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Diophantine Approximation in Projective Space

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1. Introduction

Let k be an algebraic number field and k_v the completion of k at the place v. If α belongs to k_v then Dirichlet's Theorem establishes the existence of a point β in k such that the height of β is bounded by a suitable parameter and $|\alpha - \beta|_v$ is relatively small. And for special numbers α it is a basic problem of Diophantine approximation to show that $|\alpha - \beta|_v$ cannot be too small if the height of β is bounded. In a recent paper [**2**] such problems were reformulated in projective space over k_v by replacing the flat metric determined by $|\quad|_v$ with a projective metric δ_v. Our purpose here is to give a proof of the projective form of Dirichlet's Theorem and to prove a useful inequality for the projective metric. We also discuss some open problems suggested by these results.

At each place v of k we use two absolute values $|\quad|_v$ and $\|\quad\|_v$ which are determined as in [**1**], [**2**], or [**3**]. Thus we have $|x|_v = \|x\|_v^{d_v/d}$ for all x in k_v, where $d = [k : \mathbb{Q}]$ and $d_v = [k_v : \mathbb{Q}_v]$. These absolute values have unique extensions to Ω_v, the completion of an algebraic closure of k_v. We extend $|\quad|_v$ to a norm on finite dimensional vector spaces over Ω_v as follows. If

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{pmatrix}$$

is a column vector in Ω_v^N we write

$$\|\mathbf{x}\|_v = \begin{cases} \max\{\|x_n\|_v : 1 \le n \le N\} \text{ if } v \nmid \infty \\ \left\{\sum_{n=1}^{N} \|x_n\|_v^2\right\}^{1/2} \text{ if } v \mid \infty \end{cases}, \tag{1.1}$$

and $|\mathbf{x}|_v = \|\mathbf{x}\|_v^{d_v/d}$ in both cases. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ denote the standard basis vectors in Ω_v^N and for each subset $I \subseteq \{1, 2, \ldots, N\}$ let $\mathbf{e}_I$ be the corresponding

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standard basis vector in the exterior algebra

$$\bigwedge(\Omega_v^N) = \sum_{n=0}^{\infty} \bigwedge\nolimits_n(\Omega_v^N) \ .$$

We identify Ω_v^N with the subspace $\bigwedge_1(\Omega_v^N)$ so that

$$\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_n}$$

whenever $I = \{i_1 < i_2 < \cdots < i_n\} \subseteq \{1, 2, \ldots, N\}$ is not empty. Then we extend $|\quad|_v$ and $\|\quad\|_v$ to $\bigwedge(\Omega_v^N)$ by applying (1.1) to the basis $\{\mathbf{e}_I : I \subseteq \{1, 2, \ldots, N\}\}$.

If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in Ω_v^N we set

$$\delta_v(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{x} \wedge \mathbf{y}|_v}{|\mathbf{x}|_v |\mathbf{y}|_v} \ . \tag{1.2}$$

As $\delta_v(a\mathbf{x}, b\mathbf{y}) = \delta_v(\mathbf{x}, \mathbf{y})$ for all $a \neq 0$ and $b \neq 0$ in Ω_v, it is clear that δ_v is well defined as a map

$$\delta_v : \mathbb{P}^{N-1}(\Omega_v) \times \mathbb{P}^{N-1}(\Omega_v) \to [0, 1] \ .$$

It can be shown, as in Rumely [**8**], that δ_v is a metric on $\mathbb{P}^{N-1}(\Omega_v)$ and the resulting metric topology coincides with the quotient topology determined by the norm $|\quad|_v$ on Ω_v^N.

We define an absolute height on points $\boldsymbol{\beta}$ in $\mathbb{P}^{N-1}(k)$ by

$$H(\boldsymbol{\beta}) = \prod_w |\boldsymbol{\beta}|_w \ ,$$

where the product is taken over all places w of k. It is obvious from the product formula that this height is well defined on $\mathbb{P}^{N-1}(k)$. Now suppose that $\boldsymbol{\alpha}$ belongs to $\mathbb{P}^{N-1}(k_v)$ for some place v of k. Then we may try to establish the existence of a point $\boldsymbol{\beta}$ in $\mathbb{P}^{N-1}(k)$ such that $H(\boldsymbol{\beta})$ is bounded by a suitable parameter and the projective distance $\delta_v(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is relatively small. In order to state such a result we let

$$c_k(N) = 2|\Delta_k|^{1/2d} \prod_{w|\infty} r_w(N)^{d_w/d} \ ,$$

where Δ_k is the discriminant of k and

$$r_w(N) = \begin{cases} \pi^{-1/2}\Big\{\Gamma\big(\tfrac{1}{2}N + 1\big)\Big\}^{1/N} & \text{if } w \text{ is real} \\ (2\pi)^{-1/2}\big\{\Gamma(N+1)\big\}^{1/2N} & \text{if } w \text{ is complex.} \end{cases} \tag{1.3}$$

Then Dirichlet's Theorem for $\mathbb{P}^{N-1}(k_v)$ can be formulated as follows.

THEOREM 1. *Let $\boldsymbol{\alpha}$ belong to $\mathbb{P}^{N-1}(k_v)$, let τ belong to k_v with $1 \leq |\tau|_v$. Then there exists $\boldsymbol{\beta}$ in $\mathbb{P}^{N-1}(k)$ such that*
(i) $H(\boldsymbol{\beta}) \leq c_k(N)|\tau|_v^{N-1}$,
(ii) $\delta_v(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq c_k(N)\{|\tau|_v H(\boldsymbol{\beta})\}^{-1}$.

If $\boldsymbol{\alpha}$ belongs to $\mathbb{P}^{N-1}(k_v)$ but not to $\mathbb{P}^{N-1}(k)$ then it follows from (i) and (ii) that there exist infinitely many distinct $\boldsymbol{\beta}$ in $\mathbb{P}^{N-1}(k)$ such that

$$\delta_v(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq c_k(N)^{N/(N-1)} H(\boldsymbol{\beta})^{-N/(N-1)} \ . \tag{1.4}$$

Alternatively, if $\boldsymbol{\alpha}$ belongs to $\mathbb{P}^{N-1}(k_v)$ but not to $\mathbb{P}^{N-1}(k)$ we define

$$\nu_v(\boldsymbol{\alpha}) = \liminf_{H(\boldsymbol{\beta})\to\infty} H(\boldsymbol{\beta})^{N/(N-1)}\delta_v(\boldsymbol{\alpha},\boldsymbol{\beta}) \ . \tag{1.5}$$

Then we have $\nu_v(\boldsymbol{\alpha}) \le c_k(N)^{N/(N-1)}$.

In the special case $k = \mathbb{Q}$, $k_v = \mathbb{Q}_\infty = \mathbb{R}$ and $N = 2$, we can write $\boldsymbol{\alpha} = \left[\begin{smallmatrix}\alpha\\1\end{smallmatrix}\right]$ in homogeneous coordinates with α an irrational real number. In this case it can be shown that

$$\nu_\infty(\boldsymbol{\alpha}) = \liminf_{q\to\infty} q\|\alpha\, q\| \ , \tag{1.6}$$

where $\|x\|$ is the distance from the real number x to the nearest integer. It follows that the set of values

$$\big\{\nu_\infty(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathbb{P}^1(\mathbb{R}) \setminus \mathbb{P}^1(\mathbb{Q})\big\}$$

is the *Lagrange spectrum*, as considered by Cusick [**5**] or (with a slightly different definition) by Cusick and Flahive [**6**]. By a well known result of Hurwitz [**7**] (see also Cassels [**4**] or Schmidt [**9**]) the largest point in the Lagrange spectrum is $5^{-1/2}$, which improves on the bound $\nu_\infty(\boldsymbol{\alpha}) \le C_{\mathbb{Q}}(2)^2 = 4/\pi$.

In view of these remarks the set

$$\big\{\nu_v(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathbb{P}^1(k_v) \setminus \mathbb{P}^1(k)\big\} \tag{1.7}$$

may be regarded as a generalization of the Lagrange spectrum to the completion k_v of an algebraic number field k. It follows from the Liouville inequality, given as (2.12) of [**2**], that the value of $\nu_v(\boldsymbol{\alpha})$ is positive whenever $\boldsymbol{\alpha} = \left[\begin{smallmatrix}\alpha\\1\end{smallmatrix}\right]$ and α in k_v is algebraic over k of degree 2. In general it is an open problem to give a sharp upper bound for the elements of the set (1.7) which is analogous to Hurwitz's bound. A somewhat related problem is to give an analogue of the continued fraction algorithm for an arbitrary point $\boldsymbol{\alpha}$ in $\mathbb{P}^1(k_v) \setminus \mathbb{P}^1(k)$. Evidently such an algorithm should generate a sequence of distinct points $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots$ in $\mathbb{P}^1(k)$ which are "best approximations" to $\boldsymbol{\alpha}$ with respect the height H and the projective metric δ_v. Such a sequence of best approximations should provide a generalization to $\mathbb{P}^1(k_v)$ of the well known theorem of Lagrange (see Schmidt [**9**], Chapter 1, Theorem 5E), which characterizes the convergents in the continued fraction expansion of an irrational real number.

The statement of Theorem 1 can be generalized in several ways. Let S be a finite, nonempty set of places of k. Then at each place v in S let $\mathfrak{X}_v \subseteq (k_v)^N$ be a linear subspace of dimension L_v, $1 \le L_v < N$.

Theorem 2. *At each place v in S let τ_v belong to k_v with $1 \le |\tau_v|_v$. Then there exist linearly independent points $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_N$ in $\mathbb{P}^{N-1}(k)$ such that*

(i) $$\prod_{n=1}^{M} H(\boldsymbol{\beta}_n) \le \Big\{c_k(N)\prod_{v\in S}|\tau_v|_v^{N-L_v}\Big\}^M ,$$

(ii) $$\prod_{n=1}^{M}\prod_{v\in S}\Big(\min\{\delta_v(\mathbf{x},\boldsymbol{\beta}_n) : \mathbf{x}\in\mathfrak{X}_v, \mathbf{x}\neq\mathbf{0}\}\Big) \le \Big\{c_k(N)\prod_{v\in S}|\tau_v|_v^{-L_v}\Big\}^M \prod_{n=1}^{M} H(\boldsymbol{\beta}_n)^{-1} ,$$

for each M, $1 \le M \le N$.

If S consists of one place v, if $\mathfrak{X}_v \subseteq (k_v)^N$ has dimension $L_v = 1$ and is spanned by the nonzero vector $\boldsymbol{\alpha}$, and $M = 1$, then Theorem 2 plainly reduces to

Theorem 1. It is an open problem to give an analogue of Theorem 2 in which the linear subspaces $\mathfrak{X}_v$ are replaced by more general projective varieties.

Let A be an $N \times M$ matrix over Ω_v. We extend $|\ \ |_v$ to such matrices A by setting

$$|A|_v = \sup\{|A\mathbf{x}|_v : \mathbf{x} \in \Omega_v^M\ ,\ |\mathbf{x}|_v \le 1\} = \sup\left\{\frac{|A\mathbf{x}|_v}{|\mathbf{x}|_v} : \mathbf{x} \in \mathbb{P}^{M-1}(\Omega_v)\right\}\ . \tag{1.8}$$

Now suppose that

$$2 \le M = \text{rank } A \le N \tag{1.9}$$

and define

$$\eta_v(A) = \sup\left\{\frac{|A\mathbf{x}|_v|\mathbf{y}|_v}{|A\mathbf{y}|_v|\mathbf{x}|_v} : \mathbf{x} \in \mathbb{P}^{M-1}(\Omega_v)\ ,\ \mathbf{y} \in \mathbb{P}^{M-1}(\Omega_v)\right\}\ . \tag{1.10}$$

Clearly we have $1 \le \eta_v(A)$. In work with the projective metric δ_v it is often useful to have an inequality between $\delta_v(A\mathbf{x}, A\mathbf{y})$ and $\delta_v(\mathbf{x}, \mathbf{y})$. In section 3 we give a proof of the following result.

Theorem 3. *Let A be an $N \times M$ matrix over Ω_v satisfying* (1.9). *Then for each $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}^{M-1}(\Omega_v)$ we have*

$$\eta_v(A)^{-1}\delta_v(\mathbf{x}, \mathbf{y}) \le \delta_v(A\mathbf{x}, A\mathbf{y}) \le \eta_v(A)\delta_v(\mathbf{x}, \mathbf{y})\ . \tag{1.11}$$

Moreover, both inequalities in (1.11) *are sharp in the sense that $\eta_v(A)$ cannot be replaced by a smaller number.*

If $M = N$ then A is nonsingular and we have $\eta_v(A) = |A|_v|A^{-1}|_v$. In this case it is clear that η_v is well defined as a map

$$\eta_v : PGL(N, \Omega_v) \to [1, \infty)\ .$$

Because the inequality (1.11) is sharp, A in $PGL(N, \Omega_v)$ acts as an isometry for the projective metric if and only if $\eta_v(A) = 1$.

If A belongs to $PGL(2, k)$ and $\eta_w(A) = 1$ at all places $w \ne v$ of k, then S. Tyler [**10**] has shown that $\nu_v(A\boldsymbol{\alpha}) = \nu_v(\boldsymbol{\alpha})$ for all points $\boldsymbol{\alpha}$ in $\mathbb{P}^1(k_v)$ but not in $\mathbb{P}^1(k)$. This generalizes a well known result (see Cassels [**4**], Chapter 1, section 3, Corollary) from the case $k = \mathbb{Q}$ and $k_v = \mathbb{Q}_\infty = \mathbb{R}$.

2. Proof of Theorem 2

We will make use of results from the geometry of numbers over the product of adele spaces $(k_{\mathbb{A}})^N$ and orthogonality in local fields. These subjects are developed in [**3**], sections 3 and 4.

If $v \mid \infty$ we let γ_v denote Haar measure on the Borel subsets of k_v normalized so that γ_v is Lebesgue measure if v is real and twice Lebesgue measure if v is complex. It follows that the product measure γ_v^N on $(k_v)^N$ satisfies

$$\gamma_v^N\{\mathbf{x} \in (k_v)^N : \|\mathbf{x}\|_v < r_v(N)\} = 1\ ,$$

where $r_v(N)$ is given by (1.3). If $v \nmid \infty$ we let γ_v denote Haar measure on the Borel subsets of k_v normalized so that

$$\gamma_v\{x \in k_v : \|x\|_v \le 1\} = |\mathcal{D}_v|_v^{d/2}\ ,$$

where $\mathcal{D}_v$ is the local different at v. We note that

$$\prod_{v\nmid\infty} |\mathcal{D}_v|_v^{d/2} = |\Delta_k|^{-1/2} \tag{2.1}$$

where Δ_k is the discriminant of k, and

$$\gamma_v^N\{\mathbf{x} \in (k_v)^N : \|\mathbf{x}\|_v \le 1\} = |\mathcal{D}_v|_v^{Nd/2} .$$

At each place v in S let

$$X_v = (\mathbf{x}_1^{(v)} \quad \mathbf{x}_2^{(v)} \quad \cdots \quad \mathbf{x}_N^{(v)})$$

be an $N \times N$ matrix having entries in k_v, orthogonal columns, and such that

$$\mathfrak{X}_v = \operatorname{span}_{k_v}\{\mathbf{x}_1^{(v)}, \dots, \mathbf{x}_{L_v}^{(v)}\} .$$

Because the columns of X_v are orthogonal we have

$$|\mathbf{x}_{i_1}^{(v)} \wedge \mathbf{x}_{i_2}^{(v)} \wedge \cdots \wedge \mathbf{x}_{i_M}^{(v)}|_v = \prod_{m=1}^{M} |\mathbf{x}_{i_m}^{(v)}|_v$$

for each subset $I = \{i_1 < i_2 < \cdots < i_M\} \subseteq \{1, 2, \dots, N\}$. Also, we may plainly select X_v so that $|\mathbf{x}_n^{(v)}|_v = 1$ for each $n = 1, 2, \dots, N$. Next we define the $N \times N$ matrix

$$Y_v = (\tau_v^{N-L_v}\mathbf{x}_1^{(v)} \quad \cdots \quad \tau_v^{N-L_v}\mathbf{x}_{L_v}^{(v)} \quad \tau_v^{-L_v}\mathbf{x}_{L_v+1}^{(v)} \quad \cdots \quad \tau_v^{-L_v}\mathbf{x}_N^{(v)}) ,$$

so that $X_v^{-1}Y_v$ is an $N \times N$ diagonal matrix with $\det\{X_v^{-1}Y_v\} = 1$.

At each place v of k we define $R_v \subseteq (k_v)^N$ by

$$R_v = \{\mathbf{x} \in (k_v)^N : \|\mathbf{x}\|_v < r_v(N)\}$$

if $v \mid \infty$, and

$$R_v = \{\mathbf{x} \in (k_v)^N : \|\mathbf{x}\|_v \le 1\}$$

if $v \nmid \infty$. It follows that

$$\mathfrak{A} = \prod_{v\in S}(Y_v R_v) \prod_{w\notin S} R_w \subseteq (k_{\mathbb{A}})^N$$

is an admissible subset of the N-fold product of adele spaces. Also, the Haar measures γ_v^N determine a Haar measure V on $(k_{\mathbb{A}})^N$ and we find that

$$\begin{aligned} V(\mathfrak{A}) &= \prod_{v\in S} \gamma_v^N\{Y_v R_v\} \prod_{w\notin S} \gamma_w^N\{R_w\} \\ &= \prod_{v\in S} \|\det Y_v\|_v^{d_v} \prod_{v} \gamma_v^N\{R_v\} \\ &= |\Delta_k|^{-N/2} . \end{aligned} \tag{2.2}$$

Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N < \infty$ be the successive minima associated with $\mathfrak{A}$ and $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_N\}$ the corresponding set of linearly independent vectors in k^N. Then we have

$$\prod_{v\in S} |Y_v^{-1}\boldsymbol{\beta}_n|_v \prod_{w\notin S} |\boldsymbol{\beta}_n|_w \le \lambda_n \prod_{v\mid\infty} r_v(N)^{d_v/d} \tag{2.3}$$

for each $n = 1, 2, \dots, N$. By the adelic form of Minkowski's second theorem (as in [**1**], Theorem 3),

$$(\lambda_1 \lambda_2 \cdots \lambda_N)^d \, V(\mathfrak{A}) \le 2^{dN} \ . \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) leads to the bound:

$$\begin{aligned}\prod_{n=1}^{M} \Big\{ \prod_{v \in S} |Y_v^{-1} \boldsymbol{\beta}_n|_v \prod_{w \notin S} |\boldsymbol{\beta}_n|_w \Big\} &\le \Big(\prod_{n=1}^{M} \lambda_n \Big) \Big\{ \prod_{v|\infty} r_v(N)^{d_v/d} \Big\}^M \\ &\le \Big(\prod_{n=1}^{N} \lambda_n \Big)^{M/N} \Big\{ \prod_{v|\infty} r_v(N)^{d_v/d} \Big\}^M \\ &\le c_k(N)^M \ .\end{aligned}$$

It will be convenient to rewrite this as

$$\prod_{n=1}^{M} \Big\{ \prod_{v \in S} |Y_v^{-1} \boldsymbol{\beta}_n|_v |\boldsymbol{\beta}_n|_v^{-1} \Big\} \le c_k(N)^M \prod_{n=1}^{M} H(\boldsymbol{\beta}_n)^{-1} \ . \tag{2.5}$$

Now observe that at each place v in S we have

$$\begin{aligned}|\boldsymbol{\beta}_n|_v &= |Y_v Y_v^{-1} \boldsymbol{\beta}_n|_v \\ &\le |Y_v|_v |Y_v^{-1} \boldsymbol{\beta}_n|_v \\ &\le |\tau_v|_v^{N-L_v} |Y_v^{-1} \boldsymbol{\beta}_n|_v \ .\end{aligned} \tag{2.6}$$

Therefore (2.5) and (2.6) imply that

$$\prod_{n=1}^{M} H(\boldsymbol{\beta}_n) \le c_k(N)^M \Big\{ \prod_{v \in S} |\tau_v|_v^{N-L_v} \Big\}^M \ ,$$

which is (i) in the statement of Theorem 2. In order to verify (ii) in the statement of the theorem we will show that

$$\min\big\{ \delta_v(\mathbf{x}, \boldsymbol{\beta}_n) : \mathbf{x} \in \mathfrak{X}_v, \mathbf{x} \ne \mathbf{0} \big\} \le |\tau_v|_v^{-L_v} |Y_v^{-1} \boldsymbol{\beta}_n|_v |\boldsymbol{\beta}_n|_v^{-1} \tag{2.7}$$

for each $n = 1, 2, \dots, N$ and each place v in S. Clearly (ii) follows from (2.5) and (2.7).

To establish (2.7) write

$$\boldsymbol{\beta}_n = Y_v \boldsymbol{\varphi} \ , \qquad \boldsymbol{\varphi} \in (k_v)^N \ .$$

Here $\boldsymbol{\varphi}$ depends on n and v, but these parameters are fixed in the remainder of the proof and it will simplify the notation to suppress them. Then

$$\boldsymbol{\beta}_n = \tau_v^{N-L_v} \sum_{\ell=1}^{L_v} \varphi_\ell \mathbf{x}_\ell^{(v)} + \tau_v^{-L_v} \sum_{n=L_v+1}^{N} \varphi_n \mathbf{x}_n^{(v)}$$

and the point

$$\boldsymbol{\xi} = \sum_{\ell=1}^{L_v} \varphi_\ell \mathbf{x}_\ell^{(v)}$$

occurs in the subspace $\mathfrak{X}_v$. If $\boldsymbol{\xi} = \mathbf{0}$ then the subspace $\mathfrak{X}_v$ is orthogonal to the one dimensional subspace spanned by $\boldsymbol{\beta}_n$. Therefore *every* nonzero vector $\mathbf{x}$ in $\mathfrak{X}_v$ satisfies

$$\begin{aligned}|\mathbf{x}\wedge\boldsymbol{\beta}_n|_v &= |\mathbf{x}|_v|\boldsymbol{\beta}_n|_v\\ &= |\tau_v|_v^{-L_v}|\mathbf{x}|_v|\boldsymbol{\varphi}|_v\\ &= |\tau_v|_v^{-L_v}|\mathbf{x}|_v|Y_v^{-1}\boldsymbol{\beta}_n|_v\ ,\end{aligned}$$

and this shows that (2.7) holds with equality. Suppose then that $\boldsymbol{\xi}\neq\mathbf{0}$. It follows that

$$\boldsymbol{\xi}\wedge\boldsymbol{\beta}_n = \tau_v^{-L_v}\sum_{\ell=1}^{L_v}\sum_{n=L_v+1}^{N}\varphi_\ell\varphi_n(\mathbf{x}_\ell^{(v)}\wedge\mathbf{x}_n^{(v)})\ .$$

As $\{\mathbf{x}_1^{(v)},\mathbf{x}_2^{(v)},\dots,\mathbf{x}_N^{(v)}\}$ forms an orthogonal basis for $(k_v)^N$, it is easy to verify that

$$\{\mathbf{x}_m^{(v)}\wedge\mathbf{x}_n^{(v)} : 1\le m<n\le N\}$$

forms an orthogonal basis for the subspace $\bigwedge_2(k_v^N)$ in $\bigwedge(k_v^N)$. In case $v\nmid\infty$ this implies that

$$\begin{aligned}|\boldsymbol{\xi}\wedge\boldsymbol{\beta}_n|_v &= |\tau_v|_v^{-L_v}\max\{|\varphi_\ell\varphi_n|_v : 1\le\ell\le L_v \text{ and } L_v+1\le n\le N\}\\ &\le |\tau_v|_v^{-L_v}|\boldsymbol{\xi}|_v|\boldsymbol{\varphi}|_v\\ &= |\tau_v|_v^{-L_v}|\boldsymbol{\xi}|_v|Y_v^{-1}\boldsymbol{\beta}_n|_v\ ,\end{aligned}$$

and (2.7) follows immediately. If $v\mid\infty$ then

$$\begin{aligned}\|\boldsymbol{\xi}\wedge\boldsymbol{\beta}_n\|_v^2 &= \|\tau_v\|_v^{-2L_v}\sum_{\ell=1}^{L_v}\sum_{n=L_v+1}^{N}\|\varphi_\ell\varphi_n\|_v^2\\ &\le \|\tau_v\|_v^{-2L_v}\|\boldsymbol{\xi}\|_v^2\|\varphi\|_v^2\\ &= \|\tau_v\|_v^{-2L_v}\|\boldsymbol{\xi}\|_v^2\|Y_v^{-1}\boldsymbol{\beta}_n\|_v^2\ ,\end{aligned}$$

and again (2.7) follows. This completes the proof of Theorem 2.

3. Proof of Theorem 3

We require two lemmas.

LEMMA 4. *Let A and B be $N\times M$ and $M\times L$ matrices, respectively, with entries in Ω_v and*

$$2\le L=\text{rank } B\le M=\text{rank } A\le N\ .$$

Then $\eta_v(AB)\le\eta_v(A)\eta_v(B)$.

PROOF. We write

$$\frac{|AB\mathbf{x}|_v|\mathbf{y}|_v}{|AB\mathbf{y}|_v|\mathbf{x}|_v} = \left(\frac{|AB\mathbf{x}|_v|B\mathbf{y}|_v}{|AB\mathbf{y}|_v|B\mathbf{x}|_v}\right)\left(\frac{|B\mathbf{x}|_v|\mathbf{y}|_v}{|B\mathbf{y}|_v|\mathbf{x}|_v}\right)$$

and conclude that

$$\begin{aligned}\eta_v(AB) &\le \sup\left\{\frac{|AB\mathbf{x}|_v|B\mathbf{y}|_v}{|AB\mathbf{y}|_v|B\mathbf{x}|_v} : \mathbf{x}\in\mathbb{P}^{L-1}(\Omega_v)\ ,\ \mathbf{y}\in\mathbb{P}^{L-1}(\Omega_v)\right\}\eta_v(B)\\ &\le \sup\left\{\frac{|A\mathbf{w}|_v|\mathbf{z}|_v}{|A\mathbf{z}|_v|\mathbf{w}|_v} : \mathbf{w}\in\mathbb{P}^{M-1}(\Omega_v)\ ,\ \mathbf{z}\in\mathbb{P}^{M-1}(\Omega_v)\right\}\eta_v(B)\\ &= \eta_v(A)\eta_v(B)\ .\end{aligned}$$

Lemma 5. *Let A be an $N\times 2$ matrix with entries in Ω_v and $2 = \operatorname{rank} A \le N$. Then the inequality* (1.11) *holds for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}^1(\Omega_v)$. Moreover, both inequalities in* (1.11) *are sharp in the sense that $\eta_v(A)$ cannot be replaced by a smaller number.*

Proof. Let $A = (\mathbf{a}_1\ \mathbf{a}_2)$ so that

$$\begin{aligned}|A\mathbf{x}\wedge A\mathbf{y}|_v &= |(x_1\mathbf{a}_1 + x_2\mathbf{a}_2)\wedge(y_1\mathbf{a}_1 + y_2\mathbf{a}_2)|_v\\ &= |(x_1y_2 - x_2y_2)(\mathbf{a}_1\wedge\mathbf{a}_2)|_v\\ &= |\mathbf{x}\wedge\mathbf{y}|_v|\mathbf{a}_1\wedge\mathbf{a}_2|_v\ ,\end{aligned}$$

and therefore

$$\delta_v(A\mathbf{x}, A\mathbf{y}) = \left\{\frac{|\mathbf{x}|_v|\mathbf{y}|_v|\mathbf{a}_1\wedge\mathbf{a}_2|_v}{|A\mathbf{x}|_v|A\mathbf{y}|_v}\right\}\delta_v(\mathbf{x},\mathbf{y})\ . \tag{3.1}$$

Now suppose that $v \mid \infty$ and let A^* denote the complex conjugate transpose of A. Then A^*A is a 2×2, positive definite Hermitian matrix with eigenvalues $0 < \lambda_1 \le \lambda_2 < \infty$. It is well known that

$$\begin{aligned}\lambda_1 &= \inf\left\{\frac{\|A\mathbf{x}\|_v^2}{\|\mathbf{x}\|_v^2} : \mathbf{x}\in\mathbb{P}^1(\Omega_v)\right\}\\ &\le \sup\left\{\frac{\|A\mathbf{y}\|_v^2}{\|\mathbf{y}\|_v^2} : \mathbf{y}\in\mathbb{P}^1(\Omega_v)\right\}\\ &= \lambda_2\ .\end{aligned} \tag{3.2}$$

It follows that

$$\eta_v(A) = \left(\frac{\lambda_2}{\lambda_1}\right)^{d_v/2d}\ . \tag{3.3}$$

Also, we have

$$\lambda_1\ \lambda_2 = \det(A^*A) = \|\mathbf{a}_1\wedge\mathbf{a}_2\|_v^2\ . \tag{3.4}$$

Using (3.2) and (3.4) we obtain the inequality

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{d_v/2d} \le \left\{\frac{|\mathbf{x}|_v|\mathbf{y}|_v|\mathbf{a}_1\wedge\mathbf{a}_2|_v}{|A\mathbf{x}|_v|A\mathbf{y}|_v}\right\} \le \left(\frac{\lambda_2}{\lambda_1}\right)^{d_v/2d}\ . \tag{3.5}$$

The desired estimate (1.11) follows now by combining (3.1), (3.3) and (3.5). The fact that $\eta_v(A)$ cannot be replaced by a smaller number is clear from (3.2) and (3.3).

Next we assume that $v \nmid \infty$. If $I \subseteq \{1, 2, \ldots, N\}$ with $|I| = 2$ we write

$${}_IA = (a_{nm})$$

for the 2×2 submatrix obtained by letting $n \in I$ index rows and $m = 1, 2$, index columns. Then we select $J \subseteq \{1, 2, \ldots, N\}$ so that $|J| = 2$ and

$$|\mathbf{a}_1 \wedge \mathbf{a}_2|_v = \max\{|\det{}_I A|_v : |I| = 2\} \\ = |\det{}_J A|_v \ .$$

It follows (see the discussion after (4.8) in [**1**]) that each entry of $A({}_J A)^{-1}$ has v-adic absolute value less than or equal to 1. As ${}_J(A({}_J A)^{-1})$ is the 2×2 identity matrix we have

$$|A({}_J A)^{-1}\mathbf{x}|_v = |\mathbf{x}|_v$$

for all $\mathbf{x}$ in Ω_v^2. This shows that

$$|A\mathbf{x}|_v = |A({}_J A)^{-1}\,{}_J A\mathbf{x}|_v = |{}_J A\mathbf{x}|_v \tag{3.6}$$

for all $\mathbf{x}$ in Ω_v^2. Using (3.6) we conclude that

$$\inf\left\{\frac{|A\mathbf{x}|_v}{|\mathbf{x}|_v} : \mathbf{x} \in \mathbb{P}^1(\Omega_v)\right\} = \inf\left\{\frac{|\mathbf{y}|_v}{|({}_J A)^{-1}\mathbf{y}|_v} : \mathbf{y} \in \mathbb{P}^1(\Omega_v)\right\} \\ = |({}_J A)^{-1}|_v^{-1} \ , \tag{3.7}$$

and

$$\sup\left\{\frac{|A\mathbf{x}|_v}{|\mathbf{x}|_v} : \mathbf{x} \in \mathbb{P}^1(\Omega_v)\right\} = |{}_J A|_v \ . \tag{3.8}$$

¿From the definition of $|\quad|_v$ on 2×2 matrices we find that

$$|({}_J A)^{-1}|_v = |\det\,{}_J A|_v^{-1}|\,{}_J A|_v \\ = |\mathbf{a}_1 \wedge \mathbf{a}_2|_v^{-1}|\,{}_J A|_v \ . \tag{3.9}$$

Now we combine (3.7), (3.8) and (3.9). In this way we obtain the identity

$$\eta_v(A) = |{}_J A|_v|({}_J A)^{-1}|_v \\ = |{}_J A|_v^2|\mathbf{a}_1 \wedge \mathbf{a}_2|_v^{-1} \\ = |({}_J A)^{-1}|_v^2|\mathbf{a}_1 \wedge \mathbf{a}_2|_v \ , \tag{3.10}$$

and the inequality

$$|{}_J A|_v^{-2}|\mathbf{a}_1 \wedge \mathbf{a}_2|_v \leq \left\{\frac{|\mathbf{x}|_v|\mathbf{y}|_v|\mathbf{a}_1 \wedge \mathbf{a}_2|_v}{|A\mathbf{x}|_v|A\mathbf{y}|_v}\right\} \leq |({}_J A)^{-1}|_v^2|\mathbf{a}_1 \wedge \mathbf{a}_2|_v \ . \tag{3.11}$$

The bound (1.11) follows from (3.1), (3.10) and (3.11). Again we find that $\eta_v(A)$ cannot be replaced by a smaller number by using (3.7), (3.8) and (3.10).

We are now in position to prove the inequality (1.11) in full generality. In doing so we may assume that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent points in Ω_v^M. Then there exists an $M \times 2$ matrix $B = (\mathbf{b}_1\ \mathbf{b}_2)$ having orthogonal columns which form a basis for the subspace $\mathrm{span}_{\Omega_v}\{\mathbf{x}, \mathbf{y}\} \subseteq \Omega_v^M$. Also, we can select B so that $|\mathbf{b}_1|_v = |\mathbf{b}_2|_v = 1$. It follows that $\eta_v(B) = 1$. And there exist linearly independent points $\mathbf{w}$ and $\mathbf{z}$ in Ω_v^2 such that $\mathbf{x} = B\mathbf{w}$ and $\mathbf{y} = B\mathbf{z}$.

Now AB is an $N \times 2$ matrix. Therefore Lemma 4 and Lemma 5 imply that

$$\delta_v(A\mathbf{x}, A\mathbf{y}) = \delta_v(AB\mathbf{w}, AB\mathbf{z}) \\ \leq \eta_v(AB)\delta_v(\mathbf{w}, \mathbf{z}) \\ \leq \eta_v(A)\delta_v(\mathbf{w}, \mathbf{z}) \ . \tag{3.12}$$

A second application of Lemma 5 shows that

$$\delta_v(\mathbf{x},\mathbf{y}) = \delta_v(B\mathbf{w},B\mathbf{z}) = \delta_v(\mathbf{w},\mathbf{z}) \ . \tag{3.13}$$

The inequality on the right of (1.11) follows from (3.12) and (3.13). The inequality on the left of (1.11) is established in essentially the same manner.

Finally, we will show that $\eta_v(A)$ cannot be replaced by a smaller number on the right of (1.11). Let $\epsilon > 0$ and then select $\mathbf{x}_1$ and $\mathbf{y}_1$ in Ω_v^M so that

$$(1-\epsilon)\eta_v(A) \le \frac{|A\mathbf{x}_1|_v|\mathbf{y}_1|_v}{|A\mathbf{y}_1|_v|\mathbf{x}_1|_v} \ .$$

As before there exists an $M \times 2$ matrix $B = (\mathbf{b}_1 \ \mathbf{b}_2)$ having orthogonal columns which form a basis for the subspace $\mathrm{span}_{\Omega_v}\{\mathbf{x}_1,\mathbf{y}_1\} \subseteq \Omega_v^M$. Again we can choose B so that $|\mathbf{b}_1|_v = |\mathbf{b}_2|_v = 1$ and therefore $\eta_v(B) = 1$. Now write $\mathbf{x}_1 = B\mathbf{w}_1$ and $\mathbf{y}_1 = B\mathbf{z}_1$ so that

$$\begin{aligned}(1-\epsilon)\eta_v(A) &\le \frac{|AB\mathbf{w}_1|_v|B\mathbf{z}_1|_v}{|AB\mathbf{z}_1|_v|B\mathbf{w}_1|_v} \\ &= \frac{|AB\mathbf{w}_1|_v|\mathbf{z}_1|_v}{|AB\mathbf{z}_1|_v|\mathbf{w}_1|_v} \\ &\le \eta_v(AB) \ .\end{aligned} \tag{3.14}$$

By the last assertion in Lemma 5 there exist linearly independent points $\mathbf{w}_2$ and $\mathbf{z}_2$ in Ω_v^2 such that

$$(1-\epsilon)\eta_v(AB)\delta_v(\mathbf{w}_2,\mathbf{z}_2) \le \delta_v(AB\mathbf{w}_2, AB\mathbf{z}_2) \ . \tag{3.15}$$

Write $\mathbf{x}_2 = B\mathbf{w}_2$ and $\mathbf{y}_2 = B\mathbf{z}_2$. Then (3.14) and (3.15) imply that

$$(1-\epsilon)^2\eta_v(A)\delta_v(\mathbf{x}_2,\mathbf{y}_2) \le \delta_v(A\mathbf{x}_2, A\mathbf{y}_2) \ .$$

As $\epsilon > 0$ is arbitrary it is clear that the inequality on the right of (1.11) is sharp. The inequality on the left of (1.11) is also sharp and this can be demonstrated by a similar argument.

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