STAT 270 - Chapter 3
Probability

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Some definitions

- Experiment: Any action that produces data
- Sample space: The set of all possible outcomes of an experiment
  - Discrete, e.g.,
  - Continuous, e.g.,
- Event: A subset of the sample space

Examples: Flip a coin and roll a die,

Set Theory

- Venn diagram: A graphical tool to explain events
- "A union B" denoted by $A \cup B \equiv A$ or B:
- "A intersect B" denoted by $A \cap B \equiv AB \equiv A$ and B:
- A compliment denoted by $\bar{A}$ or A^c or A':
- "The empty set" denoted by $\emptyset$:
- "mutually exclusive" or "disjoint" sets:

de Morgan's laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Definitions of probability

Axiomatic definition (Kolmogrov 1993) P is a probability measure if:

- Axiom 1. For any event A, $P(A) \geq 0$
- Axiom 2. $P(S) = 1$
- Axiom 3. If $A_1, A_2, \ldots$ are mutually exclusive events then $P(\cup A_i) = \sum P(A_i)$ (countable additivity)

Useful properties of probabilities

can be proved from the above three axioms

- $P(\bar{A}) = 1 - P(A)$

- $P(a \cup B) = P(A) + P(B) - P(A \cap B)$

Example

Suppose that 55% of all adults regularly consume coffee, 45% regularly consume carbonated soda and 70% regularly consume at least one of these two products. What is the probability that a randomly selected adult

a. regularly consumes both coffee and soda?

b. doesn't regularly consume any of the products?

useful properties of probabilities

- $P(\emptyset) = 0$

- If $A \subseteq B$ then $P(A) \leq P(B)$

- $P(A \cup B \cup C) =$
  $P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

Symmetry definition of probability

Applicable when the experiment has **finite** number of **equally likely** outcomes

$$P(A) = \frac{\text{number of outcomes leading to } A}{\text{total number of outcomes}}$$

Example. Suppose that we flip two fair coins. What is the probability that we observe at least one head?

Criticisms to the symmetry definition

- definition is circular; **probability** is defined in terms of **equally likely** outcomes
- Restricts us to finite sample spaces while most interesting probability problems have infinite sample spaces

The definition is mostly useful for calculating probabilities in games of chance, e.g., dice, cards, coin, etc.

Frequency definition

Consider N identical trials

$$P(A) = \lim_{N \to \infty} \frac{\text{number of occurances of } A}{N}$$

Criticisms:

- Doesn't tell us how to calculate probabilities
- No mathematical reasons that the limit exists
- Does not allow interpretation

Conditional probability (important)

A and B are events, we are interested in the probability that A occures **given** that B has occured,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) \neq 0$.

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

Note that,

$$P(A) = P(A|S)$$

Example

A magazine publishes three columns Arts (A), Books (B), and Cinema (C). The probability that a randomly selected reader is interested in

- Arts is 0.14
- Books is 0.23
- Cinema is 0.37
- Arts and Books is 0.08
- Arts and Cinema is 0.09
- Books and Cinema is 0.13
- Arts, Books, and Cinema is 0.05

Calculate the probabilities that a randomly selected reader is interested in
a. Arts given that s/he is interested in Books
b. Cinema given that s/he is interested in at least one of Arts or Books
c. at least one of Cinema or Books given that s/he is interested in Arts
d. Books given that s/he is interested in at least one of the columns

The law of the total probability

Consider events A and $B_1, B_2, \ldots$ where B_is form a partition of S, i.e., they are disjoint and $S = \cup_{i=1}^{\infty} B_i$, then

$$P(A) = P(\cup_{i=1}^{\infty} AB_i) = \sum_{i=1}^{\infty} P(AB_i) = \sum_{i=1}^{\infty} P(A|B_i)P(B_i)$$

Independence

The events A and B are independent iff (if and only if)

$$P(A \cap B) = P(A)P(B)$$

meaning that the occurance of either of them does not affect the probability of the other.

Example

Toss afair coin and roll a die. Define,
A: the event that we observe a head on the coin
B: the event that we observe a 6 on the die
Are A and B independent?

Tip

probability of A **if** or **given that** $\ldots$ has occured $\equiv P(A|\ldots)$

Birthday problem

If there are n people in a room, what is the probability that at least two of them were born in the same day of the year?

Independence

Proposition. Suppose $P(B) \neq 0$, then A and B are independent iff $P(A|B) = P(A)$.
Proof.

Definition. $A_1, A_2, \ldots, A_k$ are mutually independent iff

$$P(A_1 \cap A_2 \cap \ldots \cap A_m) = P(A_1)P(A_2)\ldots P(A_m)$$

for $m = 2, 3, \ldots, k$.

Combinatorial rules

- The number of permutations (arrangements) of n distinct objects is $n! = n(n-1)\dots 1$, $0! = 1$.
  e.g.

- The number of permutations of r objects chosen from n distinct objects is $n^{(r)} = \frac{n!}{(n-r)!}$, "$n$ to r factors"
  e.g.

- The number of combinations of r objects chosen from n distinct objects (order is not important)
  (n choose r)
  $$\binom{n}{r} = \frac{n^{(r)}}{r!} = \frac{n!}{(n-r)!r!}$$
  e.g.

Combinatorial rules

Proposition. The number of ways of partitioning n distinct objects into two groups of sizes r and $n-r$ is $\binom{n}{r}$

Corollary.

$$\binom{n}{r} = \binom{n}{n-r}$$

Proposition The number of ways of partitioning n distinct objects into k groups of sizes $n_1, \ldots, n_k$ where $n = n_1 + \ldots + n_k$ is

$$\frac{n!}{n_1! \ldots n_k!}$$

proof:

Example

Example 3.10. (text) 100 students, 20 female. What is the probability that in a randomly drawn sample of size five at least two are female?

Example

An academic faculty with 5 faculty members narrowed its choice for department head to either candidate A or B. Each member then voted on a slip of paper. Suppose there are 3 votes for A and 2 for B. If the slips are selected for tallying in random order, what is the probability that A remains ahead of B?

Example

Example 3.12. (text) Roll a die; if 6 is obtained draw a ball from box A containing 3 white balls and 2 black balls, if $1, \ldots, 5$ is obtained draw a ball from box B containing 2 white balls and 4 black balls.
a. What is the probability of obtaining a white ball?
b. If the chosen ball is white what is the probability that it is chosen from box A?

Playing cards

Some information,
Ordinary deck:

- 52 cards
- 13 denominations: ace, 2, 3, . . . , 10, jack, queen, king
- 4 suits: diamond $\diamondsuit$, heart $\heartsuit$, club $\clubsuit$, spade $\spadesuit$

A little poker lesson:

Example

Example 3.13. (text) In a hand of five dealt from an ordinary deck what is the probability of

a. three of a kind?

b. two pair?

c. a straight flush?

Example

Example 3.14. (text) 649 lottery: 6 balls are chosen from an urn containing 49 balls numbered from 1-49. Particioants buy tickets and select 6 different numbers between 1 and 49. Calculate the probability of,
a. winning the jackpot (all numbers match)
b. two of the numbers match.

Example

Example 3.17 (text) 3 marbles are drawn from a bag containing 4 red marbles and 6 black marbles. What is the probability that all three marbles are red when they are drawn
a. with eplacement?
b. without replacement?

repeat the example with the bag containing 40 red marbles and 60 black marbles.