measures of Littlewood polynomials $f\in\mathfrak{L}_n$, we can prove the following result.

Corollary 1.3. *We have*

$$e^{-\gamma/2} \ge \limsup_{n\to\infty} \exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n} \log\left(\frac{M(f)}{\sqrt{n}}\right)\right).$$

For every fixed $\varepsilon>0$, we also have

$$\liminf_{n\to\infty} \exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n} \log\left(\frac{M(f)}{n^{1/2-\varepsilon}}\right)\right) \ge e^{-\gamma/2}.$$

The problem of determining whether there exists a positive number ε such that $M(f)/\|f\|_2 < 1-\varepsilon$ for every Littlewood polynomial of positive degree is commonly known as *Mahler's problem* (see for instance [1]). The largest known value of $M(f)/\|f\|_2$ for a Littlewood polynomial of positive degree is $0.98636\ldots$, achieved by $f(x) = x^{12}+x^{11}+x^{10}+x^9+x^8-x^7-x^6+x^5+x^4-x^3+x^2-x+1$. The best known asymptotic result in Mahler's problem is due to Erdélyi and Lubinsky [6], who recently proved that the Fekete polynomials, defined by $f_p(x) = \sum_{k=1}^{p-1}\left(\frac{k}{p}\right)x^{k-1}$, where p is a prime number and $\left(\frac{\cdot}{p}\right)$ denotes the usual Legendre symbol, satisfy

$$\frac{M(f_p)}{\|f_p\|_2} > \frac{1}{2} - \varepsilon \tag{1.7}$$

for arbitrarily small $\varepsilon > 0$ when p is sufficiently large. Recently, in [3], we constructed Littlewood polynomials f_n of degree n satisfying (1.7) for all positive integers n, not only for prime numbers. The Rudin-Shapiro polynomials P_n are also known to satisfy $M(P_n) > c\|P_n\|_2$ for every n with an absolute constant $c > 0$, see [5].

An analogue of Fielding's result (1.2) for Littlewood polynomials, and an improved lower bound in Mahler's problem, both follow immediately.

Corollary 1.4. *For every $\varepsilon > 0$ there exist infinitely many Littlewood polynomials f satisfying*

$$\frac{M(f)}{\|f\|_2} > e^{-\gamma/2} - \varepsilon,$$

and infinitely many Littlewood polynomials f satisfying

$$\frac{M(f)}{\|f\|_2} < e^{-\gamma/2} + \varepsilon.$$

Since the arithmetic means and geometric means of the normalized Mahler's measure of $\hat{f}$ associated with Littlewood polynomials $f \in \mathfrak{L}_n$ converge to the same limiting value as $n \to \infty$, we can deduce that the values $M(\hat{f})$ are close to each other for almost all $f \in \mathfrak{L}_n$, and hence almost all of the values $M(\hat{f})$ are close to the mean. More precisely we can prove the following result.

Theorem 1.5. *The normalized Mahler's measure $M(\hat{f})/\sqrt{n}$ converges to $e^{-\gamma/2}$ in probability. That is,*

$$\lim_{n\to\infty} \frac{1}{2^n}\left|\left\{ f \in \mathfrak{L}_n : \left|\frac{M(\hat{f})}{\sqrt{n}} - e^{-\gamma/2}\right| > \varepsilon \right\}\right| = 0 \tag{1.8}$$

for every $\varepsilon > 0$, where $|A|$ denotes the number of the elements in the set A. Also, the normalized L_p norms $\|f\|_p/\sqrt{n}$ converge to $\Gamma(1+p/2)^{1/p}$ in probability. That is,

$$\lim_{n\to\infty} \frac{1}{2^n}\left|\left\{ f \in \mathfrak{L}_n : \left|\frac{\|f\|_p}{\sqrt{n}} - \Gamma(1+p/2)^{1/p}\right| > \varepsilon \right\}\right| = 0 \tag{1.9}$$

for every $\varepsilon > 0$ and $p > 0$.

Result (1.9) is proved in Theorem 1 of [2], while (1.8) is a new result.

We may obtain more information on the distribution of the Mahler's measure and the L_p norms by studying their moments. We have the following result.

Theorem 1.6. *We have*

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \left(\frac{\|f\|_p}{\sqrt{n}}\right)^\alpha = \left(\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{\|f\|_p}{\sqrt{n}}\right)^\alpha$$

and

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \left(\frac{M(\hat{f})}{\sqrt{n}}\right)^\alpha = \left(\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{M(\hat{f})}{\sqrt{n}}\right)^\alpha$$

for every $\alpha > 0$ and $p > 0$.

We think that similar techniques permit the extension of our main results to the derivatives of Littlewood polynomials.

2. Proofs of the Main Results

For $f \in \mathfrak{L}_n$ of the form

$$f(z) := \sum_{j=0}^{n-1} a_j z^j\,, \qquad z := e^{i\theta}, \quad \theta \in [0, 2\pi)\,,$$

we define

$$U_n := U_n(f,\theta) := \frac{|f(z)|^2}{n} = \frac{1}{n}\left|\sum_{j=0}^{n-1} a_j e^{ij\theta}\right|^2.$$

For a fixed $\theta \in [0, 2\pi)$, P. Borwein and Lockhart [2] studied the distribution of U_n as a random variable. Let

$$\sigma_{n,c}^2(\theta) := \sum_{j=0}^{n-1} \cos^2(j\theta) \quad \text{and} \quad \sigma_{n,s}^2(\theta) := \sum_{j=0}^{n-1} \sin^2(j\theta)$$

and write

$$a_{k,n}(\theta) := \frac{\cos(k\theta)}{\sigma_{n,c}(\theta)}$$

and, for θ not an integer multiple of π,

$$b_{k,n}(\theta) := \frac{\sin(k\theta)}{\sigma_{n,s}(\theta)}.$$

We also define random variables

$$C_n(\theta) := \frac{\sum_{j=0}^{n-1} a_j \cos(j\theta)}{\sigma_{n,c}(\theta)} \quad \text{and} \quad S_n(\theta) := \frac{\sum_{j=0}^{n-1} a_j \sin(j\theta)}{\sigma_{n,s}(\theta)}.$$

Note that

$$U_n(f,\theta) = \frac{|f(z)|^2}{n} = \frac{\sigma_{n,c}^2(\theta)C_n^2(\theta) + \sigma_{n,s}^2(\theta)S_n^2(\theta)}{n}.$$

Let Z, Z_1, Z_2 be the standard normal distributions, and let $\mathbb{E}(X)$ be the expected value of the random variable X. We need the following lemma which records some facts observed in the proof of Theorem 1 of [2].

Lemma 2.1. *For any fixed θ not an integer multiple of π, we have*

(i) *$(C_n(\theta), S_n(\theta))$ converges in distribution to (Z_1, Z_2) as $n \to \infty$;*
(ii) *$|f(e^{i\theta})|^2/n$ converges in distribution to $(Z_1^2 + Z_2^2)/2$ as $n \to \infty$;*
(iii) *$\mathbb{E}(|C_n(\theta)|^p)$ and $\mathbb{E}(|S_n(\theta)|^p)$ converge uniformly on $(0,\pi)\cup(\pi,2\pi)$ to $\mathbb{E}(|Z|^p)$ as $n \to \infty$.*

Proof of Lemma 2.1. See pages 1466 and 1469 of [2]. □

For a fixed $n \in \mathbb{N}$ and and a fixed $z = e^{i\theta} \in \mathbb{C}$, let

$$F_n^z(x) := \frac{1}{2^n}\left|\left\{f \in \mathfrak{L}_n : \frac{|f(z)|^2}{n} \leq x\right\}\right|$$

be the distribution function of U_n, and let $F(x) = 1 - e^{-x}$ be the distribution function of the standard exponential function on $[0,\infty)$. The following is a Berry-Esseen type central limit theorem for $F_n^z(x)$. Let $\|y\| := \min_{m\in\mathbb{Z}} |2\pi m - y|$.

Proposition 2.2. *Let $0 < A < 1$ and $z := e^{i\theta}$ with $\|2\theta\| \geq 1/n^A$. We have*

$$F_n^z(x) = F(x) + O(n^{-1/2} + n^{-(1-A)}) \tag{2.10}$$

for any $0 \leq x < \infty$, where the implicit constant in the O symbol is absolute, independent of θ and x.

Proof of Proposition 2.2. For a fixed $\theta \in [0, 2\pi)$, let $g_{n,c}(t)$ and $g_{n,s}(t)$ be the probability density functions of $C_n(\theta)$ and $S_n(\theta)$ and let $G_{n,c}(t)$ and $G_{n,s}(t)$ be the corresponding distribution functions, respectively. In view of (iii) of Lemma 2.1, we have $\mathbb{E}(|C_n(\theta)|^p)$ and $\mathbb{E}(|S_n(\theta)|^p)$ converge uniformly on $(0,\pi)\cup(\pi,2\pi)$ to

$$\mathbb{E}(|Z|^p) = \int_{-\infty}^{\infty} |x|^p d\Phi(x) = \frac{2^{p/2}\Gamma\left((p+1)/2\right)}{\sqrt{\pi}},$$

where

$$\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-t^2/2}\,dt\,.$$

By (i) of Lemma 2.1, both $G_{n,c}(t)$ and $G_{n,s}(t)$ converge to the standard normal distributions. So using the Berry-Esseen's central limit theorem (see Theorem 1 on p. 542 of [7]) with either $X_n = C_n(\theta)$ or $X_n = S_n(\theta)$, where $\theta \notin \{0,\pi\}$, we have

$$|G_{n,c}(x) - \Phi(x)| \leq c_3\frac{\rho_n}{\sigma_n^3\sqrt{n}} \quad \text{and} \quad |G_{n,s}(x) - \Phi(x)| \leq c_3\frac{\rho_n}{\sigma_n^3\sqrt{n}},$$

where $c_3 > 0$ is an absolute constant,

$$\rho_n := \mathbb{E}(|X_n|^3)\,, \qquad \lim_{n\to\infty}\rho_n = \frac{2^{3/2}\,\Gamma(2)}{\sqrt{\pi}} = 2\sqrt{\frac{2}{\pi}}\,,$$

and

$$\sigma_n^2 := \mathbb{E}(|X_n|^2)\,, \qquad \lim_{n\to\infty}\sigma_n^2 = \frac{2\,\Gamma(3/2)}{\sqrt{\pi}} = 1\,.$$

It follows that

$$G_{n,c}(x) = \Phi(x) + O(n^{-1/2}) \quad \text{and} \quad G_{n,s}(x) = \Phi(x) + O(n^{-1/2})$$

for all x in $(-\infty,\infty)$. Note that

$$\begin{aligned}\frac{\sigma_{n,c}^2}{n} &= \frac{1}{n}\sum_{j=0}^{n-1}\cos^2(j\theta) = \frac{1}{2} + \frac{1}{2n}\sum_{j=0}^{n-1}\cos(2j\theta) = \frac{1}{2} + \frac{1}{2n}\Re\left(\sum_{j=0}^{n-1} e^{2ij\theta}\right)\\ &= \frac{1}{2} + O\left(\frac{1}{n\|2\theta\|}\right) = \frac{1}{2} + O\left(n^{-(1-A)}\right)\end{aligned}$$

for any $\theta \in [0, 2\pi)$ with $\|2\theta\| \geq 1/n^A$, where the implicit constant in the O symbol is absolute. Similarly, we have

$$\frac{\sigma_{n,c}^2}{n} = \frac{1}{2} + O\left(n^{-(1-A)}\right)$$

for any $\theta \in [0, 2\pi)$ with $\|2\theta\| \geq 1/n^A$, where the implicit constant in the O symbol is absolute. Therefore, for any $\theta \in [0, 2\pi)$ with $\|2\theta\| \geq 1/n^A$ and for any $x \in (-\infty, \infty)$, we have

$$\begin{aligned}
&\left|\frac{1}{2^n}\left|\left\{f \in \mathfrak{L}_n : \frac{\sigma_{n,c}C_n(\theta)}{\sqrt{n}} \leq x\right\}\right| - \Phi(x)\right| \\
\leq\ &\left|\frac{1}{2^n}\left|\left\{f \in \mathfrak{L}_n : \frac{\sigma_{n,c}C_n(\theta)}{\sqrt{n}} \leq x\right\}\right| - \frac{1}{2^n}\left|\{f \in \mathfrak{L}_n : C_n(\theta) \leq x\}\right|\right| \\
&\quad + \left|\frac{1}{2^n}\left|\{f \in \mathfrak{L}_n : C_n(\theta) \leq x\}\right| - \Phi(x)\right| \\
\leq\ &\left|\Phi\left(x\left(\frac{1}{2} + \frac{C}{n^{1-A}}\right)^{-1/2}\right) - \Phi\left(x\left(\frac{1}{2} - \frac{C}{n^{1-A}}\right)^{-1/2}\right)\right| + |G_{n,c}(x) - \Phi(x)| \\
\ll\ &\frac{x}{n^{1-A}}e^{-c_4x^2} + n^{-1/2} \ll n^{-1/2} + n^{-(1-A)}
\end{aligned}$$

with some absolute constants $C > 0$ and $c_4 > 0$, where the implicit constant in the $\ll$ symbol is also absolute. Similarly, we have

$$\left|\frac{1}{2^n}\left|\left\{f \in \mathfrak{L}_n : \frac{\sigma_{n,c}S_n(\theta)}{\sqrt{n}} \leq x\right\}\right| - \Phi(x)\right| \ll n^{-1/2} + n^{-(1-A)}.$$

We recall that

$$\frac{|f(z)|^2}{n} = \frac{\sigma_{n,c}^2 C_n(\theta)^2}{n} + \frac{\sigma_{n,s}^2 S_n(\theta)^2}{n}.$$

Therefore, we have

$$F_n^z(x) = F(x) + O(n^{-1/2} + n^{-(1-A)}).$$

□

For any $z \in \mathbb{C}$ and $|z| = 1$ we define the corresponding distribution function

$$G_n^z(t) := \frac{1}{2^n}\left|\left\{f \in \mathfrak{L}_n : \frac{|\hat{f}(z)|^2}{n} \leq t\right\}\right|.$$

Observe that $|\hat{f}(z)|^2/n \geq n^{-3}$, hence $G_n^z(t) = 0$ for $t < n^{-3}$. Observe also that $|f(z)| < n^{-1}$ and $n^{-3} \leq t$ imply $|f(z)|^2/n < n^{-3} \leq t$, and hence $\chi_{[0,t)}\left(\frac{|f(z)|^2}{n}\right) = 1$. Hence, if $t \geq n^{-3}$, then we have

$$G_n^z(t) \quad = \quad F_n^z(t)\,.$$

Therefore

$$G_n^z(t) = \begin{cases} F_n^z(t) & \text{if } t \geq n^{-3}, \\ 0 & \text{if } t < n^{-3}. \end{cases} \tag{2.11}$$

Theorem 2.3. *Suppose $z = e^{i\theta}$ with $\theta \in [0, 2\pi)$ and $\|2\theta\| > 1/n^A$ for some $0 < A < 1$. There is an absolute constant $c_5 > 0$ such that*

$$\left|\int_{0^+}^{\infty} \log t \, d\left(G_n^z(t) - F(t)\right)\right| \leq c_5(n^{-1/2} + n^{-(1-A)}) \log n \tag{2.12}$$

for every $n \in \mathbb{N}$. *Hence*

$$\left|\operatorname*{mean}_{f \in \mathfrak{L}_n} \log\left(\frac{|\hat{f}(z)|^2}{n}\right) + \gamma\right| \le c_5(n^{-1/2} + n^{-(1-A)}) \log n \tag{2.13}$$

for every $n \in \mathbb{N}$. *In particular,*

$$\lim_{n\to\infty} \operatorname*{mean}_{f \in \mathfrak{L}_n} \log\left(\frac{|\hat{f}(z)|^2}{n}\right) = -\gamma\,, \tag{2.14}$$

where the convergence is uniform on $\theta \in [0, 2\pi)$ *such that* $\|2\theta\| > 1/n^A$.

Proof of Theorem 2.3. We divide the interval of integration into three subintervals $[0, n^{-3}], [n^{-3}, n], [n, \infty)$ and estimate on them separately.

First we observe that by (2.11)

$$\begin{aligned} & \left|\int_n^\infty \log t \, d\,(G_n^z(t) - F(t))\right| \\ = & \left|\int_n^\infty \log t \, d\,(F_n^z(t) - F(t))\right| \\ = & \left|[(F_n^z(t) - F(t)) \log t]_n^\infty - \int_n^\infty \frac{F_n^z(t) - F(t)}{t}\, dt\right| \\ \le & \left|\left[\left(e^{-t}\right) \log t\right]_n^\infty\right| + \int_n^\infty \frac{e^{-t}}{t}\, dt\,, \end{aligned}$$

since $F_n^z(t) = 1$ for $t \ge n$ and hence $F_n^z(t) - F(t) = 1 - (1 - e^{-t}) = e^{-t}$ for $t \ge n$. Therefore,

$$\left|\int_n^\infty \log t \, d\,(G_n^z(t) - F(t))\right| \le \frac{\log n}{e^n} + \frac{1}{n}\int_n^\infty e^{-t}\, dt \ll \frac{\log n}{e^n}\,, \tag{2.15}$$

where the implicit constant in the $\ll$ symbol is absolute.

Now we estimate on the interval $[n^{-3}, n]$. Using (2.11) and (2.10), we obtain

$$\begin{aligned} & \left|\int_{1/n^3}^n \log t \, d\,(G_n^z(t) - F(t))\right| = \left|\int_{1/n^3}^n \log t \, d\,(F_n^z(t) - F(t))\right| \\ = & \left|[(F_n^z(t) - F(t)) \log t]_{1/n^3}^n - \int_{1/n^3}^n \frac{F_n^z(t) - F(t)}{t}\, dt\right| \\ \ll & |F_n^z(n) - F(n)| \log n + \left|F_n^z(n^{-3}) - F(n^{-3})\right| \log n + \int_{1/n^3}^n \frac{|F_n^z(t) - F(t)|}{t}\, dt \\ \ll & (n^{-1/2} + n^{-(1-A)}) \log n + (n^{-1/2} + n^{-(1-A)}) \int_{1/n^3}^n \frac{1}{t}\, dt \\ \ll & (n^{-1/2} + n^{-(1-A)}) \log n\,. \end{aligned} \tag{2.16}$$

Finally we estimate on $[0, n^{-3}]$. We have

$$\left|\int_{0^+}^{1/n^3} \log t \, d\,(G_n^z(t) - F(t))\right| = \left|[-F(t) \log t]_{0^+}^{1/n^3} + \int_{0^+}^{1/n^3} \frac{F(t)}{t}\, dt\right|$$

as $G_n^z(t) \equiv 0$ when $t < 1/n^3$. Observe that $\lim_{t\to 0^+} F(t)\log t = 0$, so using (2.10) again we obtain

$$\left|\int_{0^+}^{1/n^3} \log t \, d\,(-F(t))\right| \ll (n^{-1/2} + n^{-(1-A)}) \log n + \int_{0^+}^{1/n^3} \frac{F(t)}{t}\, dt\,.$$

Estimating the second term on the right hand side gives

$$\int_{0^+}^{1/n^3} \frac{F(t)}{t}\, dt = \int_{0^+}^{1/n^3} \frac{1-e^{-t}}{t}\, dt \le \int_{0^+}^{1/n^3} 1\, dt = \frac{1}{n^3}\,,$$

as $1 - e^{-t} \le t$ for $t \ge 0$. Hence

$$\left|\int_{0^+}^{1/n^3} \log t \, d\,(G_n^z(t) - F(t))\right| \ll (n^{-1/2} + n^{-(1-A)}) \log n\,. \tag{2.17}$$

Now, (2.12) follows from (2.15), (2.16) and (2.17). □

In the case of unimodular polynomials in [4] a result of Fielding in [8] helped the authors to succeed, but an analogue of Fielding's result is not available in the case of Littlewood polynomials. However, to prove (1.4) of our Theorem 1.2 we can use a recent result of Konyagin and Schlag in [10]. From Theorem 1.2 of [10] (with $\phi \equiv 1$), for any $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \frac{1}{2^n} \left|\left\{ f \in \mathfrak{L}_n : \min\left\{|f(z)| : z \in \mathbb{C} \text{ and } ||z|-1| < \varepsilon n^{-2}\right\} < \varepsilon n^{-1/2} \right\}\right| \le c_6 \varepsilon$$

with some absolute constant $c_6 > 0$. Hence we have

Lemma 2.4. *For $n \in \mathbb{N}$ and $\varepsilon > 0$ we define*

$$A_n(\varepsilon) = \left\{ f \in \mathfrak{L}_n : \min_{|z|=1} |f(z)| < \varepsilon n^{-1/2} \right\}.$$

There is an absolute constant $c_6 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{2^n} |A_n(\varepsilon)| \le c_6 \varepsilon \tag{2.18}$$

for every $\varepsilon > 0$.

Proof of Theorem 1.2. Integrating (2.13) over the set

$$H_n := \{\theta \in [0, 2\pi) : \, \|2\theta\| \ge n^{-1/2}\}$$

and using Theorem 2.3 with $A = 1/2$, we obtain

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{H_n} \operatorname*{mean}_{f \in \mathfrak{L}_n} \log\left(\frac{|\hat f(z)|^2}{n}\right) d\theta = -\gamma\,,$$

Observe that

$$-3\log n = \log(n^{-3}) \le \log(|\hat f(z)|^2/n) \le \log n\,, \quad z = e^{i\theta}, \quad \theta \in [0, 2\pi)\,.$$

Hence

$$-3\log n \le \operatorname*{mean}_{f \in \mathfrak{L}_n} \log\left(\frac{|\hat f(z)|^2}{n}\right) \le \log n\,, \quad \theta \in [0, 2\pi)\,,$$

and as the measure of the set $[0, 2\pi) \setminus H_n$ is $2n^{-1/2}$, we have

$$\frac{-6\log n}{\sqrt{n}} \le \int_{[0,2\pi)\setminus H_n} \operatorname*{mean}_{f \in \mathfrak{L}_n} \log\left(\frac{|\hat f(z)|^2}{n}\right) d\theta \le \frac{2\log n}{\sqrt{n}}\,.$$

This yields

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{[0,2\pi)\setminus H_n} \operatorname*{mean}_{f\in\mathfrak{L}_n} \log\left(\frac{|\hat{f}(z)|^2}{n}\right)\, d\theta = 0\,,$$

which finishes the proof of (1.3).

To prove (1.4), let $\varepsilon > 0$ be fixed and suppose $n \geq \varepsilon^{-2}$. We first observe that if $f \in \mathfrak{L}_n \setminus A_n(\varepsilon)$, then $\min_{|z|=1}|f(z)| \geq \varepsilon n^{-1/2}$. Hence $|f(z)| \geq \varepsilon n^{-1/2} \geq n^{-1}$ whenever $|z| = 1$. It then follows that $\hat{f}(z) = \max\{|f(z)|, n^{-1}\} = |f(z)|$ for all $z \in \mathbb{C}$ with $|z| = 1$. Therefore, if $f \in \mathfrak{L}_n \setminus A_n(\varepsilon)$, then $\hat{f}(z) \equiv |f(z)|$ for $z \in \mathbb{C}$ with $|z| = 1$. Thus

$$\operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{M(f)}{\sqrt{n}} \quad = \quad \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{M(\hat{f})}{\sqrt{n}} + \frac{1}{2^n} \sum_{f\in A_n(\varepsilon)} \frac{M(f) - M(\hat{f})}{\sqrt{n}}\,. \tag{2.19}$$

Combining (2.18) with the inequalities

$$1 \leq M(f) \leq M(\hat{f}) \leq \|\hat{f}\|_2 \leq \|f\|_2 + \|1/n\|_2 = n^{1/2} + n^{-1/2} \leq 2n^{1/2} \tag{2.20}$$

valid for all Littlewood polynomial $f \in \mathfrak{L}_n$, we obtain

$$\limsup_{n\to\infty} \left| \frac{1}{2^n} \sum_{f\in A_n(\varepsilon)} \frac{M(f) - M(\hat{f})}{\sqrt{n}} \right| \leq 2 \limsup_{n\to\infty} \frac{1}{2^n} |A_n(\varepsilon)| \leq 2c_6\varepsilon \tag{2.21}$$

Using the inequality between the geometric and arithmetic means (or Jensen's inequality) and (1.3) we obtain

$$\liminf_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{M(\hat{f})}{\sqrt{n}} \geq \liminf_{n\to\infty} \exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n} \log\left(\frac{M(\hat{f})}{\sqrt{n}}\right)\right) = e^{-\gamma/2}. \tag{2.22}$$

Observe that for $p \in (0,1)$, we have

$$1 \leq (M(f))^p \leq \|f\|_p^p \leq \|\hat{f}\|_p^p \leq \||f| + 1/n\|_p^p \leq \|f\|_p^p + n^{-p}$$

and this gives $1 \leq \|\hat{f}\|_p / \|f\|_p \leq (1 + n^{-p})^{1/p} \leq (1 + n^{-p} + n^{-1})^{1+1/p}$. For $p > 1$,

$$1 \leq M(f) \leq \|f\|_p \leq \|\hat{f}\|_p \leq \||f| + 1/n\|_p \leq \|f\|_p + n^{-1}$$

and this gives $1 \leq \|\hat{f}\|_p / \|f\|_p \leq 1 + n^{-1} \leq (1 + n^{-p} + n^{-1})^{1+1/p}$. Hence, we have

$$1 \leq \|\hat{f}\|_p / \|f\|_p \leq (1 + n^{-p} + n^{-1})^{1+1/p} \tag{2.23}$$

hold for all Littlewood polynomials $f \in \mathfrak{L}_n$. In [2], it was shown that

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{\|f\|_p}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}\,.$$

Combining this with (2.17), we also have

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{\|\hat{f}\|_p}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}\,.$$

Therefore,

$$\limsup_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{M(\hat{f})}{\sqrt{n}} \leq \limsup_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{\|\hat{f}\|_p}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}\,.$$

As this holds for all $p > 0$, and

$$\lim_{p\to 0^+} \Gamma(1+p/2)^{1/p} = \exp\left(\frac{1}{2}\frac{\Gamma'(1)}{\Gamma(1)}\right) = e^{-\gamma/2}\,,$$

we have

$$\limsup_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{M(\hat{f})}{\sqrt{n}} \le e^{-\gamma/2} \tag{2.24}$$

Combining (2.22) and (2.24) gives

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \frac{M(\hat{f})}{\sqrt{n}} = e^{-\gamma/2}\,. \tag{2.25}$$

Combining (2.19), (2.21), and (2.25) gives (1.4).

Formula (1.6) has already been proved as Theorem 1 in [2].

To prove (1.5), we first recall that in [2], it has been proved that $\|f\|_p/\sqrt{n}$ converges to $\Gamma(1+p/2)^{1/p}$ in probability. Hence for any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{2^n}\left|\left\{f \in \mathfrak{L}_n : \left|\frac{\|f\|_p}{\sqrt{n}} - \Gamma(1+p/2)^{1/p}\right| > \varepsilon\right\}\right| = 0.$$

Let $h_p := \Gamma(1+p/2)^{1/p}$ $p > 0$, and $h_0 := e^{-\gamma/2}$. Let $0 < \varepsilon < h_0/20$. We define

$$B_{n,\varepsilon} = \left\{f \in \mathfrak{L}_n : \left|\left\|\frac{\hat{f}}{\sqrt{n}}\right\|_p - h_p\right| < \varepsilon,\ \left|\frac{M(\hat{f})}{\sqrt{n}} - h_0\right| < \varepsilon\right\}.$$

Using the convergence in probability stated in Theorem 1.5 (that we prove later in this paper) and (2.23), we get

$$1 - \varepsilon < m_{n,\varepsilon} := \frac{|B_{n,\varepsilon}|}{2^n} \le 1 \tag{2.26}$$

for all sufficiently large n. Therefore

$$\begin{aligned}
&\left(\prod_{f\in\mathfrak{L}_n} \|\hat{f}/\sqrt{n}\|_p\right)^{1/2^n} \\
= \ &\left(\prod_{f\in B_{n,\varepsilon}} \|\hat{f}/\sqrt{n}\|_p\right)^{1/2^n} \left(\prod_{f\in\mathfrak{L}_n\setminus B_{n,\varepsilon}} \|\hat{f}/\sqrt{n}\|_p\right)^{1/2^n} \\
\ge \ &(h_p - \varepsilon)^{m_{n,\varepsilon}} \left(\prod_{f\in\mathfrak{L}_n\setminus B_{n,\varepsilon}} \|\hat{f}/\sqrt{n}\|_p\right)^{1/2^n} \\
\ge \ &\min\{h_p - \varepsilon, (h_p-\varepsilon)^{1-\varepsilon}\}\left(\prod_{f\in\mathfrak{L}_n\setminus B_{n,\varepsilon}} \|\hat{f}/\sqrt{n}\|_p\right)^{1/2^n} \\
\ge \ &\min\{h_p - \varepsilon, (h_p-\varepsilon)^{1-\varepsilon}\}\left(\prod_{f\in\mathfrak{L}_n\setminus B_{n,\varepsilon}} (M(\hat{f})/\sqrt{n})\right)^{1/2^n}.
\end{aligned} \tag{2.27}$$

Now we can use (2.26) and the convergence of the geometric means of the Mahler's measure of $\hat{f}$ of Littlewood polynomials to h_0 to estimate the second factor in this lower bound as

$$\begin{aligned}\left(\prod_{f\in\mathfrak{L}_n\setminus B_{n,\varepsilon}} (M(\hat{f})/\sqrt{n})\right)^{1/2^n} &= \frac{\left(\prod_{f\in\mathfrak{L}_n} (M(\hat{f})/\sqrt{n})\right)^{1/2^n}}{\left(\prod_{f\in B_{n,\varepsilon}} (M(\hat{f})/\sqrt{n})\right)^{1/2^n}} \\ &\geq \frac{h_0-\varepsilon}{(h_0+\varepsilon)^{m_{n,\varepsilon}}} \geq \frac{h_0-\varepsilon}{(h_0+\varepsilon)^{1-\varepsilon}} \qquad (2.28)\end{aligned}$$

for all sufficiently large n. Combining (2.27) and (2.28) we obtain

$$\left(\prod_{f\in\mathfrak{L}_n} \|\hat{f}/\sqrt{n}\|_p\right)^{1/2^n} \geq \min\{h_p-\varepsilon,(h_p-\varepsilon)^{1-\varepsilon}\}\frac{h_0-\varepsilon}{(h_0+\varepsilon)^{1-\varepsilon}} \qquad (2.29)$$

for all sufficiently large n. On the other hand, using the inequality between the geometric and arithmetic means and (2.23), and then recalling (1.6), we obtain

$$\begin{aligned}\lim_{n\to\infty}\left(\prod_{f\in\mathfrak{L}_n} \|\hat{f}/\sqrt{n}\|_p\right)^{1/2^n} &\leq \lim_{n\to\infty}\frac{1}{2^n}\left(\sum_{f\in\mathfrak{L}_n}\|\hat{f}/\sqrt{n}\|_p\right) \\ &= \lim_{n\to\infty}\frac{1}{2^n}\left(\sum_{f\in\mathfrak{L}_n}\|f/\sqrt{n}\|_p\right) \\ &\leq h_p+\varepsilon\,, \qquad (2.30)\end{aligned}$$

Combining (2.29) and (2.30) and letting $n\to\infty$ and then $\varepsilon\to 0^+$, we get

$$\lim_{n\to\infty}\left(\prod_{f\in\mathfrak{L}_n} \|\hat{f}/\sqrt{n}\|_p\right)^{1/2^n} = \lim_{n\to\infty}\exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n}\log\left(\frac{\|\hat{f}\|_p}{\sqrt{n}}\right)\right) = h_p\,.$$

This proves (1.5) and completes the proof of Theorem 1.2. □

Proof of Corollary 1.3. Let

$$A_n(\varepsilon) = \left\{f\in\mathfrak{L}_n : \min_{|z|=1}|f(z)| < \varepsilon n^{-1/2}\right\}$$

be the same as in Lemma 2.4. Observe that if $f\in\mathfrak{L}_n\setminus A_n(\varepsilon)$ and $n\geq\varepsilon^{-2}$, then $|f(z)| = \hat{f}(z)$ for all $z\in\mathbb{C}$ with $|z|=1$. Hence,

$$\begin{aligned}\operatorname*{mean}_{f\in\mathfrak{L}_n}\log\left(\frac{M(f)}{\sqrt{n}}\right) &= \frac{1}{2^n}\sum_{f\in\mathfrak{L}_n\setminus A_n(\varepsilon)}\log\left(\frac{M(f)}{\sqrt{n}}\right) + \frac{1}{2^n}\sum_{f\in A_n(\varepsilon)}\log\left(\frac{M(f)}{\sqrt{n}}\right) \\ &= \frac{1}{2^n}\sum_{f\in\mathfrak{L}_n\setminus A_n(\varepsilon)}\log\left(\frac{M(\hat{f})}{\sqrt{n}}\right) + \frac{1}{2^n}\sum_{f\in A_n(\varepsilon)}\log\left(\frac{M(f)}{\sqrt{n}}\right) \\ &= \operatorname*{mean}_{f\in\mathfrak{L}_n}\log\left(\frac{M(\hat{f})}{\sqrt{n}}\right) + \frac{1}{2^n}\sum_{f\in A_n(\varepsilon)}\log\left(\frac{M(f)}{M(\hat{f})}\right).\end{aligned}$$

Recalling (2.20), we obtain

$$1\geq\exp\left(\frac{1}{2^n}\sum_{f\in A_n(\varepsilon)}\log\left(\frac{M(f)}{M(\hat{f})}\right)\right)\geq\exp\left(-\frac{1}{2^n}|A_n(\varepsilon)|\log(2\sqrt{n})\right).$$

Combining this with (1.4) and Lemma 2.4, we obtain

$$e^{-\gamma/2} \geq \limsup_{n\to\infty} \exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n} \log\left(\frac{M(f)}{\sqrt{n}}\right)\right)$$

and

$$\liminf_{n\to\infty} \exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n} \log\left(\frac{M(f)}{n^{1/2-c_6\varepsilon}}\right)\right) \geq e^{-\gamma/2}.$$

□

Proof of Theorem 1.5. Result (1.9) has been proved in Theorem 1 of [2].

To prove (1.8), we need a quantitative form of the inequality between the geometric and arithmetic means. Let s_k be the arithmetic mean and g_k be the geometric mean of the positive numbers $a_1, a_2, \ldots, a_k$. Let

$$m = m(\varepsilon) = \left|\{j \in \{1, 2, \ldots, k\} : |a_j/s_k - 1| > \varepsilon\}\right|.$$

We claim that for every $0 < \varepsilon < 1$, we have

$$\left(\frac{e^{\varepsilon}}{1+\varepsilon}\right)^{m/k} \leq s_k/g_k.$$

Indeed, we first assume $s_k = 1$. Following Pólya's proof of the inequality between the geometric and arithmetic means, we have $e^{a_j-1}/a_j \geq 1$ for all $j = 1, 2, \ldots, k$. But when $|a_j - 1| > \varepsilon$, we can gain a non-trivial factor > 1. Consider the function $f(x) := e^{x-1}/x$ for $x > 0$. The function $f(x)$ is decreasing on $(0, 1]$ and increasing on $[1, \infty)$, and takes its absolute minimum value 1 on $(0, \infty)$ at 1. So if $|x - 1| > \varepsilon$, then $x > 1 + \varepsilon$ or $x < 1 - \varepsilon$, and hence

$$\frac{e^{x-1}}{x} \geq \min\left\{\frac{e^{\varepsilon}}{1+\varepsilon}, \frac{e^{-\varepsilon}}{1-\varepsilon}\right\} = \frac{e^{\varepsilon}}{1+\varepsilon} > 1\,.$$

Therefore $\sum_{j=1}^{k} a_j = s_k n = n$ implies

$$\frac{1}{a_1 a_2 \cdots a_k} = \frac{e^{\sum_{j=1}^{k} a_j - n}}{a_1 a_2 \cdots a_k} = \prod_{j=1}^{k} \frac{e^{a_j-1}}{a_j} \geq \left(\frac{e^{\varepsilon}}{1+\varepsilon}\right)^{m}.$$

By taking the kth root of both sides and recalling that $s_k = 1$, we have

$$s_k/g_k \geq \left(\frac{e^{\varepsilon}}{1+\varepsilon}\right)^{m/k}.$$

For the case that $s_k \neq 1$, we replace a_j by $a_j' = a_j/s_k$ so that $s_k' = \frac{1}{k}\sum_{j=1}^{k} a_j/s_k = 1$, then from above, we have

$$\frac{s_k^k}{a_1 a_2 \cdots a_k} = \frac{1}{a_1' a_2' \cdots a_k'} = \prod_{j=1}^{k} \frac{e^{a_j'-1}}{a_j'} \geq \left(\frac{e^{\varepsilon}}{1+\varepsilon}\right)^{m}.$$

This also gives

$$s_k/g_k \geq \left(\frac{e^{\varepsilon}}{1+\varepsilon}\right)^{m/k}.$$

This proves the claim.

Applying the claim to our case, we have $a_j = M(\hat{f})/\sqrt{n}$ with $f \in \mathfrak{L}_n$, $k = 2^n$, $s_k = \operatorname*{mean}_{f\in\mathfrak{L}_n}\left(M(\hat{f})/\sqrt{n}\right)$ and $g_k = \exp\left(\operatorname*{mean}_{f\in\mathfrak{L}_n}\log\left(M(\hat{f})/\sqrt{n}\right)\right)$. Note that $\lim_{n\to\infty}\log(s_k/g_k) = 0$ from Theorem 1.2 and (2.25). Therefore, for any $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{2^n}\left|\left\{f \in \mathfrak{L}_n : \left|\frac{M(\hat{f})}{\sqrt{n}} - e^{-\gamma/2}\right| > \varepsilon\right\}\right| = 0.$$

This completes the proof. □

Proof of Theorem 1.6. Let $p > 0$ and $\alpha > 0$. Let $h_p := \Gamma(1+p/2)^{1/p}$ as before. In view of (1.9) in Theorem 1.5, $\|f\|_p/\sqrt{n}$ converges to h_p in probability and hence $(\|f\|_p/\sqrt{n})^\alpha$ also converges to h_p^α in probability. We let

$$A_{n,\varepsilon}(\alpha) := \{f \in \mathfrak{L}_n : \left|\left(\|f\|_p/\sqrt{n}\right)^\alpha - h_p^\alpha\right| < \varepsilon\}$$

so that for any $\varepsilon > 0$, we have

$$\lim_{n\to\infty}\frac{|A_{n,\varepsilon}(\alpha)|}{2^n} = 1.$$

We now consider

$$\begin{aligned} &\left|\operatorname*{mean}_{f\in\mathfrak{L}_n}\left(\left(\frac{\|f\|_p}{\sqrt{n}}\right)^\alpha\right) - h_p^\alpha\right| \le \operatorname*{mean}_{f\in\mathfrak{L}_n}\left(\left|\left(\frac{\|f\|_p}{\sqrt{n}}\right)^\alpha - h_p^\alpha\right|\right) \\ = &\ \frac{1}{2^n}\sum_{f\in A_{n,\varepsilon}(\alpha)}\left|\left(\frac{\|f\|_p}{\sqrt{n}}\right)^\alpha - h_p^\alpha\right| + \frac{1}{2^n}\sum_{f\in\mathfrak{L}_n\setminus A_{n,\varepsilon}(\alpha)}\left|\left(\frac{\|f\|_p}{\sqrt{n}}\right)^\alpha - h_p^\alpha\right|. \end{aligned} \tag{2.31}$$

The first term in (2.31) is

$$\frac{1}{2^n}\sum_{f\in A_{n,\varepsilon}(\alpha)}\left|\left(\frac{\|f\|_p}{\sqrt{n}}\right)^\alpha - h_p^\alpha\right| \le \varepsilon\frac{|A_{n,\varepsilon}(\alpha)|}{2^n} \to \varepsilon$$

as $n \to \infty$. In view of [2], we know that

$$\lim_{n\to\infty}\operatorname*{mean}_{f\in\mathfrak{L}_n}\left(\frac{\|f\|_p}{\sqrt{n}}\right)^{2p} = \Gamma(1+p/2)^2, \tag{2.32}$$

and using a similar proof, one can show that

$$\lim_{n\to\infty}\operatorname*{mean}_{f\in\mathfrak{L}_n}\left(\frac{\|f\|_p}{\sqrt{n}}\right)^{mp} = \Gamma(1+p/2)^m,$$

for any integer $m \ge 1$ by considering

$$g_n^{(*)}(\theta_1,\theta_2,\cdots,\theta_m) = \operatorname*{mean}_{f\in\mathfrak{L}_n}\left(\left|\frac{f(e^{i\theta_1})}{\sqrt{n}}\,\frac{f(e^{i\theta_2})}{\sqrt{n}}\cdots\frac{f(e^{i\theta_m})}{\sqrt{n}}\right|^p\right)$$

and showing that $g_n^{(*)}(\theta_1,\theta_2,\cdots,\theta_m)$ converges almost everywhere to $\Gamma(1+p/2)^m$ on $[0,2\pi)^m$ by using Lindeberg's central limit theorem as in the proof of (2.32) in [2]. Then for $p \ge 2$, by Hölder's inequality, the second term in (2.31) is at most

$$\begin{aligned} &\left(1 - \frac{|A_{n,\varepsilon}(\alpha)|}{2^n}\right)^{1-\frac{1}{p}}\left(\operatorname*{mean}_{f\in\mathfrak{L}_n}\left(\left(\frac{\|f\|_p}{\sqrt{n}}\right)^{p\alpha}\right)\right)^{\frac{1}{p}} + \left(1 - \frac{|A_{n,\varepsilon}(\alpha)|}{2^n}\right)h_p^\alpha \\ \le &\left(1 - \frac{|A_{n,\varepsilon}(\alpha)|}{2^n}\right)^{1-\frac{1}{p}}\left(\operatorname*{mean}_{f\in\mathfrak{L}_n}\left(\left(\frac{\|f\|_p}{\sqrt{n}}\right)^{p([\alpha]+1)}\right)\right)^{\frac{1}{p}} + \left(1 - \frac{|A_{n,\varepsilon}(\alpha)|}{2^n}\right)h_p^\alpha, \end{aligned}$$

where the right hand side tends to 0 as $n \to \infty$. For $0 < p < 2$, then $\|f\|_p \leq \|f\|_2 = \sqrt{n}$ for all $f \in \mathfrak{L}_n$. Hence the second term in (2.31) is also

$$\leq \left(1 - \frac{|A_{n,\varepsilon}(\alpha)|}{2^n}\right) + \left(1 - \frac{|A_{n,\varepsilon}(\alpha)|}{2^n}\right) h_p^\alpha$$

where the right hand side tends to 0 as $n \to \infty$. Therefore, we have

$$\limsup_{n\to\infty} \left| \operatorname*{mean}_{f\in\mathfrak{L}_n} \left(\left(\frac{\|f\|_p}{\sqrt{n}} \right)^\alpha \right) - h_p^\alpha \right| \leq \varepsilon \,,$$

and hence

$$\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \left(\left(\frac{\|f\|_p}{\sqrt{n}} \right)^\alpha \right) = h_p^\alpha = \left(\lim_{n\to\infty} \operatorname*{mean}_{f\in\mathfrak{L}_n} \left(\frac{\|f\|_p}{\sqrt{n}} \right) \right)^\alpha \,.$$

The case of the Mahler's measure can be proved in the same way. Now we can use (1.8) rather than (1.9) of Theorem 1.5 and observe that $M(\hat{f})/\sqrt{n} \leq \|\hat{f}\|_2/\sqrt{n} \leq 1 + 1/n^{3/2}$ for all $f \in \mathfrak{L}_n$. □

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