SMALL PRIME SOLUTIONS OF LINEAR EQUATIONS II

KWOK-KWONG STEPHEN CHOI, MING-CHIT LIU AND KAI-MAN TSANG

1. Introduction

In one of their well-known series of papers, "Partitio Numerorum", [3] Hardy and Littlewood proved that, under the Generalized Riemann Hypothesis (GRH), every sufficiently large odd natural numbers n is a sum of 3 odd primes, i.e., if $n \geq n_0$ is odd then the equation

$$n = p_1 + p_2 + p_3 \tag{1.1}$$

is soluble in odd primes p_1, p_2, p_3. Later in 1937 I.M. Vinogradov was able to prove this result without assuming the GRH. Initiated by a diophantine problem considered by A. Baker [1, Lemma 6], the last two authors took a further step by introducing coefficients to each term on the right side of (1.1) and consider the equation

$$b = a_1p_1 + a_2p_2 + a_3p_3, \tag{1.2}$$

where a_1, a_2, a_3 are nonzero integers satisfying

$$(a_1, a_2, a_3) = 1 \tag{1.3}$$

and b is any integer satisfying

$$(b, a_i, a_j) = 1 \text{ for } 1 \leq i < j \leq 3 \tag{1.4}$$

and

$$b \equiv a_1 + a_2 + a_3 \pmod 2. \tag{1.5}$$

Here (1.3) to (1.5) are the usual conditions of congruent solubility of (1.2). Baker's problem led not only to a generalization of the Vinogradov three primes theorem (ie., to obtain a lower bound b_0 for b in terms of a_1, a_2, a_3 such that the equation (1.2) is soluble if $b \geq b_0$) but also to a pioneering investigation on the size of small prime solutions of (1.2) (ie, to obtain an upper bound in terms of a_1, a_2, a_3, b for p_1p_2, p_3 in (1.2)). The latest results on these two problems were obtained by Liu and Tsang [5]. They proved:

Theorem LT. *Subject to the conditions (1.3), (1.4) and (1.5) there exists an effective absolute constant $A > 1$ such that*

(i) *if a_1, a_2, a_3 are all positive, then the equation (1.2) is soluble in primes p_1, p_2, p_3 whenever*

$$b \geq (3\max(a_1, a_2, a_3))^A; \tag{1.6}$$

(ii) *if a_1, a_2, a_3 are not all of the same sign, then (1.2) has a solution p_1, p_2, p_3 satisfying*

$$\max(p_1, p_2, p_3) \leq 3|b| + (3\max(|a_1|, |a_2|, |a_3|))^A. \tag{1.7}$$

The constant A in Theorem LT has an interesting connection with the Linnik constant L. This can be seen by considering the following 2 examples. Let l, q be coprime positive integers with $l \leq q$. If we take $a_1 = 1, a_2 = a_3 = -q$ and $b = l$ or $l + q$ according as l is odd or even, then a solution of (1.2) satisfying the bound (1.7) gives a prime p_1 in the arithmetical progression, $l + kq, k = 0, 1, 2 \cdots$ such that $p_1 \ll q^A$. So the A in (1.7) is not less than the Linnik constant L. On the other hand, for the above l and q, applying Theorem LT (i) to the example, $a_1 = a_2 = q, a_3 = q + 1$ and $b = l + kq$ where k is the smallest integer such that (1.5) and (1.6) hold, we see that the p_3 in each solution of (1.2) is congruent to $l \pmod{q}$ and $p_3 < ba_3^{-1} < 2(3^A(q+1)^{A-1})$. This shows that the constant A in (1.6) is not less than $L + 1$.

Based on these 2 examples we see that the bounds in (1.6) and (1.7) are of the right order of infinity and it remains only to determine the numerical values of the A's in (1.6) and (1.7). Recently the first author has shown that the A's in Theorem LT do not exceed 4191 and we believe that the exact value of A is close to L. In this paper we shall investigate this problem under the GRH. We shall prove the following.

Theorem. *Assuming the GRH, we can replace the bounds in (1.6) and (1.7) respectively by*

$$b \gg f(a_1, a_2, a_3) \tag{1.8}$$

and

$$\max_{1 \le j \le 3} (|a_j| p_j) \ll |b| + f(a_1, a_2, a_3) \tag{1.9}$$

where

$$f(a_1, a_2, a_3) := |a_1 a_2|^2 |a_3| \log^{10} \left(3 \max_{1 \le j \le 3} (|a_j|) \right).$$

Applying (1.9) to the above first example we deduce $L \le 3$, which is just slightly weaker than the known estimate $L \le 2$ for the Linnik constant under the GRH.

Another remarkable feature of the bounds in (1.8) and (1.9) is that, the coefficients a_3 occurs only to the first power. If we keep the coefficients a_1, a_2 bounded and let a_3 vary, then by our theorem the equation (1.2) has solutions in which p_3 grows not faster than $\log^{10} 3|a_3|$, which is much slower than $|a_3|$.

The proof of our theorem, like that of Theorem LT, is based on the circle method. While the proof of Theorem LT pays little attention to the economy of the value of A, the emphasis of the present work is on the reduction of the value of A. In order to achieve the sharpest possible bounds, we have to restructure much of the previous arguments and, among other things, the major arcs are chosen in a more delicate manner.

2. Notation and Some Preliminary Lemmas

Throughout this paper, p, with or without suffixes, always denotes a prime. ε is a sufficiently small positive number and $N \ge N_0(\varepsilon)$ is a large number which is the main parameter in the whole proof. We use $c_1, c_2, \cdots$ etc. to denote positive absolute constants. The constants implied in the O and $\ll$ symbols are also absolute. Without loss of generality, we may assume that $a_3 > 0$. Write $e(x)$ for $e^{2\pi i x}$ and $e_q(x) = e(x/q)$. Let

$$\lambda(n) := \begin{cases} \log p & \text{if } n = p, \text{ a prime,} \\ 0 & \text{otherwise.} \end{cases}$$

For any Dirichlet character $\chi \pmod{q}$ and any integer m we let

$$C_\chi(m) := \sum_{l=1}^{q} \chi(l) e_q(ml).$$

When $\chi = \chi_0$, the principal character modulo q, we write $C_q(m)$ for $C_{\chi_0}(m)$. It is well known that [4, Theorem 272] $C_q(m)$ is multiplicative in q and

$$C_q(m) = \mu(q/(m,q))\phi(q)\phi(q/(m,q))^{-1}. \tag{2.1}$$

Lemma 1. *Let $x = hq^{-1} + \eta$ where h, q are integers, $1 \le q \ll Y$ and $|\eta| \ll (\log Y)^{-2}$. Let $0 < c < 1$ be any constant. Then under the GRH*

$$\sum_{cY<n\le Y} \lambda(n)e(nx) = C_q(h)\phi(q)^{-1}\int_{cY}^{Y} e(y\eta)dy + O\left((Yq)^{\frac12}\log^2 Y + Y(q|\eta|)^{\frac12}\log Y\right) \tag{2.2}$$

and

$$\sum_{cY<n\le Y} \lambda(n)e_q(nh) = (1-c)YC_q(h)\phi(q)^{-1} + \\ + \phi(q)^{-1}\sum_{\chi \pmod q} C_{\overline{\chi}}(h)\Phi_\chi(Y) + O(\log^2 Y) \tag{2.3}$$

where $\Phi_\chi(Y)$ is independent of h and

$$\Phi_\chi(Y) \ll Y^{\frac12}\log^2 Y. \tag{2.4}$$

Proof. First of all, by the orthogonality relation of $\chi \pmod q$ we have

$$\sum_{cY<n\le Y} \lambda(n)e(nx) = \phi(q)^{-1}\sum_{\chi \pmod q} C_{\overline{\chi}}(h)\sum_{cY<n\le Y}\lambda(n)\chi(n)e(ny) + O(\log^2 Y). \tag{2.5}$$

It is well known that [2, Chapter 19] if $2 \le T \le y$ that

$$\psi(y,\chi) := \sum_{n\le y}\Lambda(n)\chi(n) = \delta_\chi y - \sum_{|\gamma|\le T} y^\rho\rho^{-1} + O(yT^{-1}\log^2(qy)), \tag{2.6}$$

where $\Lambda(n)$ is the von Mangoldt function, $\rho = \beta + i\gamma$ denotes the nontrivial zeros of the L-function $L(s,\chi)$ and $\delta_\chi = 1$ if $\chi = \chi_0$ and $\delta_\chi = 0$ otherwise. Taking now $T = cY$ in (2.6) and using the Riemann Stieltjes integration, we find that

$$\begin{aligned} \sum_{cY<n\le Y}\lambda(n)\chi(n)e(n\eta) &= \sum_{cY<n\le Y}\Lambda(n)\chi(n)e(n\eta) + O(Y^{\frac12}) \\ &= \int_{cY}^{Y} e(y\eta)d\psi(y,\chi) + O(Y^{\frac12}) \\ &= \delta_\chi\int_{cY}^{Y} e(y\eta)dy - \sum_{|\gamma|\le cY}\int_{cY}^{Y} y^{\rho-1}e(y\eta)dy \\ &\quad + O((1+Y|\eta|)\log^2 Y) + O(Y^{\frac12}). \end{aligned} \tag{2.7}$$

To bound the second term above, we use the following result from [5, Lemma 3.2]:

$$\int_{cY}^{Y} y^{\rho-1}e(y\eta)dy \ll \begin{cases} Y^\beta|\gamma|^{-1} & \text{if } |\gamma| \ge |\eta|4\pi Y := u_1, \\ Y^\beta|\gamma|^{-\frac12} & \text{if } u_2 \le |\gamma| < u_1, \\ Y^{\beta-1}|\eta|^{-1} & \text{if } |\gamma| < |\eta|c\pi Y := u_2. \end{cases}$$

Under the GRH, $\beta = \frac{1}{2}$. Hence, when $Y|\eta| \geq 1$

$$\begin{aligned}\sum_{|\gamma|\leq cY} \int_{cY}^{Y} y^{\rho-1} e(y\eta)dy &\ll \sum_{|\gamma|\ll Y|\eta|} Y^{-\frac{1}{2}}|\eta|^{-1} + \sum_{|\gamma|\sim Y|\eta|} Y^{\frac{1}{2}}|\gamma|^{-\frac{1}{2}} + \sum_{Y|\eta|\ll|\gamma|\leq cY} Y^{\frac{1}{2}}|\gamma|^{-1} \\ &\ll Y^{\frac{1}{2}}\log^2 Y + Y|\eta|^{\frac{1}{2}}\log Y.\end{aligned}$$

Here we need the well-known zero counting formula for $L(s,\chi)$ [2, (1) in Chapter 16], ie., for $t \geq 2$

$$\sum_{|\gamma|\leq t} 1 = \frac{t}{\pi}\log\left(\frac{qt}{2\pi}\right) + O(t\log q). \tag{2.8}$$

When $Y|\gamma| < 1$ the same estimate on $\sum_{|\gamma|\leq cY} \int_{cY}^{Y} y^{\rho-1}e(y\eta)dy$ still holds, by writing $\sum_{|\gamma|\leq cY} = \sum_{|\gamma|\ll 1} + \sum_{1\ll|\gamma|\leq cY}$ and using the trivial bound $\int_{cY}^{Y} y^{-\frac{1}{2}+i\gamma}e(y\eta)dy \ll Y^{\frac{1}{2}}$ for the first sum $\sum_{|\gamma|\leq 1}$. Hence (2.7) becomes

$$\sum_{cY<n\leq Y} \lambda(n)\chi(n)e(n\eta) = \delta_\chi \int_{cY}^{Y} e(y\eta)dy + O(Y^{\frac{1}{2}}\log^2 Y + Y|\eta|^{\frac{1}{2}}\log Y). \tag{2.9}$$

In the proof of (2.2) we also need the following equality. For any integer h we have

$$\sum_{\chi \pmod q} |C_\chi(h)|^2 = \sum_{\substack{l_1=1,l_2=1\\(l_1,q)=1=(l_2,q)}}^{q} e_q((l_1-l_2)h)\sum_{\chi}\chi(l_1)\overline{\chi}(l_2) = \phi^2(q), \tag{2.10}$$

by the orthogonality relation of characters.

Substituting (2.9) into (2.5), and then using estimate $\sum_{\chi}|C_\chi(h)| \ll \phi(q)^{\frac{3}{2}}$ which follows from (2.10) by applying the Schwarz inequality, we obtain (2.2).

When $\eta = 0$ substituting (2.7) with $\eta = 0$ directly into (2.5), we arrive at (2.3) with $\Phi_\chi(Y) = \sum_{|\gamma|\leq cY}((cY)^\rho - Y^\rho)\rho^{-1} + O(Y^{\frac{1}{2}})$ which, by the GRH and (2.8), is $\ll Y^{\frac{1}{2}}\log^2 Y$. This completes the proof of Lemma 1. □

Lemma 2. *Let u, v, v' be integers. For any coprime positive integers s, m we have*

$$\sum_{\substack{h=1\\(h,m)=1}}^{ms} e_{ms}(uh)C_{ms}(vh)C_{ms}(v'h) = Z_s(u,v,v')Z'_m(u,v,v') \tag{2.11}$$

where

$$\begin{aligned}Z_s(u,v,v') &:= \sum_{h=1}^{s} e_s(uh)C_s(vh)C_s(v'h) \\ &= s \times \mathit{card}\{l,l' : 1\leq l,l'\leq s, (l,s)=1=(l',s), vl+v'l'+u\equiv 0 \pmod s\}, \\ Z'_m(u,v,v') &:= \sum_{\substack{k=1\\(k,m)=1}}^{m} e_m(uk)C_m(k)C_m(v'k) = C_m(u)C_m(v)C_m(v').\end{aligned}$$

Furthermore, Z_s and Z'_m are multiplicative functions in s and m respectively.

Proof. Write $h = r_1m + r_2s$. Then when r_1 runs through a complete residue system modulo s and r_2 runs through a reduced residue system modulo m, h will run through the set

$$\{1 \le n \le ms : (n, m) = 1\}$$

modulo ms. Hence the sum in (2.11) is equal to

$$\sum_{r_1=1}^{s} \sum_{\substack{r_2=1 \\ (r_2,m)=1}}^{m} e_{ms}(u(r_1m + r_2s))C_{ms}(v(r_1m + r_2s))C_{ms}(v'(r_1m + r_2s)).$$

Since $(m, s) = 1$, by (2.1) $C_{ms}(t) = C_m(t)C_s(t)$. Also $C_m(t+nm) = C_m(t)$ and $C_m(tl) = C_m(t)$ if $(m, l) = 1$. Using this, we easily see that the above sum splits into $Z_sZ'_m$ as in (2.11). That Z_s and Z'_m are multiplicative functions can be proved by similar arguments. □

For any positive integer s, let

$$\mathcal{N}(s) := \{l_1, l_2 : 1 \le l_1, l_2 \le s, (l_1, s) = 1 = (l_2, s), a_1l_1 + a_2l_2 \equiv b \pmod{s}\}. \tag{2.12}$$

Clearly, $\mathcal{N}(s) = s^{-1}Z_s(-b, a_1, a_2)$ so that $\mathcal{N}(s)$ is a multiplicative function of s. Suppose $p^\alpha \| a_3$ $(\alpha \ge 1)$. Then by (1.3) we may assume $p \nmid a_1$, say. Since each l_2 determines exactly one l_1 from the congruence $l_1 \equiv a_1^{-1}(b - a_2l_2) \pmod{p^\alpha}$, we see that $\mathcal{N}(p^\alpha) = \phi(p^\alpha) - n$ where n is the number of l_2 satisfying $1 \le l_2 \le p^\alpha, (l_2, p) = 1$ and $a_2l_2 \equiv b \pmod{p}$. In view of (1.4) (ie., $(a_2, a_3, b) = 1$) it is easily seen that $n = p^{\alpha-1}$ if $p \nmid a_2b$ and $n = 0$ otherwise. Hence $\phi(p^\alpha) \ge \mathcal{N}(p^\alpha) \ge p^{\alpha-1}(p - 2) = p^{-\alpha}\phi(p^\alpha)^2(1 - (p - 1)^{-2})$. If $p = 2|a_3$ then by (1.5), $n = 0$ so that $\mathcal{N}(2^\alpha) = 2^{\alpha-1}$. Combining these estimates, we have

Lemma 3. *We have*

$$\phi(a_3) \ge \mathcal{N}(a_3) \gg a_3^{-1}\phi(a_3)^2.$$

Lemma 4. *We have*

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{a_3q} e_{a_3q}(uh) = \begin{cases} a_3C_q(u/a_3) & \text{if } a_3|u, \\ \\ 0 & \text{if } a_3 \nmid u. \end{cases}$$

Proof. The first case follows immediately from (2.1). Suppose now that $a_3 \nmid u$. Writing $a_3q = sm$ where s is the largest divisor of a_3 that is coprime with q, then by (2.11) with $v = v' = 0$, we have

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{a_3q} e_{a_3q}(uh) = \phi(a_3q)^{-2}Z_s(u, 0, 0)Z'_m(u, 0, 0) = C_m(u)\sum_{h=1}^{s} e_s(uh). \tag{2.13}$$

The sum on the right hand side vanishes if$s \nmid u$. By (2.1), $C_m(u)$ will also vanish if $m/(m, u)$ is not square-free. Let $m' = m/q$, the largest divisor of a_3 whose prime factors all appear in q. Then $m/(m, u) = m'q/(m'q, u)$. It is not square-free if $m' \nmid (m'q, u)$ ie., if $m' \nmid u$. Now $a_3 = sm' \nmid u$. Hence either $C_m(u)$ or the sum on the right side of (2.13) vanishes. This proves our lemma. □

3. Proof of Theorem

Let

$$\tau := \sqrt{a_3/N}\log N, \qquad Q := |a_1a_2|^{\frac{1}{2}}\log^2 N, \tag{3.1}$$

and assume throughout that

$$N\log^{-10} N \geq \varepsilon^{-4}(a_1a_2)^2a_3. \tag{3.2}$$

So

$$2Q\tau < 1 \text{ and } N\tau \geq a_3Q. \tag{3.3}$$

For $j = 1,2,3$ let

$$S_j(x) := \sum_{N_j' < n \leq N_j} \lambda(n)e(na_jx)$$

where $N_j := c_jN|a_j|^{-1}, N_j' := c_j'N|a_j|^{-1}$ and $c_j > c_j' > 0$ are constants to be determined later in (3.9) and (3.10). Put

$$I(N) := \int_{\tau a_3^{-1}}^{1+\tau a_3^{-1}} S_1(x)S_2(x)S_3(x)e(-bx)dx. \tag{3.4}$$

The aim of the proof below is to show that $I(N) \gg N^2|a_1a_2a_3|^{-1}$. As usual with the circle method, we begin by specifying the major arcs $\mathcal{M}$ to be the union of the intervals $m(h,q) = \left[\frac{h-\tau}{a_3q}, \frac{h+\tau}{a_3q}\right)$, where $1 \leq h, q \leq Q$ are coprime integers such that $h \leq a_3q$. With the help of (3.3) we see that these $m(h,q)$ are pairwise disjoint subintervals of $[\tau a_3^{-1}, 1+\tau a_3^{-1})$. Let $\mathcal{M}'$ be the complement of $\mathcal{M}$ in $[\tau a_3^{-1}, 1+\tau a_3^{-1})$.

For any $x \in [\tau a_3^{-1}, 1+\tau a_3^{-1})$, by Dirichlet's theorem on diophantine approximation there exist coprime integers h, q such that $1 \leq q \leq \tau^{-1}$ and $\eta := a_3x - hq^{-1}$ satisfies $|\eta| < \tau q^{-1}$. In view of (3.2) all the hypotheses in Lemma 1 with $Y = N_3$ are satisfied. Applying Lemma 1 to $S_3(x)$, we deduce from (2.2) that

$$S_3(x) = C_q(h)\phi(q)^{-1}\int_{N_3'}^{N_3} e(y\eta)dy + O((N_3q)^{\frac{1}{2}}\log^2 N + N_3(q|\eta|)^{\frac{1}{2}}\log N). \tag{3.5}$$

Since $\tau \leq a_3x$ we see that $1 \leq h$. Now if $x \in \mathcal{M}'$ then $q > Q$. Indeed, if $q \leq Q$ then, by $a_3x \leq a_3 + \tau$, (3.3) and $-\tau/q < a_3x - hq^{-1}$, we have $h \leq a_3q$ which contradicts that $x \notin \mathcal{M}$. Using (3.5), (2.1) and (3.1) we have

$$S_3(x) \ll N_3\phi(q)^{-1} + (N_3/\tau)^{\frac{1}{2}}\log^2 N + N_3\tau^{\frac{1}{2}}\log N \ll N_3Q^{-1}\log\log N + (N_3)^{\frac{3}{4}}\log^{\frac{3}{2}} N$$

for $x \in \mathcal{M}'$. Hence

$$\int_{\mathcal{M}'} S_1(x)S_2(x)S_3(x)e(-bx)dx \ll (N_3Q^{-1}\log\log N + (N_3)^{\frac{3}{4}}\log^{\frac{3}{2}} N) \times \times \int_{\mathcal{M}'} |S_1(x)S_2(x)|dx. \tag{3.6}$$

Clearly, by Schwarz's inequality, for any subset J of $[\tau a_3^{-1}, 1+\tau a_3^{-1})$,

$$\int_J |S_1(x)S_2(x)|dx \leq \left\{\int_{\tau a_3^{-1}}^{1+\tau a_3^{-1}} |S_1(x)|^2dx \int_{\tau a_3^{-1}}^{1+\tau a_3^{-1}} |S_2(x)|^2dx\right\}^{\frac{1}{2}} \ll N|a_1a_2|^{-\frac{1}{2}}\log N, \tag{3.7}$$

since

$$\int_{\tau a_3^{-1}}^{1+\tau a_3^{-1}} |S_j(x)|^2 dx = \sum_{\substack{N_j' < n,k \le N_j \\ n=k}} \lambda(n)\lambda(k) \ll N_j \log N.$$

Thus, in view of (3.1) and (3.2), (3.6) and (3.7) give the bound

$$\int_{\mathcal{M}'} S_1(x)S_2(x)S_3(x)e(-bx)dx \ll \varepsilon N^2 |a_1a_2a_3|^{-1}.$$

So we can now write (3.4) as

$$I(N) = \sum_{q \le Q} \sum_{\substack{h=1 \\ (h,q)=1}}^{a_3 q} \int_{m(h,q)} S_1(x)S_2(x)S_3(x)e(-bx)dx + O(\varepsilon N^2 |a_1a_2a_3|^{-1}).$$

For each $x \in m(h,q)$ write $x = h(a_3q)^{-1} + \eta a_3^{-1}$ with $|\eta| \le \tau q^{-1}$. Then (3.5) still holds. Similar to the argument for (3.6), the contribution to $I(N)$ from the O-term in (3.5) is

$$\ll ((N_3Q)^{\frac{1}{2}} \log^2 N + N_3 \tau^{\frac{1}{2}} \log N) \int_{\mathcal{M}} |S_1(x)S_2(X)|dx \ll \varepsilon N^2 |a_1a_2a_3|^{-1}.$$

Hence we have

$$\begin{aligned} I(N) &= a_3^{-1} \sum_{q \le Q} \sum_{\substack{h=1 \\ (h,q)=1}}^{a_3 q} e_{a_3 q}(-bh) \int_{-\tau/q}^{\tau/q} S_1(h(a_3q)^{-1} + \eta a_3^{-1}) S_2(h(a_3q)^{-1} + \eta a_3^{-1}) \times \\ &\quad \times C_q(h)\phi(q)^{-1} \int_{N_3'}^{N_3} e(y\eta)dy e_{a_3}(-b\eta)d\eta + O(\varepsilon N^2 |a_1a_2a_3|^{-1}) \\ &= a_3^{-1} \sum_{q \le Q} \mu(q)\phi(q)^{-1} \sum_{N_1' < n_1 \le N_1} \sum_{N_2' < n_1 \le N_2} \lambda(n_1)\lambda(n_2) \times \\ &\quad \times \sum_{\substack{h=1 \\ (h,q)=1}}^{a_3 q} e_{a_3 q}(h(a_1n_1 + a_2n_2 - b)) \times \\ &\quad \times \int_{N_3'}^{N_3} \int_{-\tau/q}^{\tau/q} e_{a_3}(\eta(a_1n_1 + a_2n_2 + a_3y - b)) d\eta dy + O(\varepsilon N^2 |a_1a_2a_3|^{-1}). \end{aligned} \tag{3.8}$$

The double integral above is equal to $\pi^{-1} \int_\alpha^\beta t^{-1} \sin t dt$ where

$$\alpha := 2\pi\tau(a_1n_1 + a_2n_2 + a_3N_3' - b)(a_3q)^{-1},$$

$$\beta := 2\pi\tau(a_1n_1 + a_2n_2 + a_3N_3 - b)(a_3q)^{-1}.$$

We now choose the constants c_j, c_j' as follows:

(3.9) (i) When a_1, a_2, a_3 are all positive, put $b = 3N/2, c_j' = \frac{1}{4}$ for $j = 1, 2, 3, c_3 = \frac{5}{4}$ and $c_1c_2 = \dfrac{1}{2}$;

(3.10) (ii) when a_1 or a_2 is negative, say $a_1 < 0$ then put $c_1 = 32, c_1' = 28, c_2 = 2$ $c_2' = 1, c_3 = 48, c_3' = 12$ and assume $|b| \le 12N$.

This choice of the c_j, c_j' and b ensure that, in both cases,

$$\beta \ge \pi\tau N(2a_3q)^{-1} \text{ and } \alpha \le -\pi\tau N(2a_3q)^{-1}.$$

Hence,

$$\pi^{-1}\int_\alpha^\beta t^{-1}\sin t dt = 1 + O(|\alpha|^{-1}+|\beta|^{-1}) = 1 + O(a_3q(\tau N)^{-1}).$$

Substituting this into (3.8), we obtain

$$\begin{aligned}(3.11)\qquad I(N) \;=\; & q_3^{-1}\sum_{q\le Q}\mu(q)\phi(q)^{-1}\sum_{\substack{h=1\\(h,q)=1}}^{a_3q} e_{a_3q}(-bh)S_1(h(a_3q)^{-1})S_2(h(a_3q)^{-1}) + \\ & +O\Bigg((\tau N)^{-1}\sum_{q\le Q}q\phi(q)^{-1}\sum_{N_1'<n_1\le N_1}\sum_{N_2'<n_1\le N_2}\lambda(n_1)\lambda(n_2)\times \\ & \times\Bigg|\sum_{\substack{h=1\\(h,q)=1}}^{a_3q} e_{a_3q}(h(a_1n_1+a_2n_1-b))\Bigg|\Bigg) + O(\varepsilon N^2|a_1a_2a_3|^{-1}).\end{aligned}$$

Let E denote the last 3 sums on n_1, n_2, h in the above first O-term. Then by Lemma 4

$$\begin{aligned}(a_3\phi(q))^{-1}E \;&\ll \sum_{\substack{N_1'<n_1\le N_1, N_2'<n_1\le N_2\\ a_3|a_1n_1+a_2n_2-b}}\lambda(n_1)\lambda(n_2) \\ &= \sum_{\substack{N_j'<n_j\le N_j\\(n_j,a_3)=1, j=1,2,\\ a_3|a_1n_1+a_2n_2-b}}\lambda(n_1)\lambda(n_2) + O((N_1+N_2)\log^2 N) \\ &\ll \sum_{\substack{l_1,l_2=1\\(l_1,a_3)=1=(l_2,a_3)\\ a_3|a_1l_1+a_2l_2-b}}^{a_3}\sum_{\substack{N_j'<n_j\le N_j, j=1,2\\ n_j\equiv l_j \pmod{a_3}}}\log^2 N + (N_1+N_2)\log^2 N \\ &\ll \sum_{\substack{l_1,l_2=1\\(l_1,a_3)=1=(l_2,a_3)\\ a_3|a_1l_1+a_2l_2-b}}^{a_3} N_1N_2a_3^{-2}\log^2 N + (N_1+N_2)\log^2 N \\ &\ll (\mathcal{N}(a_3)a_3^{-2}N_1N_2 + N_1 + N_2)\log^2 N,\end{aligned}$$

by (2.12). Using the upper bound $\mathcal{N}(a_3) \le \phi(a_3)$ in Lemma 3, and (3.1), (3.2), we find that

$$(3.12)\qquad \text{the first } O\text{-term on (3.11)} \ll \varepsilon N^2|a_1a_2a_3|^{-1}.$$

The main term for $I(N)$ in (3.11) is to be treated by applying (2.3) to $S_j(h(a_3q)^{-1}), j = 1, 2$. In view of (2.3) we write, for $j = 1, 2$

$$S_j(h(a_3q)^{-1}) = M_j + R_j$$

where

$$(3.13)\qquad \begin{cases} M_j := (c_j - c_j')N|a_j|^{-1}C_{a_3q}(a_jh)\phi(a_3q)^{-1}, \\ R_j := \phi(a_3q)^{-1}\displaystyle\sum_{\chi \pmod{a_3q}} C_{\overline{\chi}}(a_jh)\Phi_\chi(c_jN/|a_j|) + O(\log^2 N). \end{cases}$$

So the main term in (3.11) is

$$a_3^{-1}\sum_{q\le Q}\mu(q)\phi(q)^{-1}\sum_{\substack{h=1\\(h,q)=1}}^{a_3q} e_{a_3q}(-bh)\{M_1M_2+M_1R_2+M_2R_1+R_1R_2\}. \tag{3.14}$$

We shall see that M_1M_2 will contribute as the main term in (3.14) and M_1R_2, M_2R_1, R_1R_2 will form the error terms. Actually, we shall prove that

$$\text{the error terms in (3.14)} \ll \varepsilon N^2|a_1a_2a_3|^{-1}. \tag{3.15}$$

Hence in view of (3.14), (3.13) and (3.12) we can rewrite (3.11) as

$$I(N)=\frac{(c_1-c_1')(c_2-c_2')}{|a_1a_2a_3|}N^2\sum_{q\le Q}\frac{\mu(q)}{\phi(q)\phi(a_3q)^2}A(q)+O(\varepsilon N^2|a_1a_2a_3|^{-1}) \tag{3.16}$$

where

$$A(q):=\sum_{\substack{h=1\\(h,q)=1}}^{a_3q} e_{a_3q}(-bh)C_{a_3q}(a_1h)C_{a_3q}(a_2h). \tag{3.17}$$

We are now going to prove (3.15). Write $a_3q=t_1t_2$ where t_1,t_2 are positive integers satisfying $(t_1,q)=1$ and $p|q$ whenever $p|t_2$. So $(t_1,t_2)=1$ and $\phi(t_1)\le\phi(a_3)$ as $t_1|a_3$. If $(h,q)=1$ so that $(h,t_2)=1$ then

$$C_{a_3q}(a_jh)=C_{t_1}(a_jh)C_{t_2}(a_jh)=C_{t_1}(a_jh)C_{t_2}(a_j),$$

since $C_q(m)$ is multiplicative in q. Then by (3.13)

$$\sum_{\substack{h=1\\(h,q)=1}}^{a_3q}|M_j|^2\ll(N|C_{t_2}(a_j)||a_j|^{-1}\phi(a_3q)^{-1})^2\sum_{h=1}^{a_3q}|C_{t_1}(a_jh)|^2$$

and the above last sum over h is equal to

$$\sum_{\substack{l_1,l_2=1\\(t_1,l_j)=1}}^{t_1}\sum_{h=1}^{a_3q}e_{t_1}(a_jh(l_1-l_2))=a_3q\sum_{\substack{l_1,l_2=1\\(t_1,l_j)=1,t_1|a_j(l_1-l_2)}}^{t_1}1\le a_3q\phi(t_1)(a_j,t_1).$$

So

$$\begin{aligned}\left(\sum_{\substack{h=1\\(h,q)=1}}^{a_3q}|M_j|^2\right)^{\frac12} &\ll N(|a_j|\phi(q))^{-1}\{a_3q(a_j,t_1t_2)(a_j,t_2)\phi(a_3)^{-1}\}^{\frac12}\\ &\le N\phi(q)^{-1}(a_3q/\phi(a_3))^{\frac12}.\end{aligned} \tag{3.18}$$

Here we have used

$$|C_{t_2}(a_j)|\le(a_j,t_2)\text{ and }\phi(t_1)\le\phi(a_3).$$

On the other hand, by (3.13) and (2.4) with $Y = c_j N|a_j|^{-1}$ we have

$$\begin{aligned}\sum_{h=1}^{a_3q}|R_j|^2 &\ll \phi(a_3q)^{-2}\sum_{\chi,\chi' \pmod{a_3q}}\Phi_\chi(c_jN|a_j|^{-1})\times\\ &\quad\times\overline{\Phi}_{\chi'}(c_jN|a_j|^{-1})\sum_{h=1}^{a_3q}C_{\overline{\chi}}(a_jh)C_{\chi'}(-a_jh)+\sum_{h=1}^{a_3q}\log^4 N\\ &\ll N(|a_j|\phi(a_3q)^2)^{-1}\log^4 N\times \qquad (3.19)\\ &\quad\times\sum_{\chi,\chi' \pmod{a_3q}}\left|\sum_{h=1}^{a_3q}C_{\overline{\chi}}(a_jh)C_{\chi'}(-a_jh)\right|+a_3q\log^4 N.\end{aligned}$$

For any m with $(m, a_3q) = 1$ the above last sum over h satisfies

$$\sum_{h=1}^{a_3q}C_{\overline{\chi}}(a_jh)C_{\chi'}(-a_jh)=\sum_{h=1}^{a_3q}C_{\overline{\chi}}(a_jhm)C_{\chi'}(-a_jhm)=\chi\overline{\chi}'(m)\sum_{h=1}^{a_3q}C_{\overline{\chi}}(a_jh)C_{\chi'}(-a_jh)$$

since h and hm run through the same set modulo a_3q. If $\overline{\chi}\chi' \neq \chi_0 \pmod{a_3q}$ we can choose m such that $\overline{\chi}\chi'(m) \neq 1$ and $(m, a_3q) = 1$. So

$$\sum_{h=1}^{a_3q}C_{\overline{\chi}}(a_jh)C_{\chi'}(-a_jh)=0$$

if $\chi' \neq \chi \pmod{a_3q}$. Thus

$$\begin{aligned}\sum_{\chi,\chi' \pmod{a_3q}}\left|\sum_{h=1}^{a_3q}C_{\overline{\chi}}(a_jh)C_{\chi'}(-a_jh)\right| &= \sum_{\chi \pmod{a_3q}}\sum_{h=1}^{a_3q}|C_{\overline{\chi}}(a_jh)|^2\\ &= \sum_{h=1}^{a_3q}\sum_{l_1=1}^{a_3q}\sum_{l_2=1}^{a_3q}e_{a_3q}(a_jh(l_1-l_2))\sum_{\chi \pmod{a_3q}}\overline{\chi}(l_1)\chi(l_2)=a_3q\phi(a_3q)^2. \qquad (3.20)\end{aligned}$$

It follows from (3.19) and (3.20) that

$$\begin{aligned}\left(\sum_{\substack{h=1\\(h,q)=1}}^{a_3q}|R_j|^2\right)^{\frac12} &\ll \{Na_3q|a_j|^{-1}\}^{\frac12}\log^2 N+\{a_3q\log^4 N\}^{\frac12} \qquad (3.21)\\ &\ll \{Na_3q|a_j|^{-1}\}^{\frac12}\log^2 N.\end{aligned}$$

Using Schwarz's inequality, (3.18) and (3.21) we have

$$\begin{aligned}a_3^{-1}\sum_{q\le Q}\mu(q)\phi(q)^{-1}\sum_{\substack{h=1\\(h,q)=1}}^{a_3q}e_{a_3q}(-bh)M_1R_2 &\ll N^{\frac32}(\log^2 N)(\phi(a_3)|a_2|)^{-\frac12}\sum_{q\le Q}q\phi(q)^{-2}\\ &\ll N^{\frac32}|a_2a_3|^{-\frac12}\log^4 N \ll \varepsilon N^2|a_1a_2a_3|^{-1}.\end{aligned}$$

Here the last two inequalities follow from (3.1) and (3.2). This proves (3.15) for the error term arising from M_1R_2. The same arguments can be applied to prove (3.15) for the error terms corresponding to M_2R_1 and R_1R_2.

We come now to consider the main term in (3.16). Note that if $(a_3, q) > 1$ then by (3.17), $A(q) = 0$. Indeed if p is a prime dividing (a_3, q) then $p \nmid (a_1h, a_2h)$ since $(q, h) = 1$ and

$(a_1, a_2, a_3) = 1$. Let $p \nmid a_1 h$, say. Then by (2.1) $C_{a_3 q}(a_1 h) = 0$ since $a_3 q/(a_3 q, a_1 h)$ is not square-free. For q coprime with a_3, applying Lemma 2 with $s = a_3, m = q, u = -b, v = a_1, v' = a_2$, we have,

$$A(q) = Za_3(-b, a_1, a_2)C_q(-b)C_q(a_1)C_q(a_2) = a_3\mathcal{N}(a_3)C_q(-b)C_q(a_1)C_q(a_2),$$

by (2.12). Hence (3.16) can be written as

$$\begin{aligned} I(N) \quad &= \quad \frac{(c_1 - c_1')(c_2 - c_2')}{|a_1 a_2 a_3|} N^2 \frac{a_3 \mathcal{N}(a_3)}{\phi(a_3)^2} \sum_{\substack{q \le Q \\ (q, a_3)=1}} \mu(q)\phi(q)^{-3} C_q(-b)C_q(a_1)C_q(a_2) \\ &\qquad + O(\varepsilon N^2 |a_1 a_2 a_3|^{-1}). \end{aligned}$$

Let

$$F(q) := \mu(q)\phi(q)^{-3}C_q(-b)C_q(a_1)C_q(a_2).$$

In view of the lower bound in Lemma 3, our proof of $I(N) \gg N^2|a_1a_2a_3|^{-1}$ will be completed if we show that the above sum $\sum_{q \le Q, (q,a_3)=1} F(q) \ge c_4 > 0$. Clearly $F(q)$ is multiplicative and, by (2.1), $F(p) = (-1)^\lambda \phi(p)^{-3+\lambda}$ if p divides exactly λ members of $\{a_1, a_2, b\}$. Note that $0 \le \lambda \le 2$ and $\lambda = 0$ for all sufficiently large p. Therefore $\sum_p |F(p)| < \infty$ and $\sum_{(q,a_3)=1} F(q)$ converges to $\prod_{p \nmid a_3} (1 + F(p)) := c_5$. Note also that if $p = 2 \nmid a_3$ then, by (1.5), $\lambda = 0$ or 2 and whence $F(2) > 0$. So $c_5 \ge \prod_{3 \le p \nmid a_3} (1 - \phi(p)^{-2}) > 0$. Since $|C_q(k)| \le (k, q)$ and $(a_1, a_2, b) = 1$, we see easily that

$$\begin{aligned} |F(q)| \quad &\le \quad \phi(q)^{-3}(a_1a_2b, q)^2 \le (a_1a_2b, q)q^{-2}(\log\log 10|a_1a_2b|)^3 \\ &\ll \quad (a_1a_2b, q)q^{-2}(\log\log N)^3. \end{aligned}$$

Hence

$$\sum_{q>Q} |F(q)| \ll (\log\log N)^3 Q^{-1} d(a_1a_2b)$$

and $\sum_{q \le Q, (q,a_3)=1} F(q) \ge c_5/2$ when$Q$ is sufficiently large. This proves that

$$I(N) \gg N^2|a_1a_2a_3|^{-1}.$$

Now, clearly, by (3.4)

$$\begin{aligned} N^2|a_1a_2a_3|^{-1} \ll I(N) \quad &= \quad \sum_{\substack{N_j' < n_j \le N_j, j=1,2,3 \\ a_1n_1 + a_2n_2 + a_3n_3 = b}} \lambda(n_1)\lambda(n_2)\lambda(n_3) \\ &\le \quad (\log N)^3 \text{ card } \{p_1, p_2, p_3 : N_j' < p_j \le N_j, a_1p_1 + a_2p_2 + a_3p_3 = b\}. \end{aligned}$$

In view of (3.2), (3.9) and (3.10), this proves (1.8) and (1.9) and hence our Theorem.

References

[1] A. Baker, "On Some Diophantine Inequalities Involving Primes," *J. reine angew. Math.*, **228** (1967), 166-181.

[2] H. Davenport, *Multiplicative Number Theory*, 2nd edition, Graduate Text in Math., Vol 74, Springer-Verlag.

[3] G.H. Hardy and J.E. Littlewood, "Some Problems of "Partitio Numerorum": III, on the Expression of a Number as a Sum of Primes," *Acta Math* . **44** (1923), 1-70.

[4] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 5th edition, Oxford University Press (1979).

[5] M.C. Liu and K.M. Tsang, "Small Prime Solutions of Linear Equations," *Théorie des nombres*, (1989), 595-624, Walter de Gruyter, Berlin.

Department of Mathematics, University of Hong Kong, Pokfulam Road, Hong Kong