BOUNDS ON AUTOCORRELATION COEFFICIENTS OF RUDIN-SHAPIRO POLYNOMIALS II

STEPHEN CHOI

(aperiodic)

Abstract. We study the (~~periodic~~) autocorrelation coefficients of the Rudin-Shapiro polynomials and prove that

Theorem. *If P_n and Q_n are the n-th Rudin-Shapiro polynomials and*

$$|P_n(z)|^2 = \sum_{k=-L_n+1}^{L_n-1} a_k z^k, \quad |z| = 1$$

($a_0 = L_n$, $a_k = a_{-k}$, $k \geq 1$), then

$$0.27771487\cdots(1+o(1))|\lambda|^n \leq \max_{1\leq k\leq L_n-1} |a_k| \leq (3.78207844\cdots)|\lambda|^n$$

where λ is the real root of $x^3 - x^2 - 2x + 4 = 0$.

Also, if we let

$$\left(\overline{P_n} Q_n\right)(z) = \left(P_n \overline{Q_n}\right)(1/z) = \sum_{k=-L_n+1}^{L_n-1} b_k z^k, \quad |z| = 1$$

($b_0 = 2 - L_n$, $b_k = \overline{b_{-k}}$, $k \geq 1$), then

$$0.46071984\cdots(1+o(1))|\lambda|^n \leq \max_{1\leq k\leq L_n-1} |b_k| \leq (3.78207844\cdots)|\lambda|^n,$$

which was conjectured by B. Saffari [7] 40 years ago. This improves previous results in [1] and makes the upper bound of the correct order of infinity.

Keywords — Golay-Rudin-Shapiro polynomials, autocorrelation coefficients, trigonometric polynomials

1. Introduction

The Rudin-Shapiro polynomials were discovered by H. S. Shapiro in 1951 [8] (also see the paper [5] of M. J. E. Golay the same year, and the footnote on the first page of [3]). They were studied later by W. Rudin in a 1959 paper [6] as recalled, e.g., in [2]. These polynomials, also called the Shapiro-Rudin polynomials, are constructed as follows.

Let $P_0(z) = 1$ and $Q_0(z) = 1$. For any integer $n \geq 1$, define

$$\begin{cases} P_n(z) := P_{n-1}(z) + z^{2^{n-1}} Q_{n-1}(z), \\ Q_n(z) := P_{n-1}(z) - z^{2^{n-1}} Q_{n-1}(z). \end{cases}$$

Polynomials $P_n(z)$ and $Q_n(z)$ are called Rudin-Shapiro polynomials. Note that $P_n(z)$ and $Q_n(z)$ are polynomials with ± 1 coefficients of degree $L_n - 1$ where $L_n = 2^n$. It is well-known that

$$\left|P_n(e^{it})\right|^2 + \left|Q_n(e^{it})\right|^2 = L_{n+1} = 2^{n+1}, \quad \text{for any } t \in [0, 2\pi).$$

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By the Fourier coefficients of f at k, we mean the coefficient for the term z^k, or simply

$$\widehat{f}(k) = (f)^\wedge(k) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-ikt}dt.$$

By the definition we have

$$\begin{aligned}
|P_n|^2(z) &= z^{2^{n-1}}\overline{P_{n-1}(z)}Q_{n-1}(z) + \overline{z}^{2^{n-1}}P_{n-1}(z)\overline{Q_{n-1}(z)} + 2^n, \\
\left(\overline{P_n}Q_n\right)(z) &= 2|P_{n-1}(z)|^2 - z^{2^{n-1}}\overline{P_{n-1}(z)}Q_{n-1}(z) \\
&\qquad + \overline{z}^{2^{n-1}}P_{n-1}(z)\overline{Q_{n-1}(z)} - 2^n, \\
\left(P_n\overline{Q_n}\right)(z) &= 2|P_{n-1}(z)|^2 + z^{2^{n-1}}\overline{P_{n-1}(z)}Q_{n-1}(z) \\
&\qquad - \overline{z}^{2^{n-1}}P_{n-1}(z)\overline{Q_{n-1}(z)} - 2^n.
\end{aligned} \tag{1.1}$$

People are interested in estimating $\max_k |\left(|P_n|^2\right)^\wedge(k)|$. If we write

$$|P_n(e^{it})|^2 = \sum_{k=-L_n+1}^{L_n-1} a_k z^k,$$

then

$$\begin{cases} \left(|P_n|^2\right)^\wedge(k) = a_k, & \text{when } -L_n+1 \le k \le L_n - 1, \\ \left(|P_n|^2\right)^\wedge(k) = 0, & \text{when } |k| > L_n. \end{cases}$$

Saffari conjectured that

$$K_1|\lambda|^n = K_1 2^{0.7302852598\cdots n} \le \max_{1\le k\le L_n-1}|a_k| \le K_2 2^{0.7302852598\cdots n} = K_2|\lambda|^n$$

with positive constants K_1 and K_2 where $2^{0.7302852598\cdots n} = |\lambda|^n$ and λ is the real root of the equation $x^3 - x^2 - 2x + 4 = 0$. Recently, in [1], the following theorem is proved along with some other results.

Theorem [ACDES]. *For $n \ge 1$, we have*

$$K_1|\lambda|^n \le \max_{1\le k\le L_n-1}|a_k| \le K_2((1.00000100000025)|\lambda|)^n$$

with positive constants K_1 and K_2.

We should remark that λ in [1] is $|\lambda|^2$ in this paper.

Their lower bound is of correct order and the upper bound is very close to the correct order. In this article, we continue the methodology of [1] and prove Theorem 1.1 below.

Theorem 1.1. *If P_n and Q_n are the n-th Rudin-Shapiro polynomials and*

$$|P_n(z)|^2 = \sum_{k=-L_n+1}^{L_n-1} a_k z^k, \quad |z| = 1$$

($a_0 = L_n$, $a_k = a_{-k}$, $k \ge 1$), then

$$0.27771487\cdots(1+o(1))|\lambda|^n \le \max_{1\le k\le L_n-1}|a_k| \le (3.78207844\cdots)|\lambda|^n$$

where

$$\lambda = -\frac{(44+3\sqrt{177})^{1/3} + (44-3\sqrt{177})^{1/3} - 1}{3} = -1.658967081916\cdots$$

is the real root of $x^3 - x^2 - 2x + 4 = 0$. Note that $|\lambda|^n = 2^{0.7302852598\cdots n}$.

Also, if we let

$$\left(\overline{P_n}Q_n\right)(z) = \left(P_n\overline{Q_n}\right)(1/z) = \sum_{k=-L_n+1}^{L_n-1} b_k z^k, \quad |z| = 1$$

($b_0 = 2 - L_n$, $b_k = \overline{b_{-k}}$, $k \geq 1$), then

$$0.46071984\cdots(1+o(1))|\lambda|^n \leq \max_{1\leq k\leq L_n-1} |b_k| \leq (3.78207844\cdots)|\lambda|^n.$$

Theorem 1.1 proves Saffari's conjecture.

We should remark that T. Høholdt, H. E. Jensen and J. Justesen in [4] proved a bound of $O(2^{0.9n})$ for the *aperiodic* autocorrelation coefficients of Rudin-Shapiro polynomials ~~while we study the *periodic* autocorrelation coefficients here~~.

2. Upper bound for the autocorrelation coefficients

By induction on (1.1), for positive integers k, we have

$$\left(|P_n|^2\right)^\wedge(2k) = \left(\overline{P_n}Q_n\right)^\wedge(2k) = \left(P_n\overline{Q_n}\right)^\wedge(2k) = 0.$$

For $n \geq 1$, let S_n be the set of all odd integers k with $-L_n < k < L_n$ and let

$$S_n^\tau := \left\{k \in S_n : (\tau-3)2^{n-1} < k \leq (\tau-2)2^{n-1}\right\}$$

so that S_n is the disjoint union of S_n^1, S_n^2, S_n^3 and S_n^4. For $n \geq 2$, let k_n be an odd integer in S_n. We define k_{n-1} and k_n' from k_n as follows

$$k_n' := \begin{cases} k_n + 2^n & \text{if } k_n \in S_n^1 \cup S_n^2, \\ k_n - 2^n & \text{if } k_n \in S_n^3 \cup S_n^4, \end{cases}$$

and

$$k_{n-1} := \begin{cases} k_n & \text{if } k_n \in S_n^2 \cup S_n^3, \\ k_n' & \text{if } k_n \in S_n^1 \cup S_n^4. \end{cases}$$

It is easy to see that if $k_n \in S_n$, then $k_{n-1} \in S_{n-1}$ and $k_n' \in S_n$.

Let

$$\omega_n(k_n) := \begin{pmatrix} \left(|P_n|^2\right)^\wedge(k_n) \\ \left(\overline{P_n}Q_n\right)^\wedge(k_n') \\ \left(P_n\overline{Q_n}\right)^\wedge(k_n') \end{pmatrix}. \tag{2.1}$$

Formula (2.2) below gives a recursive relation for $\omega_n(k_n)$. In [1], we apply the recursive relation and write $\omega_n(k_n) = M\omega_1(k_1)$ for some 3×3 matrix and prove a nice factorization of M as follows.

Lemma 2.1. *For $n \geq 2$, we have*

$$\omega_n(k_n) = M\omega_1(k_1), \tag{2.2}$$

where M is 3×3 matrix in the form of

$$M = T^{\delta_1}B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\cdots B^{k_L}M_1^{\ell_L}B^{k_{L+1}}T^{\delta_2} \tag{2.3}$$

for $\ell_1, \ell_2, \ldots, \ell_L \geq 1$, $k_2, k_3, \cdots, k_L \in \{1,2,3\}$, $, k_1, k_{L+1} \in \{0,1,2,3\}$ and $\delta_1, \delta_2 \in \{0,1\}$ and

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ 2 & 0 & 1 \end{pmatrix}, \omega_1(k_1) = \pm\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Moreover, we have

$$\sum_{j=1}^{L}(k_j + \ell_j) + k_{L+1} \leq n. \tag{2.4}$$

Proof. This follows from Lemmas 1 and 2 in [1]. □

Note that $T^2 = I, B^3 = B^5$ and $B^4 = B^2$ where I is the 3×3 identity matrix.

The characteristic polynomial of M_1 is $g(x) = x^3 - x^2 - 2x + 4$. Then $g(x) = 0$ has one real root λ and two complex roots λ' and $\overline{\lambda'}$ with $|\lambda| > |\lambda'|$ and

$$\lambda = -\frac{(44+3\sqrt{177})^{1/3} + (44-3\sqrt{177})^{1/3} - 1}{3} = -1.65896708\cdots$$

and

$$\begin{aligned}\lambda' &= \frac{(44+3\sqrt{177})^{1/3} + (44-3\sqrt{177})^{1/3} + 2}{6} \\ &\quad + i\sqrt{3}\left(\frac{(44+3\sqrt{177})^{1/3} - (44-3\sqrt{177})^{1/3}}{6}\right) \\ &= 1.32948354\cdots + 0.80225455\cdots i.\end{aligned}$$

Let $\lambda' = re^{i\theta}$ where $r = 1.55278422\cdots, \theta = 0.54294011\cdots$ and $\mu := \frac{\lambda'}{\lambda}$ with $|\mu| = 0.93599459\cdots$. Since these eigenvalues are distinct, there is a nonsingular matrix S such that $S^{-1}M_1S = \Lambda$ with $\Lambda = \mathrm{diag}[\lambda, \lambda', \overline{\lambda'}]$, the diagonal matrix with diagonal elements λ, λ' and $\overline{\lambda'}$ respectively. For any $k \geq 1$, we have

$$M_1^k = S\Lambda^k S^{-1}$$

where

$$S := \begin{pmatrix} s_1 & s_3 & \overline{s_3} \\ s_2 & s_4 & \overline{s_4} \\ 1 & 1 & 1 \end{pmatrix},$$

$$s_1 := \frac{1}{66}\left((10-\sqrt{177})(71+6\sqrt{177})^{1/3} - (10+\sqrt{177})(-71+6\sqrt{177})^{1/3} - 22\right),$$

$$s_2 := \frac{1}{66}\left(\left(1+\sqrt{177}\right)(71+6\sqrt{177})^{1/3} - \left(1-\sqrt{177}\right)(-71+6\sqrt{177})^{1/3} + 44\right),$$

$$\begin{aligned}s_3 := -\frac{1}{132}\Big(&\left(1-i\sqrt{3}\right)\left(10-\sqrt{177}\right)(71+6\sqrt{177})^{1/3} \\ &- \left(1+i\sqrt{3}\right)\left(10+\sqrt{177}\right)(-71+6\sqrt{177})^{1/3} + 44\Big)\end{aligned}$$

and

$$\begin{aligned}s_4 := \frac{1}{132}\Big(&-\left(1-i\sqrt{3}\right)\left(1+\sqrt{177}\right)(71+6\sqrt{177})^{1/3} \\ &+ \left(1+i\sqrt{3}\right)\left(1-\sqrt{177}\right)(-71+6\sqrt{177})^{1/3} + 88\Big).\end{aligned}$$

For any matrix A, let $\|A\|$ be the spectral norm of A induced by the L_2 norm, that is, if A is an $m \times m$ matrix, A^* is its conjugate transpose and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of A^*A (the eigenvalues are all real), then $\|A\| = \sqrt{\max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_m|\}}$. It is known that the spectral norm satisfies the submultiplicativity, i.e, $\|A_1A_2\| \leq \|A_1\|\|A_2\|$.

The idea of the proof is as follows. In view of the factorization of M in (2.3) and the submultiplicativity of the norms, it is tempting to show

$$\|B^{k_1}M_1^{\ell_1}\| \leq |\lambda|^{k_1+\ell_1} \tag{2.5}$$

and then the upper bounds in Theorem 1.1 follows instantly because $\|T\| = 1$ and $\|B\| \leq |\lambda|$. However, the inequality (2.5) is true for all k_1, ℓ_1 but 4 exceptions (see Corollary 2.6 below). Due to these 4 exceptions and no control of the number of these exceptions appearing in the factorization of M, this does not prove our theorem. We need to study products of more matrices such as $\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\|$. Unfortunately,

$$\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_1}\| \leq |\lambda|^{k_1+\ell_1+k_2+\ell_2}$$

is true for all k_1, k_2, ℓ_1, ℓ_2 but 5 exceptions (see Lemma 2.10 below). This makes us to study

$$\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\| \leq |\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4} \tag{2.6}$$

and finally we are able to show that (2.6) is true for all k_j, ℓ_j without any exception. As a result, we can use (2.6) to obtain our upper bound of the theorem. Since $\|A\| = \sqrt{\max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_m|\}}$ and λ_j are the eigenvalues of A^*A. With the aid of `Maple`, we compute the characteristic polynomial of A^*A and hence find out the eigenvalue with the largest absolute value. This allows us to obtain an exact formula for, e.g. $\|B^{k_1}M_1^{\ell_1}\|$, in terms of $|\lambda|$ and $|\lambda'|$ and get very good upper bounds for $\|B^{k_1}M_1^{\ell_1}\|$. In order to show (2.6), we first need to estimate the norms of some basic factors in the factorization (2.3). These are Lemmas 2.2 - 2.9 below.

We should warn the readers that from now on, the values of $C_1, C_2, b_j, a_j, c_j, d_j$ may be different in different lemmas but their values are defined and the same within the individual lemma and its proof.

Lemma 2.2. *For $\ell \geq 1$, we have*

$$\|M_1^\ell\| = \sqrt{a}(1+o(1)) = (1.13871099\cdots)(1+o(1))|\lambda|^\ell$$

and

$$\|M_1^\ell\| \leq 1.70493263\cdots|\lambda|^\ell \tag{2.7}$$

where $a := 1.29666272\cdots$.

Proof. We first compute the characteristic polynomial $P_{M_1^\ell}(x)$ for the matrix $(M_1^\ell)^*(M_1^\ell)$. By using `Maple`, we find

$$P_{M_1^\ell}(x) = x^3 - C_2x^2 + C_1x + C_0$$

where $C_0 = -16^\ell$,

$$C_1 := a|\lambda'|^{4\ell} + (b_1\sin(\ell\theta) - b_2\cos(\ell\theta))(\lambda|\lambda'|^3)^\ell + (d - c_1\sin(2\ell\theta) - c_2\cos(2\ell\theta))|\lambda\lambda'|^{2\ell},$$

$$C_2 := a|\lambda|^{2\ell} - (b_1\sin(\ell\theta) + b_2\cos(\ell\theta))(\lambda|\lambda'|)^\ell + (d + c_1\sin(2\ell\theta) - c_2\cos(2\ell\theta))|\lambda'|^{2\ell}$$

and $d := 2.52552883\cdots$,

$$b_1 := 0.12119555\cdots, b_2 := 0.5933254\cdots, c_1 := 0.0605977\cdots, c_2 := 0.22886611\cdots.$$

Note that $|\lambda|^2 = 2.75217177\cdots, |\lambda\lambda'|^2 = 6.63586832\cdots, |\lambda'|^2 = 2.41113886\cdots$ and

$$|\lambda|^2 > |\lambda\lambda'| > |\lambda'|^2 \quad \text{and} \quad |\lambda\lambda'|^2 > |\lambda\lambda'^3| > |\lambda'|^4$$

so that

$$C_1 = (d - c_1\sin(2\ell\theta) - c_2\cos(2\ell\theta))(1+o(1))|\lambda\lambda'|^{2\ell} \text{ and } C_2 = a(1+o(1))|\lambda|^{2\ell}. \tag{2.8}$$

Let $y := x - \frac{C_2}{3}$. Then the equation $P_{M_1^\ell}(x) = 0$ becomes $y^3 + py + q = 0$ with

$$p := \frac{-C_2^2 + 3C_1}{3}, \quad \text{and} \quad q := \frac{-2C_2^3 + 9C_1C_2 + 27(-16)^\ell}{27}$$

and the three roots are

$$2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - \frac{2\pi k}{3}\right), \quad k = 0, 1, 2.$$

From (2.8), $-p = \frac{a^2}{3}(1+o(1))|\lambda|^{4\ell}$. Hence the largest root in absolute value is $\frac{C_2}{3} + 2\left(\sqrt{-\frac{p}{3}}\right)(1+o(1)) = a(1+o(1))|\lambda|^{2\ell}$. Recall that $\|A\|$ is the square root of the large absolute value of the eigenvalues of A^*A. Therefore,

$$\|M_1^\ell\| = \sqrt{a}(1+o(1))|\lambda|^\ell = (1.13871099\cdots)(1+o(1))(1.65896708\cdots)^\ell.$$

Since

$$\begin{aligned} C_1 &= a|\lambda'|^{4\ell} + (b_1\sin(\ell\theta) - b_2\cos(\ell\theta))(\lambda|\lambda'|^3)^\ell + (d - c_1\sin(2\ell\theta) - c_2\cos(2\ell\theta))|\lambda\lambda'|^{2\ell} \\ &\geq a|\lambda'|^{4\ell} - (b_1 + b_2)|\lambda\lambda'^3|^\ell + (d - c_1 - c_2)|\lambda\lambda'|^{2\ell} \\ &\geq 2.23606494(6.63586832\cdots)^\ell + 1.29666272(5.81359060\cdots)^\ell \\ &\quad -0.71452101(6.21113691\cdots)^\ell > 0, \end{aligned}$$

and

$$\begin{aligned}
C_1 &= a|\lambda'|^{4\ell} + (b_1\sin(\ell\theta) - b_2\cos(\ell\theta))(\lambda|\lambda'|^3)^\ell + (d - c_1\sin(2\ell\theta) - c_2\cos(2\ell\theta))|\lambda\lambda'|^{2\ell} \\
&\le a|\lambda'|^{4\ell} + (b_1+b_2)|\lambda\lambda'^3|^\ell + (d+c_1+c_2)|\lambda\lambda'|^{2\ell} \\
&\le 2.81499273(6.63586832\cdots)^\ell + 1.29666273(5.81359060\cdots)^\ell \\
&\quad +0.71452101(6.21113691\cdots)^\ell
\end{aligned}$$

and

$$\begin{aligned}
C_2 &= a|\lambda|^{2\ell} - (b_1\sin(\ell\theta) + b_2\cos(\ell\theta))(\lambda|\lambda'|)^\ell + (d + c_1\sin(2\ell\theta) - c_2\cos(2\ell\theta))|\lambda'|^{2\ell} \\
&\ge a|\lambda|^{2\ell} - (b_1+b_2)|\lambda\lambda'|^\ell + (d-c_1-c_2)|\lambda'|^{2\ell} \\
&\ge 2.23606494(2.41113886\cdots)^\ell + 1.29666272(2.75217177\cdots)^\ell \\
&\quad -0.71452101(2.5760179\cdots)^\ell > 0
\end{aligned}$$

for $\ell \ge 1$, so

$$\begin{aligned}
p &= \frac{-C_2^2 + 3C_1}{3} \\
&\le -\frac{1}{3}(a|\lambda|^{2\ell} - (b_1+b_2)|\lambda\lambda'|^\ell + (d-c_1-c_2)|\lambda'|^{2\ell})^2 \\
&\quad +(a|\lambda'|^{4\ell} + (b_1+b_2)|\lambda\lambda'^3|^\ell + (d+c_1+c_2)|\lambda\lambda'|^{2\ell}) \\
&\le -\frac{1}{3}(a-(b_1+b_2))^2|\lambda|^{4\ell} + (a+b_1+b_2+d+c_1+c_2)|\lambda\lambda'|^{2\ell}
\end{aligned}$$

because $|\lambda'| < |\lambda|$. Hence

$$\begin{aligned}
p &\le -0.11296299(7.57444950\cdots)^\ell + 4.82617645(6.63586832\cdots)^\ell \\
&< 4.82617645(7.57444950\cdots)^\ell\left((0.87608588\cdots)^\ell - 0.02340631\right) < 0
\end{aligned}$$

for $\ell \ge 29$. For $1 \le \ell \le 28$, we use the exact formula $p = \frac{-C_2^2+3C_1}{3}$ to verify that $p < 0$ directly. Hence $|p| < \frac{C_2^2}{3}$ and

$$\left|2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - \frac{2\pi k}{3}\right)\right| \le 2\sqrt{-\frac{p}{3}} \le \frac{2}{3}C_2.$$

It then follows that the largest root in absolute value of $P_{M_1^\ell}(x) = 0$ is $\le \frac{1}{3}C_2 + \frac{2}{3}C_2 = C_2$ and

$$\begin{aligned}
\|M_1^\ell\| &\le \sqrt{C_2} \quad \le \sqrt{a|\lambda|^{2\ell} + (b_1+b_2)|\lambda\lambda'|^\ell + (d+c_1+c_2)|\lambda'|^{2\ell}} \\
&= \sqrt{a + (b_1+b_2)|\mu|^\ell + (d+c_1+c_2)|\mu|^{2\ell}}|\lambda|^\ell \\
&\le 1.69158015\cdots|\lambda|^\ell.
\end{aligned}$$

for $\ell \ge 7$. For $1 \le \ell \le 6$, we compute the exact values of $\|M_1^\ell\|$ and find that inequality (2.7) holds with equality when $\ell = 1$. This proves the lemma. □

Lemma 2.3. *For $\ell \ge 1$, the characteristic polynomial for $(BM_1^\ell)^*(BM_1^\ell)$ is $x(x^2 - C_2x + C_1)$ and*

$$\|BM_1^\ell\| = \sqrt{\frac{C_2 + \sqrt{C_2^2 - 4C_1}}{2}} \tag{2.9}$$

$$\le \left((1.82283467\cdots) + (1.89404583\cdots)|\mu|^\ell + (3.16896344\cdots)|\mu|^{2\ell}\right)^{\frac{1}{2}}|\lambda|^\ell, \tag{2.10}$$

where C_1, C_2 and $a_1, a_2, a_3, b_3, a_4, b_4, c_1, c_2, c_3, d_3, c_4, d_4$ are defined in (2.14) and (2.15) respectively. Also

$$\|BM_1^\ell\| \le |\lambda|^{\ell+1} \tag{2.11}$$

for all $\ell \geq 1$ except $\ell \in \{1, 4, 6, 11\}$. The values of the exceptional cases are

$$\begin{aligned}\|BM_1\| &= (1.28074503\cdots)|\lambda|^2\\ \|BM_1^4\| &= (1.15409372\cdots)|\lambda|^5\\ \|BM_1^6\| &= (1.15708225\cdots)|\lambda|^7\\ \|BM_1^{11}\| &= (1.04665404\cdots)|\lambda|^{12}.\end{aligned} \tag{2.12}$$

Hence for $\ell \geq 1$, we have

$$\|BM_1^\ell\| \leq (1.28074503\cdots)|\lambda|^{\ell+1}. \tag{2.13}$$

Proof. Using `Maple`, the characteristic polynomial for $(BM_1^\ell)^*(BM_1^\ell)$ is $x(x^2 - C_2x + C_1)$ where

$$\begin{aligned}C_1 &:= a_1(|\lambda'|^4)^\ell + a_2(|\lambda\lambda'|^2)^\ell - (a_3\cos(2\ell\theta) + b_3\sin(2\ell\theta))(|\lambda\lambda'|^2)^\ell\\ &\quad -(a_4\cos(\ell\theta) - b_4\sin(\ell\theta))(\lambda|\lambda'|^3)^\ell,\\ C_2 &:= c_1(|\lambda|^2)^\ell + c_2(|\lambda'|^2)^\ell - (c_3\cos(2\ell\theta) + d_3\sin(2\ell\theta))(|\lambda'|^2)^\ell\\ &\quad -(c_4\cos(\ell\theta) - d_4\sin(\ell\theta))(\lambda|\lambda'|)^\ell,\end{aligned} \tag{2.14}$$

and

$$\begin{array}{lll} a_1 := 1.83307335\cdots & a_2 := 2.56903005\cdots & a_3 := 0.57883606\cdots\\ b_3 := 0.35542042\cdots & a_4 := 1.82326734\cdots & b_4 := 0.47473146\cdots\\ c_1 := 1.82283467\cdots & c_2 := 2.93380493\cdots & c_3 := 0.11097026\cdots\\ d_3 := 0.12418824\cdots & c_4 := 1.64566934\cdots & d_4 := 0.24837649\cdots. \end{array} \tag{2.15}$$

Since

$$\begin{aligned}C_1 &\geq a_1|\lambda'|^{4\ell} + (a_2 - a_3 - b_3)|\lambda\lambda'|^{2\ell} - (a_4 + b_4)|\lambda\lambda'^3|^\ell\\ &= 1.83307335\cdots(5.81359060\cdots)^\ell + 1.63477356\cdots(6.63586832\cdots)^\ell\\ &\quad -2.29799880\cdots(6.21113691\cdots)^\ell > 0\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}C_2 &\geq c_1|\lambda|^{2\ell} + (c_2 - c_3 - d_3)|\lambda'|^{2\ell} - (c_4 + d_4)(|\lambda\lambda'|)^\ell\\ &= 1.82283467\cdots(2.75217177\cdots)^\ell + 2.69864642\cdots(2.41113886\cdots)^\ell\\ &\quad -1.89404583\cdots(2.57601792\cdots)^\ell > 0\end{aligned} \tag{2.17}$$

for all $\ell \geq 1$, so

$$\begin{aligned}C_2^2 - 4C_1 &\geq \left(c_1|\lambda|^{2\ell} + (c_2 - c_3 - d_3)|\lambda'|^{2\ell} - (c_4 + d_4)(|\lambda\lambda'|)^\ell\right)^2\\ &\quad -4\left(a_1|\lambda'|^{4\ell} + (a_2 + a_3 + b_3)|\lambda\lambda'|^{2\ell} + (a_4 + b_4)|\lambda\lambda'^3|^\ell\right)\\ &= c_1^2|\lambda|^{4\ell} + (2c_1(c_2 - c_3 - d_3) + (c_4 + d_4)^2 - 4(a_2 + a_3 + b_3))|\lambda\lambda'|^{2\ell}\\ &\quad +((c_2 - c_3 - d_3)^2 - 4a_1)|\lambda'|^{4\ell} - 2c_1(c_4 + d_4)|\lambda^3\lambda'|^\ell\\ &\quad -(2(c_2 - c_3 - d_3)(c_4 + d_4) + 4(a_4 + b_4)|\lambda\lambda'^3|^\ell\\ &\geq 3.32272623|\lambda|^{4\ell} - 0.58736404|\lambda\lambda'|^{2\ell} - 0.049600899|\lambda'|^{4\ell}\\ &\quad -6.90506483|\lambda^3\lambda'|^\ell - 19.41471526|\lambda\lambda'^3|^\ell\\ &> 3.32272623|\lambda|^{4\ell} - 26.95674502|\lambda^3\lambda'|^\ell\\ &> 26.95674502|\lambda|^{4\ell}\left(0.12326140 - |\mu|^\ell\right) > 0\end{aligned} \tag{2.18}$$

for $\ell \geq 32$. For $1 \leq \ell \leq 31$, we use (2.14) to verify $C_2^2 - 4C_1 > 0$ directly. Thus the largest absolute value of these two roots is $\frac{C_2 + \sqrt{C_2^2 - 4C_1}}{2}$ and hence

$$\|BM_1^\ell\| = \sqrt{\frac{C_2 + \sqrt{C_2^2 - 4C_1}}{2}} \leq \sqrt{C_2}.$$

Since

$$\begin{aligned} C_2 &\leq c_1\lambda^{2\ell} + c_2|\lambda'|^{2\ell} + (c_3+d_3)|\lambda'|^{2\ell} + (c_4+d_4)(|\lambda\lambda'|)^{\ell} \\ &= (1.82283467\cdots)|\lambda|^{2\ell} + (1.89404583\cdots)|\lambda\lambda'|^{\ell} + (3.16896344\cdots)|\lambda'|^{2\ell}, \end{aligned} \tag{2.19}$$

so

$$\|BM_1^{\ell}\| \leq \sqrt{(1.82283467\cdots) + (1.89404583\cdots)|\mu|^{\ell} + (3.16896344\cdots)|\mu|^{2\ell}}|\lambda|^{\ell}.$$

Using this upper bound, we can show that for $\ell \geq 18$,

$$\|BM_1^{\ell}\| \leq |\lambda|^{\ell+1}$$

for $\ell \geq 18$. For $1 \leq \ell \leq 17$ and $\ell \notin \{1,4,6,11\}$, we use (2.9) to verify (2.11) directly. This proves (2.11). □

Lemma 2.4. *For $\ell \geq 1$, the characteristic polynomial for $(B^2M_1^{\ell})^*(B^2M_1^{\ell})$ is $x^2(x-\alpha)$ and*

$$\begin{aligned} \|B^2M_1^{\ell}\| &= \sqrt{\alpha} \\ &\leq \sqrt{a_1|\lambda|^{2\ell} + (a_3+b_3)|\lambda\lambda'|^{\ell} + (a_2+a_4+b_4)|\lambda'|^{2\ell}} \\ &= \sqrt{(2.47940687\cdots) + (3.3569727\cdots)|\mu|^{\ell} + (3.11924482\cdots)|\mu|^{2\ell}}|\lambda|^{\ell}, \end{aligned} \tag{2.20}$$

where α and a_1,a_2,a_3,b_3,a_4,b_4 are defined in (2.22) *and* (2.23) *respectively. Also, we have*

$$\|B^2M_1^{\ell}\| \leq |\lambda|^{\ell+2} \tag{2.21}$$

for all $\ell \geq 1$

Proof. Using `Maple`, the characteristic polynomial for $(B^2M_1^{\ell})^*(B^2M_1^{\ell})$ is $x^2(x-\alpha)$ where

$$\alpha := a_1|\lambda|^{2\ell} - (a_3\cos(\ell\theta) - b_3\sin(\ell\theta))\lambda^{\ell}|\lambda'|^{\ell} + (a_2 + a_4\cos(2\ell\theta) - b_4\sin(2\ell\theta))|\lambda'|^{2\ell}, \tag{2.22}$$

and

$$\begin{array}{lll} a_1 := 2.47940687\cdots & a_2 := 2.09279978\cdots & a_3 := 2.19449470\cdots \\ b_3 := 1.16247810\cdots & a_4 := 0.62228803\cdots & b_4 := 0.40415701\cdots. \end{array} \tag{2.23}$$

It follows that

$$\begin{aligned} \alpha &\geq a_1|\lambda|^{2\ell} - (a_3+b_3)|\lambda\lambda'|^{\ell} + (a_2-a_4-b_4)|\lambda'|^{2\ell} \\ &= 2.47940687\cdots(2.75217177\cdots)^{\ell} + 1.06635473\cdots(2.41113886\cdots)^{\ell} \\ &\quad -3.35697280\cdots(2.57601792\cdots)^{\ell} > 0 \end{aligned}$$

for all $\ell \geq 1$. Hence

$$\|B^2M_1^{\ell}\| = \sqrt{\alpha} \leq \sqrt{a_1 + (a_3+b_3)|\mu|^{\ell} + (a_2+a_4+b_4)|\mu|^{2\ell}}|\lambda|^{\ell}$$

for $\ell \geq 1$. Using this upper bound, we can show that $\|B^2M_1^{\ell}\| \leq |\lambda|^{\ell+2}$ for $\ell \geq 3$. For $\ell = 1,2$, we can verify (2.21) directly. This proves the lemma. □

Lemma 2.5. *For $\ell \geq 1$,*

$$\|B^3M_1^{\ell}\| \leq (0.94843708\cdots)|\lambda|^{\ell+3}.$$

Proof. Since $M_1 = S\Lambda S^{-1}$, so

$$\begin{aligned} \|B^3M_1^{\ell}\| &= \|B^3S\Lambda^{\ell}S^{-1}\| \leq \|B^3S\|\cdot\|\Lambda^{\ell}\|\cdot\|S^{-1}\| \\ &= (4.16100894\cdots)(1.04069431\cdots)|\lambda|^{\ell} \\ &= (4.33033835\cdots)|\lambda|^{\ell} = (0.94843708\cdots)|\lambda|^{\ell+3}. \end{aligned}$$

□

Corollary 2.6. *We have*

$$\|B^kM_1^{\ell}\| \leq |\lambda|^{k+\ell}$$

for $k = 1,2,3$ and $\ell \geq 1$ except $(k,\ell) \in E_1 := \{(1,1),(1,4),(1,6),(1,11)\}$.

Proof. It follows from Lemmas 2.3, 2.4 and 2.5. □

Lemma 2.7. *For $\ell \geq 1$, the characteristic polynomial for $(M_1^\ell B)^*(M_1^\ell B)$ is $x(x^2 - C_2 x + C_1)$ and*

$$\|M_1^\ell B\| = \sqrt{\frac{C_2 + \sqrt{C_2^2 - 4C_1}}{2}} \tag{2.24}$$

$$\leq \left((1.04617217\cdots) + (1.02408285\cdots)|\mu|^\ell + (3.43481377\cdots)|\mu|^{2\ell}\right)^{\frac{1}{2}} |\lambda|^\ell, \tag{2.25}$$

where C_1, C_2 and $a_1, a_2, a_3, b_3, a_4, b_4, c_1, c_2, c_3, d_3, c_4, d_4$ are defined in (2.27) *and* (2.28) *respectively. Also*

$$\|M_1^\ell B\| \leq |\lambda|^{\ell+1} \tag{2.26}$$

for all $\ell \geq 2$ and $\|MB\| = 2.82842712\cdots = 1.02770733\cdots|\lambda|^2$.

Proof. Using `Maple`, the characteristic polynomial for $(M_1^\ell B)^*(M_1^\ell B)$ is $x(x^2 - C_2 x + C_1)$ where

$$\begin{aligned} C_1 &:= a_1(|\lambda'|^4)^\ell + a_2(|\lambda\lambda'|^2)^\ell - (a_3\cos(2\ell\theta) + b_3\sin(2\ell\theta))(|\lambda\lambda'|^2)^\ell \\ &\quad -(a_4\cos(\ell\theta) - b_4\sin(\ell\theta))(\lambda|\lambda'|^3)^\ell \\ C_2 &:= c_1(|\lambda|^2)^\ell + c_2(|\lambda'|^2)^\ell - (c_3\cos(2\ell\theta) + d_3\sin(2\ell\theta))(|\lambda'|^2)^\ell \\ &\quad -(c_4\cos(\ell\theta) - d_4\sin(\ell\theta))(\lambda|\lambda'|)^\ell \end{aligned} \tag{2.27}$$

and

$$\begin{array}{lll} a_1 := 1.74580388\cdots & a_2 := 1.91991483\cdots & a_3 := 0.01699032\cdots \\ b_3 := 0.68050491\cdots & a_4 := 1.64872839\cdots & b_4 := 1.12490044\cdots \\ c_1 := 1.04617217\cdots & c_2 := 2.99225661\cdots & c_3 := 0.18176539\cdots \\ d_3 := 0.26079176\cdots & c_4 := 0.85666339\cdots & d_4 := 0.16741945\cdots. \end{array} \tag{2.28}$$

This characteristic polynomial has the same form of the characteristic polynomial for $(BM_1^\ell)^*(BM_1^\ell)$ so that the proof is similar to Lemma 2.3. Similar to (2.16) and (2.17), we have

$$C_1 \geq 1.74580388|\lambda'|^{4\ell} + 1.22241959|\lambda\lambda'|^{2\ell} - 2.77362885|\lambda\lambda'^3|^\ell > 0$$

and

$$C_2 \geq 1.04617217|\lambda|^{2\ell} - 1.02408286|\lambda\lambda'|^\ell + 2.54969945|\lambda'|^{2\ell} > 0$$

for all $\ell \geq 1$. Hence similar to (2.18), we have

$$\begin{aligned} C_2^2 - 4C_1 \quad &\geq \quad 1.09447621|\lambda|^{4\ell} - 4.086045385|\lambda\lambda'|^{2\ell} - 0.48224821|\lambda'|^{4\ell} \\ &\qquad -2.14273397(|\lambda^3\lambda'|)^\ell - 16.31672236(|\lambda\lambda'^3|)^\ell \\ &> \quad 1.09447621|\lambda|^{4\ell} - 23.02774990(|\lambda^3\lambda'|)^\ell > 0 \end{aligned}$$

for $\ell \geq 47$. For $1 \leq \ell \leq 46$, use (2.27) to verify $C_2^2 - 4C_1 > 0$ directly. Thus the largest absolute value of these two roots is $\frac{C_2 + \sqrt{C_2^2 - 4C_1}}{2}$ and hence

$$\|M_1^\ell B\| = \sqrt{\frac{C_2 + \sqrt{C_2^2 - 4C_1}}{2}} \leq \sqrt{C_2}.$$

Similar to (2.19), we have

$$C_2 \quad \leq \quad \left((1.04617217\cdots) + (1.02408285\cdots)|\mu|^\ell + (3.43481377\cdots)|\mu|^{2\ell}\right) |\lambda|^{2\ell},$$

and hence

$$\|M_1^\ell B\| \leq \sqrt{(1.04617217\cdots) + (1.02408285\cdots)|\mu|^\ell + (3.43481377\cdots)|\mu|^{2\ell}}|\lambda|^\ell.$$

Using this upper bound, we can show that for $\ell \geq 9$, $\|M_1^\ell B\| \leq |\lambda|^{\ell+1}$. For $2 \leq \ell \leq 9$, we use (2.24) to verify (2.26) directly. This proves (2.26). □

Lemma 2.8. *For $\ell \geq 1$, the characteristic polynomial for $(BM_1^\ell B)^*(BM_1^\ell B)$ is $x(x^2 - C_2 x + C_1)$ and*

$$\|BM_1^\ell B\| = \sqrt{\frac{C_2+\sqrt{C_2^2-4C_1}}{2}} \tag{2.29}$$

$$\leq \sqrt{(1.4706977\cdots)+(8.33075603\cdots)|\mu|^\ell+(5.34967703\cdots)|\mu|^{2\ell}}|\lambda|^\ell, \tag{2.30}$$

where C_1, C_2 and $a_1, a_2, a_3, b_3, a_4, b_4, c_1, c_2, c_3, d_3, c_4, d_4$ are defined in (2.32) and (2.33) respectively. Also

$$\|BM_1^\ell B\| \leq |\lambda|^{\ell+2} \tag{2.31}$$

for all $\ell \geq 1$.

Proof. Using `Maple`, the characteristic polynomial for $(BM_1^\ell B)^*(BM_1^\ell B)$ is $x(x^2 - C_2 x + C_1)$. where

$$\begin{aligned} C_1 &:= a_1|\lambda'|^{4\ell} + a_2|\lambda\lambda'|^{2\ell} + (a_3\cos(2\ell\theta) - b_3\sin(2\ell\theta))|\lambda\lambda'|^{2\ell} \\ &\quad +(-a_4\cos(\ell\theta) + b_4\sin(\ell\theta))(\lambda|\lambda'|^3)^\ell \\ C_2 &:= c_1\lambda^{2\ell} + c_2|\lambda'|^{2\ell} + (c_3\cos(2\ell\theta) - d_3\sin(2\ell\theta))|\lambda'|^{2\ell} \\ &\quad -(c_4\cos(\ell\theta) - d_4\sin(\ell\theta))(\lambda|\lambda'|)^\ell \end{aligned} \tag{2.32}$$

and

$$\begin{array}{lll} a_1 := 2.46801772\cdots & a_2 := 1.95298460\cdots & a_3 := 0.51503312\cdots \\ b_3 := 1.88384970\cdots & a_4 := 4.93603544\cdots & b_4 := 3.76769941\cdots \\ c_1 := 1.47069771\cdots & c_2 := 3.47598376\cdots & c_3 := 0.10557346\cdots \\ d_3 := 0.19712721\cdots & c_4 := 2.05225495\cdots & d_4 := 1.24217879\cdots \end{array} \tag{2.33}$$

This characteristic polynomial has the same form of the characteristic polynomial for $(BM_1^\ell)^*(BM_1^\ell)$ so that the proof is similar to Lemma 2.3.. Similar to (2.17), we have

$$C_2 \geq 1.47069771|\lambda|^{2\ell} - 3.29443375|\lambda\lambda'|^\ell + 3.17328308|\lambda'|^{2\ell} > 0$$

for all $\ell \geq 1$ and similar to (2.18)

$$\begin{aligned} C_2^2 - 4C_1 &\geq 2.16295177|\lambda|^{4\ell} + 2.77970436|\lambda\lambda'|^{2\ell} + 0.19765463|\lambda'|^{4\ell} \\ &\quad -9.69023240|\lambda^3\lambda'|^\ell - 55.72328124|\lambda\lambda'^3|^\ell \\ &> 2.16295177|\lambda|^{4\ell} - 9.69023240|\lambda^3\lambda'|^\ell - 55.72328124|\lambda\lambda'^3|^\ell \\ &> 2.16295177|\lambda|^{4\ell} - 65.41351364|\lambda^3\lambda'|^\ell > 0 \end{aligned}$$

for $\ell \geq 52$. For $1 \leq \ell \leq 51$, use (2.32) to verify $C_2^2 - 4C_1 > 0$ directly. Thus the largest absolute value of these two roots is $\frac{C_2+\sqrt{C_2^2-4C_1}}{2}$ and hence

$$\|BM_1^\ell\| = \sqrt{\frac{C_2+\sqrt{C_2^2-4C_1}}{2}}.$$

Similar to (2.19), we have

$$C_2 \leq \left((1.470697718\cdots) + (3.29443374\cdots)|\mu|^\ell + (3.77868444\cdots)|\mu|^{2\ell}\right)|\lambda|^{2\ell}.$$

In this case, although we suspect $C_1 > 0$ for all $\ell \geq 1$, however, we are unable to prove it as before. Instead we give a slightly weaker upper bound as follows. We have

$$\begin{aligned} \sqrt{|C_1|} &\leq (a_1|\lambda'|^{4\ell} + (a_2+a_3+b_3)|\lambda\lambda'|^{2\ell} + (a_4+b_4)|\lambda\lambda'^3|^\ell)^{1/2} \\ &\leq \sqrt{a_1|\lambda'|^{4\ell}} + \sqrt{(a_2+a_3+b_3)|\lambda\lambda'|^{2\ell}} + \sqrt{(a_4+b_4)|\lambda\lambda'^3|^\ell} \\ &\leq \sqrt{a_1}|\lambda'|^{2\ell} + (\sqrt{a_2+a_3+b_3} + \sqrt{a_4+b_4})|\lambda\lambda'|^\ell \\ &\leq (1.57099259\cdots)|\lambda'|^{2\ell} + (5.03632228\cdots)|\lambda\lambda'|^\ell \end{aligned}$$

so

$$\begin{aligned}\|BM_1^\ell B\| &= \sqrt{\frac{C_2+\sqrt{C_2^2-4C_1}}{2}} \le \sqrt{\frac{C_2+\sqrt{C_2^2+4|C_1|}}{2}} \le \sqrt{C_2+\sqrt{|C_1|}}\\ &\le \sqrt{(1.4706977\cdots)+(8.33075603\cdots)|\mu|^\ell+(5.34967703\cdots)|\mu|^{2\ell}}|\lambda|^\ell.\end{aligned}$$

Using this upper bound, we can show that for $\ell \ge 10$, $\|BM_1^\ell B\| \le |\lambda|^{\ell+2}$. For $1 \le \ell \le 9$, we use (2.29) to verify (2.31) directly. This proves (2.31). ☐

Lemma 2.9. *For $\ell \ge 1$, the characteristic polynomial for $(BM_1BM_1^\ell)^*(BM_1BM_1^\ell)$ is $x(x^2-C_2x+C_1)$ and*

$$\|BM_1BM_1^\ell\| = \sqrt{\frac{C_2+\sqrt{C_2^2-4C_1}}{2}} \tag{2.34}$$

$$\le \Big((3.01929157\cdots)+(6.10271497\cdots)|\mu|^\ell+(27.480227244\cdots)|\mu|^{2\ell}\Big)^{\frac{1}{2}}|\lambda|^\ell, \tag{2.35}$$

where C_1, C_2 and $a_1,a_2,a_3,b_3,a_4,b_4,c_1,c_2,c_3,d_3,c_4,d_4$ are defined in (2.37) and (2.38) respectively. Also

$$\|BM_1BM_1^\ell\| \le |\lambda|^{\ell+3} \tag{2.36}$$

for all $\ell \ge 1$ except $\ell = 3$ and $\|BM_1BM_1^3\| = 21.42027291\cdots = 1.02753917\cdots|\lambda|^6$.

Proof. Using `Maple`, the characteristic polynomial for $(BM_1BM_1^\ell)^*(BM_1BM_1^\ell)$ is $x(x^2-C_2x+C_1)$. where

$$\begin{aligned}C_1 &:= a_1|\lambda'|^{4\ell}+a_2|\lambda\lambda'|^{2\ell}-(a_3\cos(2\ell\theta)+b_3\sin(2\ell\theta))|\lambda\lambda'|^{2\ell}\\ &\quad+(-a_4\cos(\ell\theta)+b_4\sin(\ell\theta))(\lambda|\lambda'|^3)^\ell\\ C_2 &:= c_1\lambda^{2\ell}+c_2|\lambda'|^{2\ell}-(c_3\cos(2\ell\theta)+d_3\sin(2\ell\theta))|\lambda'|^{2\ell}\\ &\quad-(c_4\cos(\ell\theta)+d_4\sin(\ell\theta))(\lambda|\lambda'|)^\ell\end{aligned} \tag{2.37}$$

and

$$\begin{array}{lll} a_1 := 29.32917370\cdots & a_2 := 41.10448091\cdots & a_3 := 9.26137709\cdots\\ b_3 := 5.68672677\cdots & a_4 := 29.17227752\cdots & b_4 := 7.59570337\cdots\\ c_1 := 3.01929157\cdots & c_2 := 21.18864105\cdots & c_3 := 6.29047801\cdots\\ d_3 := 0.001108174\cdots & c_4 := 2.91745461\cdots & d_4 := 3.18526036\cdots \end{array} \tag{2.38}$$

This characteristic polynomial has the same form of the characteristic polynomial for $(BM_1^\ell)^*(BM_1^\ell)$ so that the proof is similar to Lemma 2.3. Similar to (2.16) and (2.17), we have

$$C_1 \ge 29.32917370|\lambda'|^{4\ell}+26.15637704|\lambda\lambda'|^{2\ell}-36.76798090|\lambda\lambda'^3|^\ell > 0$$

and

$$C_2 \ge 3.01929157|\lambda|^{2\ell}+14.89705486|\lambda\lambda'|^\ell-6.10271498|\lambda'|^{2\ell} > 0$$

for all $\ell \ge 1$ and similar to (2.18) we have

$$\begin{aligned}C_2^2-4C_1 &\ge 9.11612160\lambda^{4\ell}-97.01010461|\lambda\lambda'|^{2\ell}+104.60554883|\lambda'|^{4\ell}\\ &\quad-36.85175181|\lambda^3\lambda'|^\ell-328.89688323|\lambda\lambda'^3|^\ell\\ &> 9.11612160\lambda^{4\ell}-462.75873965|\lambda^3\lambda'|^\ell > 0\end{aligned}$$

for $\ell \ge 60$. For $1 \le \ell \le 59$, use (2.37) to verify $C_2^2-4C_1 > 0$ directly. Thus the largest absolute value of these two roots is $\frac{C_2+\sqrt{C_2^2-4C_1}}{2}$ and hence

$$\|BM_1BM_1^\ell\| = \sqrt{\frac{C_2+\sqrt{C_2^2-4C_1}}{2}} \le \sqrt{C_2}.$$

Similar to (2.19), we have

$$C_2 \leq \Big((3.01929157\cdots) + (6.10271497\cdots)|\mu|^{\ell} + (27.480227244\cdots)|\mu|^{2\ell}\Big)\,|\lambda|^{\ell},$$

so

$$\|BM_1BM_1^{\ell}\| \leq \sqrt{(3.01929157\cdots) + (6.10271497\cdots)|\mu|^{\ell} + (27.480227244\cdots)|\mu|^{2\ell}}|\lambda|^{\ell}.$$

Using this upper bound, we can show that for $\ell \geq 6$,

$$\|BM_1BM_1^{\ell}\| \leq |\lambda|^{\ell+3}$$

For $1 \leq \ell \leq 5$, we use (2.34) to verify (2.36) directly except when $\ell = 3$. This proves (2.36). ☐

Lemma 2.10. *We have*

$$\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\| \leq |\lambda|^{k_1+\ell_1+k_2+\ell_2} \tag{2.39}$$

for any $k_j = 1,2,3$ and $\ell_j \geq 1$ for $j = 1,2$ except $(k_1,\ell_1,k_2,\ell_2) \in E_2$ where

$$E_2 := \{(1,1,1,3),(1,2,1,1),(1,4,1,1),(1,4,1,3),(1,9,1,1)\}.$$

We also have

$$\begin{aligned}
\|BM_1BM_1^3\| &= 1.02753917\cdots|\lambda|^6\\
\|BM_1^2BM_1\| &= 1.13945965\cdots|\lambda|^5\\
\|BM_1^4BM_1\| &= 1.14952443\cdots|\lambda|^7\\
\|BM_1^4BM_1^3\| &= 1.04530507\cdots|\lambda|^9\\
\|BM_1^9BM_1\| &= 1.01996194\cdots|\lambda|^{12}.
\end{aligned} \tag{2.40}$$

Proof. We divide our consideration into 3 cases: (1) $k_2 = 3$, (2) $k_2 = 2$, (3) $k_2 = 1$.

Case 1: $k_2 = 3$.

We have

$$\frac{\|B^{k_1}M_1^{\ell_1}B^3M_1^{\ell_2}\|}{|\lambda|^{k_1+\ell_1+3+\ell_2}} \leq \frac{\|B^{k_1}M_1^{\ell_1}B\|}{|\lambda|^{k_1+\ell_1+1}}\cdot\frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}} \leq \frac{\|BM_1^{\ell_1}B\|}{|\lambda|^{2+\ell_1}}\cdot\frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}} \leq 1$$

by Lemmas 2.4 and 2.8 because $\|B\| \leq |\lambda|$. This proves (2.39) for $k_2 = 3$.

Case 2: $k_2 = 2$.

(i) If $(k_1,\ell_1) \notin E_1$ where E_1 is defined in Corollary 2.6, we have

$$\|B^{k_1}M_1^{\ell_1}B^2M_1^{\ell_2}\| \leq \|B^{k_1}M_1^{\ell_1}\|\cdot\|B^2M_1^{\ell_2}\| \leq |\lambda|^{k_1+\ell_1}\cdot|\lambda|^{2+\ell_2} = |\lambda|^{k_1+\ell_1+2+\ell_2}$$

by Corollary 2.6 and Lemma 2.4.

(ii) If $(k_1,\ell_1) \in E_1 = \{(1,1),(1,4),(1,6),(1,11)\}$, then $k_1 = 1$ and $\ell_1 = 1,4,6,11$ and we have

$$\begin{aligned}
&\max\left\{\frac{\|BM_1B\|}{|\lambda|^3},\frac{\|BM_1^4B\|}{|\lambda|^6},\frac{\|BM_1^6B\|}{|\lambda|^8},\frac{\|BM_1^{11}B\|}{|\lambda|^{13}}\right\}\\
= &\max\{0.77201353\cdots, 0.70946397\cdots, 0.68586629\cdots, 0.61917169\cdots\}\\
= &\,0.77201353\cdots = \frac{\|BM_1B\|}{|\lambda|^3}.
\end{aligned}$$

So

$$\frac{\|BM_1^{\ell_1}B^2M_1^{\ell_2}\|}{|\lambda|^{1+\ell_1+2+\ell_2}} \leq \frac{\|BM_1^{\ell_1}B\|}{|\lambda|^{\ell_1+2}}\cdot\frac{\|BM_1^{\ell_2}\|}{|\lambda|^{\ell_2+1}} \leq (0.77201353\cdots)(1.28074503\cdots) < 1$$

by (2.13). This proves (2.39) for $k_2 = 2$.

Case 3: $k_2 = 1$.

(i) If $(k_1, \ell_1) \notin E_1$ and $(k_2, \ell_2) \notin E_1$, then

$$\|B^{k_1} M_1^{\ell_1} B M_1^{\ell_2}\| \le \|B^{k_1} M_1^{\ell_1}\| \cdot \|B M_1^{\ell_2}\| \le |\lambda|^{k_1+\ell_1} \cdot |\lambda|^{1+\ell_2} = |\lambda|^{k_1+\ell_1+1+\ell_2}$$

by Corollary 2.6.

(ii) Suppose $(k_1, \ell_1) \in E_1 = \{(1,1), (1,4), (1,6), (1,11)\}$ and $(k_2, \ell_2) \notin E_1$.
If $(k_1, \ell_1) = (1,1)$, then

$$\|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2}\| = \|B M_1 B M_1^{\ell_2}\| \le |\lambda|^{3+\ell_2}$$

by Lemma 2.9 except $\ell_2 = 3$.
If $(k_1, \ell_1) = (1,4), (1,6), (1,11)$, then from (2.12) we have

$$\frac{\|BM_1^4\|}{|\lambda|^5}, \frac{\|BM_1^6\|}{|\lambda|^7}, \frac{\|BM_1^{11}\|}{|\lambda|^{12}} \le \frac{\|BM_1^6\|}{|\lambda|^7} = (1.15708225\cdots) = \frac{1}{0.86424279\cdots}. \tag{2.41}$$

Using the upper bound (2.10), we have

$$\|BM_1^{\ell_2}\| \le (0.86111571\cdots)|\lambda|^{1+\ell_2}$$

for $\ell_2 \ge 35$. Hence

$$\begin{aligned} \frac{\|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2}} &\le \frac{\|B^{k_1} M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}} \cdot \frac{\|B^{k_2} M_1^{\ell_2}\|}{|\lambda|^{k_2+\ell_2}} \\ &\le \frac{\|BM_1^6\|}{|\lambda|^7} \cdot \frac{\|BM_1^{\ell_2}\|}{|\lambda|^{1+\ell_2}} \le \frac{0.86111571\cdots}{0.86424279\cdots} < 1 \end{aligned}$$

for $\ell_2 \ge 35$. For $1 \le \ell_2 \le 34$, we compute the exact values of $\|BM_1^4 BM_1^{\ell_2}\|$, $\|BM_1^6 BM_1^{\ell_2}\|$ and $\|BM_1^{11} BM_1^{\ell_2}\|$ and verify (2.39) directly with the only exceptions $BM_1^4 BM_1$ and $BM_1^4 BM_1^3$.

(iii) Suppose $(k_1, \ell_1) \notin E_1$ and $(k_2, \ell_2) \in E_1 = \{(1,1), (1,4), (1,6), (1,11)\}$.
If $(k_2, \ell_2) = (1,1)$ we have

$$\begin{aligned} \frac{\|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2}} &= \frac{\|B^{k_1} M_1^{\ell_1} B M_1\|}{|\lambda|^{k_1+\ell_1+1+1}} \\ &\le \left(\frac{\|B\|}{|\lambda|}\right)^{k_1} \cdot \frac{\|M_1^{\ell_1} B\|}{|\lambda|^{\ell_1+1}} \cdot \frac{\|M_1\|}{|\lambda|} \\ &= (0.85246631\cdots)^{k_1} (1.70493263\cdots) \frac{\|M_1^{\ell_1} B\|}{|\lambda|^{\ell_1+1}} \end{aligned} \tag{2.42}$$

because $\|B\| = \sqrt{2} = (0.85246631\cdots)|\lambda|$ and $\|M_1\| = 2\sqrt{2} = (1.70493263\cdots)|\lambda|$. In view of (2.42), since $k_1 \ge 1$, so we have

$$\begin{aligned} \frac{\|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2}} &\le (0.85246631\cdots)^{k_1} (1.70493263\cdots) \frac{\|M_1^{\ell_1} B\|}{|\lambda|^{\ell_1+1}} \\ &\le (0.85246631\cdots)(1.70493263\cdots) \frac{\|M_1^{\ell_1} B\|}{|\lambda|^{\ell_1+1}} \\ &\le (1.45339765\cdots) \frac{\|M_1^{\ell_1} B\|}{|\lambda|^{\ell_1+1}} \\ &= \frac{1}{0.68804294\cdots} \frac{\|M_1^{\ell_1} B\|}{|\lambda|^{\ell_1+1}} \\ &\le \frac{0.68596169\cdots}{0.68804294\cdots} < 1 \end{aligned}$$

for $\ell_1 \geq 40$ using upper bound (2.25). For $1 \leq \ell_1 \leq 39$ and $k_1 = 1, 2, 3$, we compute the exact values of $\|B^{k_1} M_1^{\ell_1} BM_1\|$ and verify that (2.39) is true except $BM_1^2 BM_1, BM_1^4 BM_1$ and $BM_1^9 BM_1$.

If $(k_2, \ell_2) = (1, 4), (1, 6), (1, 11)$, then since $\|B\| \leq |\lambda|$, so

$$\begin{aligned} \|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2}\| &\leq \|B\|^{k_1-1} \cdot \|BM_1^{\ell_1}\| \cdot \|BM_1^{\ell_2}\| \\ &\leq |\lambda|^{k_1-1} \|BM_1^{\ell_1}\| \cdot \|BM_1^{\ell_2}\| \\ &\leq \frac{0.86111571\cdots}{0.86424279\cdots} |\lambda|^{k_1+\ell_1+k_2+\ell_2} < |\lambda|^{k_1+\ell_1+k_2+\ell_2} \end{aligned}$$

for $\ell_1 \geq 35$ by (2.10) and (2.41). For $1 \leq \ell_1 \leq 34, k_1 = 1, 2, 3$, and $(k_2, \ell_2) = (1, 4), (1, 6), (1, 11)$, we verify (2.39) directly.

(iv) It remains to consider the case that $(k_1, \ell_1), (k_2, \ell_2) \in E_1$. There are only 16 such cases. We verify (2.39) for these 16 cases directly and find that (2.39) holds except the case $BM_1^4 BM_1$. This proves (2.39) for $k_1 = 1$ and hence completes the proof of the lemma. □

Lemma 2.11. *We have*

$$\|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2} B^{k_3} M_1^{\ell_3}\| \leq |\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3} \tag{2.43}$$

for any $k_j = 1, 2, 3$ *and* $\ell_j \geq 1$ *for* $j = 1, 2, 3$ *except* $BM_1^4 BM_1 BM_1$ *and* $BM_1^4 BM_1 BM_1^3$ *with*

$$\begin{aligned} \|BM_1^4 BM_1 BM_1\| &= 1.02962397\cdots |\lambda|^9 \\ \|BM_1^4 BM_1 BM_1^3\| &= 1.00541478\cdots |\lambda|^{11}. \end{aligned}$$

Proof.
(i) If $(k_1, \ell_1, k_2, \ell_2) \notin E_2$ and $(k_3, \ell_3) \notin E_1$, then

$$\frac{\|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2} B^{k_3} M_1^{\ell_3}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3}} \leq \frac{\|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2}} \cdot \frac{\|B^{k_3} M_1^{\ell_3}\|}{|\lambda|^{k_3+\ell_3}} \leq 1$$

by Corollary 2.6 and Lemma 2.10.

(ii) If $(k_1, \ell_1, k_2, \ell_2) \in E_2$ and $(k_3, \ell_3) \in E_1$, then since there are only 20 such cases we can verify (2.43) directly, and we find that the only exception is $BM_1^4 BM_1 BM_1$.

(iii) Suppose $(k_1, \ell_1, k_2, \ell_2) \notin E_2$ and $(k_3, \ell_3) \in E_1$.

If $(k_2, \ell_2, k_3, \ell_3) \notin E_2$ and if $(k_1, \ell_1) \notin E_1$, then we have

$$\frac{\|B^{k_1} M_1^{\ell_1} B^{k_2} M_1^{\ell_2} B^{k_3} M_1^{\ell_3}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3}} \leq \frac{\|B^{k_1} M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}} \cdot \frac{\|B^{k_2} M_1^{\ell_2} B^{k_3} M_1^{\ell_3}\|}{|\lambda|^{k_2+\ell_2+k_3+\ell_3}} \leq 1$$

by Corollary 2.6 and Lemma 2.10.

If $(k_2, \ell_2, k_3, \ell_3) \notin E_2$ and $(k_1, \ell_1) \in E_1$, then now we have $(k_1, \ell_1), (k_3, \ell_3) \in E_1$. We compute

$$\begin{aligned} &\max \left\{ \frac{\|B^{k_1} M_1^{\ell_1} B^{k_2}\|}{|\lambda|^{k_1+\ell_1+k_2}} \cdot \frac{\|M_1\|^{\ell_3}}{|\lambda|^{\ell_3}} : (k_1, \ell_1), (k_3, \ell_3) \in E_1, k_2 = 1, 2, 3 \right\} \\ = \; & \frac{\|BM_1 B\|}{|\lambda|^3} \cdot \frac{\|M_1\|}{|\lambda|} = (0.77201353\cdots)(1.70493263\cdots) \\ = \; & 1.31623106\cdots. \end{aligned}$$

Then

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3}} &\le \frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}\|}{|\lambda|^{k_1+\ell_1+k_2}}\cdot\frac{\|M_1^{\ell_2}B^{k_3}\|}{|\lambda|^{\ell_2+k_3}}\cdot\frac{\|M_1\|^{\ell_3}}{|\lambda|^{\ell_3}}\\
&\le (1.31623106\cdots)\frac{\|M_1^{\ell_2}B^{k_3}\|}{|\lambda|^{\ell_2+k_3}}\\
&= \frac{1}{0.75974502\cdots}\frac{\|M_1^{\ell_2}B\|}{|\lambda|^{\ell_2+1}}\\
&\le \frac{0.75594244\cdots}{0.75974502\cdots}<1
\end{aligned}$$

for $\ell_2 \ge 29$ by (2.25) because $\|B\| \le |\lambda|$. For $k_2 = 1,2,3, 1 \le \ell_2 \le 28, (k_1,\ell_1),(k_3,\ell_3) \in E_1$, we can verify (2.43) directly with the only exception $BM_1^4BM_1BM_1$.

If $(k_2,\ell_2,k_3,\ell_3) \in E_2$, then since $(k_3,\ell_3) \in E_1$, so $(k_2,\ell_2,k_3,\ell_3) = (1,2,1,1),(1,4,1,1),(1,9,1,1)$. If $(k_1,\ell_1) \in E_1$, then since there are only 12 such cases. We can verify (2.43) directly with only exception $BM_1^4BM_1BM_1$. If $(k_1,\ell_1) \notin E_1$, then from Lemma 2.10, we have

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3}} &\le \frac{\|B^{k_1}M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}}\cdot\frac{\|B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}\|}{|\lambda|^{k_2+\ell_2+k_3+\ell_3}}\\
&\le \frac{\|B^{k_1}M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}}(1.14952443\cdots)\\
&= \frac{1}{0.86992496\cdots}\frac{\|BM_1^{\ell_1}\|}{|\lambda|^{1+\ell_1}}\\
&\le \frac{0.86863950\cdots}{0.86992496\cdots}<1
\end{aligned}$$

for $\ell_1 \ge 33$ by (2.10) because $\|B\| \le |\lambda|$. For $k_1 = 1,2,3, 1 \le \ell_1 \le 32, (k_2,\ell_2,k_3,\ell_3) = (1,2,1,1),(1,4,1,1),(1,9,1,1)$, we verify (2.43) directly.

(iv) Suppose $(k_1,\ell_1,k_2,\ell_2) \in E_2$ and $(k_3,\ell_3) \notin E_1$. Then from Lemma 2.10, we have

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3}} &\le \frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\|}{|\lambda|^{k_1+\ell_1k_2+\ell_2}}\cdot\frac{\|B^{k_3}M_1^{\ell_3}\|}{|\lambda|^{k_3+\ell_3}}\\
&\le (1.14952443\cdots)\frac{\|B^{k_3}M_1^{\ell_3}\|}{|\lambda|^{k_3+\ell_3}}\\
&\le (1.14952443\cdots)\frac{\|BM_1^{\ell_3}\|}{|\lambda|^{1+\ell_3}}\\
&\le \frac{0.86863950\cdots}{0.86992496\cdots}<1
\end{aligned}$$

for $\ell_3 \ge 33$ by (2.10) as above. For $(k_1,\ell_1,k_2,\ell_2) \in E_2, k_3 = 1,2,3, 1 \le \ell_3 \le 32$, we verify (2.43) directly with the only exceptions $BM_1^4BM_1BM_1$ and $BM_1^4BM_1BM_1^3$. This completes the proof of the lemma. $\square$

Lemma 2.12. *We have*

$$\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\| \le |\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4} \tag{2.44}$$

for $k_j = 1,2,3$ *and* $\ell_j \ge 1$ *for* $1 \le j \le 4$.

Proof. If $(k_1,\ell_1,k_2,\ell_2),(k_3,\ell_3,k_4,\ell_4) \notin E_2$, then

$$\begin{aligned}
\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\| &\le \|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\|\cdot\|B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|\\
&\le |\lambda|^{k_1+\ell_1+k_2+\ell_2}\cdot|\lambda|^{k_3+\ell_3+k_4+\ell_4}\\
&= |\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}
\end{aligned}$$

by Lemma 2.10.

We assume either $(k_1,\ell_1,k_2,\ell_2) \in E_2$ or $(k_3,\ell_3,k_4,\ell_4) \in E_2$.

(a) If $(k_1,\ell_1,k_2,\ell_2)\in E_2$, then we have

$$\begin{aligned}
&\max\left\{\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+1}} : (k_1,\ell_1,k_2,\ell_2)\in E_2\right\}\\
= &\quad \frac{\|BM_1^4BM_1B\|}{|\lambda|^8} = 0.69291575\cdots . \qquad (2.45)
\end{aligned}$$

If $k_3\geq 2$, then

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\leq \frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+1}}\cdot\frac{\|B^{k_3-1}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_3-1+\ell_3+k_4+\ell_4}}\\
&\leq (0.69291575\cdots)(1.14952443\cdots)<1
\end{aligned}$$

by (2.40) and (2.45).

If $k_3=1$ and $(k_4,\ell_4)\notin E_1$, then from Corollary 2.6 and (2.40), we have

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\leq \frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2}}\cdot\frac{\|BM_1^{\ell_3}\|}{|\lambda|^{1+\ell_3}}\cdot\frac{\|B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_4+\ell_4}}\\
&\leq (1.14952443\cdots)\frac{\|BM_1^{\ell_3}\|}{|\lambda|^{1+\ell_3}}\\
&= \frac{0.86863950\cdots}{0.86992496\cdots}<1
\end{aligned}$$

for $\ell_3\geq 33$ by (2.10) since $\|B\|<\lambda$. If $1\leq\ell_3\leq 32$ and $\ell_3\neq 1,4,6,11$, then from Corollary 2.6 and (2.40) again, we have

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\leq \frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2}}\cdot\frac{\|BM_1^{\ell_3}\|}{|\lambda|^{1+\ell_3}}\cdot\frac{\|B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_4+\ell_4}}\\
&\leq (1.14952443\cdots)\frac{\|BM_1^{\ell_3}\|}{|\lambda|^{1+\ell_3}}\cdot\frac{\|BM_1^{\ell_4}\|}{|\lambda|^{1+\ell_4}}\\
&\leq (1.14952443\cdots)\frac{\|BM_1^{\ell_4}\|}{|\lambda|^{1+\ell_4}}\\
&= \frac{0.86863950\cdots}{0.86992496\cdots}<1
\end{aligned}$$

for $\ell_4\geq 33$ by (2.40) and (2.10). For $(k_1,\ell_2,k_2,\ell_2)\in E_2, k_3=1, k_4=1,2,3$ and $1\leq\ell_3,\ell_4\leq 32$, we verify (2.44) directly.

If $k_3=1$ and $(k_4,\ell_4)\in E_1$, then from (2.40) and $\|M_1\|=(1.70493263\cdots)|\lambda|$, we have

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\leq \frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+1}}\cdot\frac{\|M_1^{\ell_3}B\|}{|\lambda|^{\ell_3+1}}\cdot\frac{\|M_1\|}{|\lambda|}\\
&\leq (0.69291575\cdots)(1.70493263\cdots)\frac{\|M_1^{\ell_3}B\|}{|\lambda|^{\ell_3+1}}\\
&= (1.18137469\cdots)\frac{\|M_1^{\ell_3}B\|}{|\lambda|^{\ell_3+1}}\\
&= \frac{1}{0.84647149\cdots}\frac{\|M_1^{\ell_3}B\|}{|\lambda|^{\ell_3+1}}\\
&= \frac{0.84522348\cdots}{0.84647149\cdots}<1
\end{aligned}$$

if $\ell_3\geq 21$ by (2.25). We verify (2.44) directly for $(k_1,\ell_1,k_2,\ell_2)\in E_2$, $k_3=1$ and $1\leq\ell_3\leq 20$ and $(k_4,\ell_4)\in E_1$.

(b) If $(k_3,\ell_3,k_4,\ell_4)\in E_2$, then $k_4=1,\ell_4=1,3$ and $(k_3,\ell_3)=(1,1),(1,2),(1,4),(1,9)$.

(i) If $(k_2,\ell_2,k_3,\ell_3)\in E_2$, then $(k_2,\ell_2,k_3,\ell_3)=(1,2,1,1),(1,4,1,1),(1,9,1,1)$. Then from (2.45), we have

$$\begin{aligned}\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\le \frac{\|B^{k_1}M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}}\cdot\frac{\|B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B\|}{|\lambda|^{k_2+\ell_2+k_3+\ell_3+1}}\cdot\frac{\|M_1^{\ell_4}\|}{|\lambda|^{\ell_4}}\\ &\le (0.69291575\cdots)(1.70493263\cdots)\frac{\|B^{k_1}M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}}\\ &\le (1.1837469\cdots)\frac{\|BM_1^{\ell_1}\|}{|\lambda|^{1+\ell_1}}\\ &= \frac{0.84445490\cdots}{0.84647149\cdots}<1\end{aligned}$$

by (2.10) for $\ell_1\ge 41$ because $\frac{\|M_1^3\|}{|\lambda|^3}<\frac{\|M_1\|}{|\lambda|}=1.70493263\cdots$ and $\frac{\|B\|}{|\lambda|}\le 1$. For $1\le\ell_1\le 40, k_1=1,2,3,(k_2,\ell_2)=(1,2),(1,4),(1,9)$ and $(k_3,\ell_3,k_4,\ell_4)\in E_2$, we verify (2.44) directly.

(ii) If $(k_2,\ell_2,k_3,\ell_3)\notin E_2$, then by Lemma 2.10,

$$\begin{aligned}\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\le \frac{\|B^{k_1}M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}}\cdot\frac{\|B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}\|}{|\lambda|^{k_2+\ell_2+k_3+\ell_3}}\cdot\frac{\|BM_1^{\ell_4}\|}{|\lambda|^{1+\ell_4}}\\ &\le \frac{\|B^{k_1}M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}}\cdot\frac{\|BM_1^{\ell_4}\|}{|\lambda|^{1+\ell_4}}\end{aligned}$$

where $\ell_4=1,3$.

If $k_1\ge 2$, then from (2.13), we have

$$\begin{aligned}\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\le \frac{\|B^{k_1}M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}}\cdot\frac{\|BM_1^{\ell_4}\|}{|\lambda|^{1+\ell_4}}\\ &\le (1.28074503\cdots)\frac{\|B^2M_1^{\ell_1}\|}{|\lambda|^{2+\ell_1}}\\ &\le \frac{1}{0.78079553\cdots}\frac{\|B^2M_1^{\ell_1}\|}{|\lambda|^{2+\ell_1}}\\ &= \frac{0.76724174\cdots}{0.78079553\cdots}<1\end{aligned}$$

for $\ell_1\ge 13$ by (2.20) because $\frac{\|BM_1^3\|}{|\lambda|^{1+3}}\le\frac{\|BM_1\|}{|\lambda|^{1+1}}=1.28074503\cdots$ and $\frac{\|B\|}{|\lambda|}\le 1$. If $1\le\ell_1\le 12$, then by Lemmas 2.4, 2.5 and (2.40),

$$\begin{aligned}\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\le \frac{\|B^{k_1}M_1^{\ell_1}\|}{|\lambda|^{k_1+\ell_1}}\cdot\frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}}\cdot\frac{\|B^{k_3}M_1^{\ell_3}BM_1^{\ell_4}\|}{|\lambda|^{k_3+\ell_3+1+\ell_4}}\\ &\le \frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}}\cdot\frac{\|B^{k_3}M_1^{\ell_3}BM_1^{\ell_4}\|}{|\lambda|^{k_3+\ell_3+1+\ell_4}}\\ &\le \frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}}(1.14952443\cdots)\\ &= \frac{0.85518256\cdots}{0.86992496\cdots}<1\end{aligned}$$

for $\ell_2\ge 8$ by (2.20). For $1\le\ell_1\le 12, 1\le\ell_2\le 7, k_1=2,3,k_2=1,2,3,(k_3,\ell_3,k_4,\ell_4)\in E_2$, we verify (2.44) directly for these cases.

If $k_1 = 1$, then if $k_2 \geq 2$, then by (2.13), (2.40) and $\|B\| \leq |\lambda|$,

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\leq \frac{\|BM_1^{\ell_1}\|}{|\lambda|^{1+\ell_1}} \cdot \frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}} \cdot \frac{\|B^{k_3}M_1^{\ell_3}BM_1^{\ell_4}\|}{|\lambda|^{k_3+\ell_3+1+\ell_4}} \\
&\leq (1.28074503\cdots)\frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}}(1.14952443\cdots) \\
&= (1.47224771\cdots)\frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}} \\
&\leq \frac{1}{0.67923352\cdots}\frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}} \\
&= \frac{0.67318960\cdots}{0.67923352\cdots} < 1
\end{aligned}$$

for $\ell_2 \geq 22$ by (2.20). For $1 \leq \ell_2 \leq 21$, then by (2.21) and (2.40),

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\leq \frac{\|BM_1^{\ell_1}\|}{|\lambda|^{1+\ell_1}} \cdot \frac{\|B^2M_1^{\ell_2}\|}{|\lambda|^{2+\ell_2}} \cdot \frac{\|B^{k_3}M_1^{\ell_3}BM_1^{\ell_4}\|}{|\lambda|^{k_3+\ell_3+1+\ell_4}} \\
&\leq \frac{\|BM_1^{\ell_1}\|}{|\lambda|^{1+\ell_1}}(1.14952443\cdots) \\
&= \frac{1}{0.86992496\cdots}\frac{\|BM_1^{\ell_1}\|}{|\lambda|^{1+\ell_1}} \\
&= \frac{0.86863950\cdots}{0.86992496\cdots} < 1
\end{aligned}$$

for $\ell_1 \geq 33$ by (2.10). For $1 \leq \ell_1 \leq 32, 1 \leq \ell_2 \leq 21, k_1 = 1, k_2 = 2, 3, (k_3, \ell_3, k_4, \ell_4) \in E_2$, we verify (2.44) directly for these cases. If $k_2 = 1$, then by (2.7) and (2.40), we have

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\leq \frac{\|BM_1^{\ell_1}B\|}{|\lambda|^{1+\ell_1+1}} \cdot \frac{\|M_1^{\ell_2}\|}{|\lambda|^{\ell_2}} \cdot \frac{\|BM_1^{\ell_3}BM_1^{\ell_4}\|}{|\lambda|^{1+\ell_3+1+\ell_4}} \\
&\leq (1.70493263\cdots)\frac{\|BM_1^{\ell_1}B\|}{|\lambda|^{1+\ell_1+1}}(1.14952443\cdots) \\
&\leq (1.95986172\cdots)\frac{\|BM_1^{\ell_1}B\|}{|\lambda|^{1+\ell_1+1}} \\
&\leq \frac{1}{0.51024007\cdots}\frac{\|BM_1^{\ell_1}B\|}{|\lambda|^{1+\ell_1+1}} \\
&= \frac{0.50611335\cdots}{0.51024007\cdots} < 1
\end{aligned}$$

for $\ell_1 \geq 44$ by (2.30). If $1 \leq \ell_1 \leq 43$, then since $(k_3, \ell_3, k_4, \ell_4) \in E_2$, so we have

$$\max\left\{\frac{\|M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{\ell_3+k_4+\ell_4}} : (k_3, \ell_3, k_4, \ell_4) \in E_2\right\} = \frac{\|M_1BM_1^3\|}{|\lambda|^5} = 1.46332489\cdots$$

Then from (2.12)

$$\begin{aligned}
\frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}B^{k_3}M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{k_1+\ell_1+k_2+\ell_2+k_3+\ell_3+k_4+\ell_4}} &\le \frac{\|BM_1^{\ell_1}\|}{|\lambda|^{1+\ell_1}}\cdot\frac{\|BM_1^{\ell_2}B\|}{|\lambda|^{1+\ell_2+1}}\cdot\frac{\|M_1^{\ell_3}B^{k_4}M_1^{\ell_4}\|}{|\lambda|^{\ell_3+k_4+\ell_4}} \\
&\le (1.28074503\cdots)(1.46332489\cdots)\frac{\|BM_1^{\ell_2}B\|}{|\lambda|^{1+\ell_2+1}} \\
&= (1.87415580\cdots)\frac{\|BM_1^{\ell_2}B\|}{|\lambda|^{1+\ell_2+1}} \\
&\le \frac{1}{0.5335735\cdots}\frac{\|BM_1^{\ell_2}B\|}{|\lambda|^{1+\ell_2+1}} \\
&= \frac{0.53065091\cdots}{0.53357356\cdots} < 1
\end{aligned}$$

for $\ell_2 \ge 39$ by (2.30). We verify (2.44) directly for $1 \le \ell_1 \le 43, 1 \le \ell_2 \le 38$ and $(k_3,\ell_3,k_4,\ell_4) \in E_2$.

This completes the proof of the lemma. □

Proof of Upper Bounds in Theorem 1.1. For $n \ge 2$, we have

$$\omega_n(k_n) = M\omega_1(k_1),$$

where M is 3×3 matrix in the form of

$$M = T^{\delta_1}B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\cdots B^{k_L}M_1^{\ell_L}B^{k_{L+1}}T^{\delta_2}$$

for $\ell_1,\ell_2,\dots,\ell_L \ge 1,\ k_2,k_3,\cdots,k_L \in \{1,2,3\},\ ,k_1,k_{L+1} \in \{0,1,2,3\}$ and $\delta_1,\delta_2 \in \{0,1\}$. Note that $\|T\| = 1$ and $\|B\| \le |\lambda|$. For $L = 1$, we have

$$\begin{aligned}
\frac{\|M\|}{|\lambda|^{k_1+\ell_1+k_2}} &= \frac{\|T^{\delta_1}B^{k_1}M_1^{\ell_1}B^{k_2}T^{\delta_2}\|}{|\lambda|^{k_1+\ell_1+k_2}} \le \frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}\|}{|\lambda|^{k_1+\ell_1+k_2}} \\
&\le \frac{\|B\|^{k_1}}{|\lambda|^{k_1}}\cdot\frac{\|M_1^{\ell_1}\|}{|\lambda|^{\ell_1}}\cdot\frac{\|B\|^{k_2}}{|\lambda|^{k_2}} \le \frac{\|M_1^{\ell_1}\|}{|\lambda|^{\ell_1}} \\
&\le 1.70493263\cdots
\end{aligned}$$

by Lemma 2.2.

For $L \ge 2$ we have

$$\begin{aligned}
\frac{\|M\|}{|\lambda|^{\sum_{j=1}^{L}(k_j+\ell_j)+k_{L+1}}} &= \frac{\|T^{\delta_1}B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\cdots B^{k_L}M_1^{\ell_L}B^{k_{L+1}}T^{\delta_2}\|}{|\lambda|^{\sum_{j=1}^{L}(k_j+\ell_j)+k_{L+1}}} \\
&\le \frac{\|B^{k_1}M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\cdots B^{k_L}M_1^{\ell_L}B^{k_{L+1}}\|}{|\lambda|^{\sum_{j=1}^{L}(k_j+\ell_j)+k_{L+1}}} \\
&\le \frac{\|M_1^{\ell_1}B^{k_2}M_1^{\ell_2}\cdots B^{k_L}M_1^{\ell_L}\|}{|\lambda|^{\ell_1+\sum_{j=2}^{L}(k_j+\ell_j)}} \\
&\le \frac{\|M_1^{\ell_1}\|}{|\lambda|^{\ell}}\cdot\frac{\|B^{k_2}M_1^{\ell_2}\cdots B^{k_L}M_1^{\ell_L}\|}{|\lambda|^{\sum_{j=2}^{L}(k_j+\ell_j)}} \qquad (2.46)
\end{aligned}$$

because $\|B\| \le |\lambda|$. We now write $L-1 = 4q+r$ with $q \ge 0$ and $r = 1,2,3,4$. Then we have

$$\begin{aligned}
\frac{\|B^{k_2}M_1^{\ell_2}\cdots B^{k_L}M_1^{\ell_L}\|}{|\lambda|^{\sum_{j=2}^{L}(k_j+\ell_j)}} &\le \left(\prod_{j=0}^{q-1}\frac{\|B^{k_{4j+2}}M_1^{\ell_{4j+2}}B^{k_{4j+3}}M_1^{\ell_{4j+3}}B^{k_{4j+4}}M_1^{\ell_{4j+4}}B^{k_{4j+5}}M_1^{\ell_{4j+5}}\|}{|\lambda|^{k_{4j+2}+\ell_{4j+2}+k_{4j+3}+\ell_{4j+3}+k_{4j+4}+\ell_{4j+4}+\ell_{4j+5}+k_{4j+5}}}\right) \\
&\quad\times\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\cdots B^{k_{4q+r+1}}M_1^{\ell_{4q+r+1}}\|}{|\lambda|^{k_{4q+1}+\ell_{4q+1}+\cdots+k_{4q+r+1}+\ell_{4q+r+1}}} \\
&\le \frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\cdots B^{k_{4q+r+1}}M_1^{\ell_{4q+r+1}}\|}{|\lambda|^{k_{4q+1}+\ell_{4q+1}+\cdots+k_{4q+r+1}+\ell_{4q+r+1}}} \qquad (2.47)
\end{aligned}$$

by Lemma 2.12. We now treat the last factor in (2.47). If $r=1$,

$$\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\cdots B^{k_{4q+r+1}}M_1^{\ell_{4q+r+1}}\|}{|\lambda|^{k_{4q+2}+\ell_{4q+2}+\cdots+k_{4q+r+1}+\ell_{4q+r+1}}}=\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\|}{|\lambda|^{k_{4q+2}+\ell_{4q+2}}}\leq 1.28074503\cdots \tag{2.48}$$

by Corollary 2.6 and (2.12). If $r=2$, then

$$\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\cdots B^{k_{4q+r+1}}M_1^{\ell_{4q+r+1}}\|}{|\lambda|^{k_{4q+2}+\ell_{4q+2}+\cdots+k_{4q+r+1}+\ell_{4q+r+1}}}=\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}B^{k_{4q+3}}M_1^{\ell_{4q+3}}\|}{|\lambda|^{k_{4q+2}+\ell_{4q+2}+k_{4q+3}+\ell_{4q+3}}}\leq 1.14952443\cdots \tag{2.49}$$

by Lemma 2.10. If $r=3$, then

$$\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\cdots B^{k_{4q+r+1}}M_1^{\ell_{4q+r+1}}\|}{|\lambda|^{k_{4q+2}+\ell_{4q+2}+\cdots+k_{4q+r+1}+\ell_{4q+r+1}}}\leq 1.02962397\cdots \tag{2.50}$$

by Lemma 2.11. If $r=4$, then

$$\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\cdots B^{k_{4q+r+1}}M_1^{\ell_{4q+r+1}}\|}{|\lambda|^{k_{4q+2}+\ell_{4q+2}+\cdots+k_{4q+r+1}+\ell_{4q+r+1}}}=\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\cdots B^{k_{4q+5}}M_1^{\ell_{4q+5}}\|}{|\lambda|^{k_{4q+2}+\ell_{4q+2}+\cdots+k_{4q+5}+\ell_{4q+51}}}\leq 1 \tag{2.51}$$

by Lemma 2.12. In view of (2.48) - (2.51), we have

$$\frac{\|B^{k_{4q+2}}M_1^{\ell_{4q+2}}\cdots B^{k_{4q+r+1}}M_1^{\ell_{4q+r+1}}\|}{|\lambda|^{k_{4q+2}+\ell_{4q+2}+\cdots+k_{4q+r+1}+\ell_{4q+r+1}}}\leq 1.28074503\cdots \tag{2.52}$$

for $q\geq 0$ and $r=1,2,3,4$.

Therefore by (2.46), (2.47) and Lemma 2.2, we have

$$\begin{aligned}\frac{\|M\|}{|\lambda|^{\sum_{j=1}^{L}(k_j+\ell_j)+k_{L+1}}} &\leq \frac{\|M_1^{\ell_1}\|}{|\lambda|^{\ell_1}}(1.28074503\cdots)\\ &\leq (1.70493263\cdots)(1.28074503\cdots)=2.18358400\cdots\end{aligned}$$

and

$$\|M\|\leq(2.18358400\cdots)|\lambda|^{\sum_{j=1}^{L}(k_j+\ell_j)+k_{L+1}}\leq(2.18358400\cdots)|\lambda|^n$$

by (2.4). Recall (2.1) that

$$\omega_n(k_n)=\begin{pmatrix}\left(|P_n|^2\right)^\wedge(k_n)\\ \left(\overline{P_n}Q_n\right)^\wedge(k_n')\\ \left(P_n\overline{Q_n}\right)^\wedge(k_n')\end{pmatrix}.$$

We have proved that

$$\begin{aligned}\left|\left(|P_n|^2\right)^\wedge(k_n)\right|,\left|\left(\overline{P_n}Q_n\right)^\wedge(k_n')\right| &\leq \|\omega_n(k_n)\|=\|M\omega_1(k_1)\|\leq\|M\|\cdot\|\omega_1(k_1)\|\\ &\leq \sqrt{3}(2.18358400\cdots)|\lambda|^n\leq(3.78207844\cdots)|\lambda|^n.\end{aligned}$$

This proves the upper bounds in Theorem 1.1.

3. Lower bound for the autocorrelation coefficients

This is in fact the proof for the lower bounds in [1]. Since now the values of K_1 are relevant, we carry out the same proof to compute the values of K_1. Let

$$k_n=\frac{1}{3}(2L_n+(-1)^\ell)\quad\text{and}\quad k_n':=k_n-L_n$$

and define

$$\omega_n(k_n):=\begin{pmatrix}\left(|P_n|^2\right)^\wedge(k_n)\\ \left(\overline{P_n}Q_n\right)^\wedge(k_n')\\ \left(P_n\overline{Q_n}\right)^\wedge(k_n')\end{pmatrix}.$$

In view of (3.2), (3.3) and (3.4) in [1], we have

$$\omega_n=M_2\omega_{n-2}$$

for $n \geq 2$ where $M_2 = \begin{pmatrix} 2 & -1 & 0 \\ 2 & 1 & 2 \\ -2 & -1 & 2 \end{pmatrix}$ and $\Lambda^2 = {S'}^{-1} M_2 S'$ where Λ is the 3×3 diagonal matrix with diagonal entries $\lambda, \lambda', \overline{\lambda'}$. Here

$$S' := \begin{pmatrix} s_5 & s_6 & \overline{s_6} \\ s_7 & s_8 & \overline{s_8} \\ 1 & 1 & 1 \end{pmatrix},$$

and

$$s_5 := \frac{-11\left(1+\sqrt{177}\right)\left(71+6\sqrt{177}\right)^{1/3} - \left(103-7\sqrt{177}\right)\left(71+6\sqrt{177}\right)^{2/3} + 242}{1452},$$

$$s_6 := \frac{11\left(1-i\sqrt{3}\right)\left(\sqrt{177}+1\right)\left(71+6\sqrt{177}\right)^{1/3} + \left(1+i\sqrt{3}\right)\left(103-7\sqrt{177}\right)\left(71+6\sqrt{177}\right)^{2/3} + 484}{2904},$$

$$s_7 := \frac{11\left(-21+\sqrt{177}\right)\left(71+6\sqrt{177}\right)^{1/3} + \left(-39+5\sqrt{177}\right)\left(71+6\sqrt{177}\right)^{2/3}}{726},$$

$$s_8 := \frac{11\left(1-i\sqrt{3}\right)\left(21-\sqrt{177}\right)\left(71+6\sqrt{177}\right)^{1/3} + \left(1+i\sqrt{3}\right)\left(39-5\sqrt{177}\right)\left(71+6\sqrt{177}\right)^{2/3}}{1452}.$$

If n is even, then inductively, we have

$$\omega_n = M_1^{n/2}\omega_0$$

where $\omega_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ because $|P_0|^2(z) = (\overline{P_0}Q_0)(z) = (P_0\overline{Q_0})(z) = 1$ and $k_0 = 1, k_0' = 0$. We have

$$\omega_n = S'(\Lambda^2)^{n/2}{S'}^{-1}\omega_0 = S'\Lambda^n {S'}^{-1}\omega_0.$$

Therefore, by computing $S'\Lambda^n {S'}^{-1}\omega_0$ explicitly using `Maple`, we have

$$\begin{aligned} & \left|\left(|P_n|^2\right)^{\wedge}\left(\frac{1}{3}(2L_n+1)\right)\right| \\ = \quad & \left|a_1\lambda^\ell + b_1(\lambda')^\ell + \overline{b_1}(\overline{\lambda'})^\ell\right| \geq |a_1|\lambda^\ell - 2|b_1||\lambda'|^\ell \geq 0.38215952\cdots(1+o(1))|\lambda^\ell| \end{aligned}$$

where $a_1 = -0.38215952\cdots$ and $b_1 = 0.19107976\cdots + i0.88541019\cdots$ and $|\lambda| > |\lambda'|$. Similarly, we have

$$\begin{aligned} & \left|\left(P_n\overline{Q_n(z)}\right)^{\wedge}(k_n')\right| \\ = \quad & \left|a_2\lambda^n + b_2(\lambda')^n + \overline{b_2}(\overline{\lambda'})^n\right| \geq |a_2|\lambda^n - 2|b_2||\lambda'|^n \geq 0.63399007\cdots(1+o(1))|\lambda|^n \end{aligned}$$

where $a_2 = 0.63399007\cdots$ and $b_1 = 0.18300496\cdots + i0.27100843\cdots$

If n is odd, then inductively, we have

$$\omega_n = M_1^{(n-1)/2}\omega_1$$

where $\omega_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ because

$$|P_1|^2(z) = \overline{z} + 1 + z, \quad (\overline{P_1}Q_1)(z) = \overline{z} - z, \quad (P_1\overline{Q_1})(z) = -\overline{z} + z$$

and $k_1 = 1, k_1' = -1$. We have

$$\omega_n = S'\Lambda^{n-1}{S'}^{-1}\omega_1.$$

Therefore, we have

$$\begin{aligned}
&\left|\left(|P_n|^2\right)^\wedge\left(\frac{1}{3}(2L_n-1)\right)\right| = \left|a'\lambda^{(n-1)} + b'(\lambda')^{(n-1)} + \overline{b'}(\overline{\lambda'})^{(n-1)}\right| \\
= \quad & \left|(a'\lambda^{-1})\lambda^n + (b'\lambda'^{-1})(\lambda')^n + (\overline{b'}\lambda'^{-1})(\overline{\lambda'})^n\right| \geq (|a'|\lambda^{-1})\lambda^n - (2|b'||\lambda'|^{-1})|\lambda'|^n \\
\geq \quad & 0.27771487\cdots(1+o(1))|\lambda|^n
\end{aligned}$$

where $a_1' = 0.46071984\cdots$ and $b_1' = 0.18300496\cdots + i0.27100843\cdots$ and

$$\begin{aligned}
&\left|\left(P_n\overline{Q_n(z)}\right)^\wedge(k_n')\right| = \left|a_2'\lambda^{(n-1)} + b_2'(\lambda')^{(n-1)} + \overline{b_2'}(\overline{\lambda'})^{(n-1)}\right| \\
= \quad & \left|(a_2'\lambda^{-1})\lambda^n + (b_2'\lambda'^{-1})(\lambda')^n + (\overline{b_2'}\lambda'^{-1})(\overline{\lambda'})^n\right| \geq (|a_2'|\lambda^{-1})\lambda^n - (2|b_2'||\lambda'|^{-1})|\lambda'|^n \\
\geq \quad & 0.46071984\cdots(1+o(1))|\lambda|^n
\end{aligned}$$

where $a_2' = -0.76431905\cdots$ and $b_2' = 0.25884331\cdots + i0.50711782\cdots$.

Therefore,

$$\max_{1\leq k\leq L_n-1}|a_k| \geq \left|\left(|P_n|^2\right)^\wedge\left(\frac{1}{3}(2L_n+1)\right)\right| \geq 0.27771487\cdots(1+o(1))|\lambda|^n.$$

and

$$\max_{1\leq k\leq L_n-1}|b_k| \geq \left|\left(P_n\overline{Q_n(z)}\right)^\wedge\left(\frac{1}{3}(-L_n+(-1)^n)\right)\right| \geq 0.46071984\cdots(1+o(1))|\lambda|^n.$$

This proves the lower bounds for Theorem 1.1.

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Stephen Choi, Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, British Columbia V5A 1S6, Canada

E-mail address: `schoia@sfu.ca`