STAT 270
Sample Final

Name:

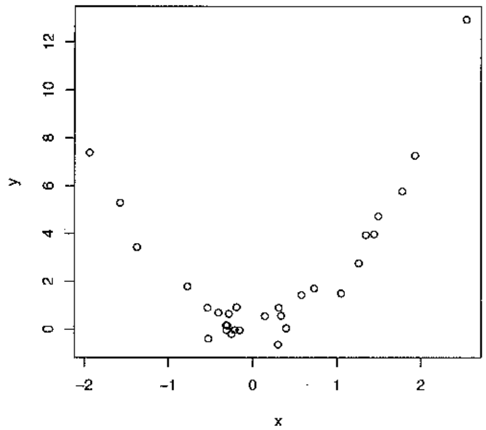
Student ID:

Instructions:

- There are 8 pages in this exam booklet, including cover page.
- There are 7 questions. Please read all questions carefully and write the answers in the space provided for each question.
- Define all variables/events used in your solutions
- The exam time is 3 hours.
- You are allowed to have a double-sided cheat sheet and a non-programmable calculator.

Good luck!

1. Consider the following scatter plot to study the relationship between two variables X and Y. The correlation coefficient is also calculated: $r = 0.41$. Comment on the relationship between X and Y.



A quadratic relationship between X and Y can be understood from the Scatterplot. There is a decreasing relationship for $X < 0$ and an increasing relationship for $X > 0$.

$r = .41$ Shows the degree of linear relationship between X and Y which suggests that there is a moderate positive correlation between the two variables.

2. The Smiths have two children . At least one of the children is a boy.

(a) What is the probability that both children are boys?

(b) What is the probability that one of the children is a girl?

A_1: the event that the first child is a boy

A_2: the event that the second child is a boy

$A_1 \cap A_2$: ~ ~ ~ both children are boys

$A_1 \cup A_2$: ~ ~ ~ at least one of the children is a boy.

(a) $$P(A_1 \cap A_2 \mid A_1 \cup A_2) = \frac{P((A_1 \cap A_2) \cap (A_1 \cup A_2))}{P(A_1 \cup A_2)} = \frac{P(A_1 \cap A_2)}{P(A_1 \cup A_2)}$$

$$= \frac{1/4}{3/4} = 1/3$$

independence

$$P(A_1 \cap A_2) \overset{\uparrow}{=} P(A_1)P(A_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$

(b) $$P((A_1^c \cap A_2) \cup (A_1 \cap A_2^c) \mid A_1 \cup A_2) = \frac{P(A_1^c \cap A_2)}{P(A_1 \cup A_2)} + \frac{P(A_1 \cap A_2^c)}{P(A_1 \cup A_2)}$$

$$= \frac{1/4}{3/4} + \frac{1/4}{3/4} = 2/3$$

or $\quad 1 - P(\text{part (a)}) = 1 - 1/3 = 2/3$

$\Downarrow$

because when at least one of the children is a boy then either they are both boys or one is a girl.

3. Let X have the following probability density function. Derive the expectation and variance of X.

$$f(x) = \begin{cases} \frac{x}{\beta^2} \exp -\frac{x}{\beta} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \rightsquigarrow \quad \text{Gamma}(2, \beta)$$

$$E(X) = \int x f(x)\, dx$$

$$= \int_0^\infty x \frac{x}{\beta^2} e^{-x/\beta}\, dx$$

$$= \Gamma(3)\beta \underbrace{\int_0^\infty \frac{x^2}{\Gamma(3)\beta^3} e^{-x/\beta}\, dx}_{\text{pdf of Gamma}(3,\beta)} = 1$$

$$= \Gamma(3)\beta$$

$$= 2\beta$$

4. Government regulations indicate that the total weight of cargo in a certain kind of airplane cannot exceed 330 kg. On a particular day a plane is loaded with 100 boxes of a particular item only. Historically, the weight distribution for the individual boxes of this variety has a mean 3.2 kg and standard deviation 0.4 kg.

   (a) what is the distribution of the sample mean (give the pdf and appropriate parameter values)?

   (b) what is the probability that the observed sample mean is larger than 3.5 kg?

   (c) what is the probability that the government regulation is met?

$X_1, \cdots, X_{100}$ iid with $E(X_i) = 3.2$ $var(X_i) = (.4)^2$

(a) By the Central limit theorem,

$$\bar{X} \sim N\left(3.2, \frac{.4}{10}\right) \text{ (approximately).}$$

$$\Rightarrow f_{\bar{X}}(x) = \frac{1}{\sqrt{2\pi}(.04)} \exp\left[-\frac{1}{2}\left(\frac{x-3.2}{.04}\right)^2\right]$$

(b) $P(\bar{X} > 3.5) = P\left(Z > \frac{3.5 - 3.2}{.04}\right) = P(Z > 7.5) \approx 0$

(c) $P(100\bar{X} \leq 330) = P(\bar{X} \leq 3.3)$

$$= P\left(Z \leq \frac{3.3 - 3.2}{.04}\right)$$

$$= P(Z \leq 2.5) = .9937$$

5. Let X and Y have the joint probability density function

$$f(x,y) = \begin{cases} cxy & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Obtain c.

(b) Let $Z = XY$. Calculate $E(Z)$.

(a) $\iint f(x,y)\, dx\, dy = 1$

$\Rightarrow \int_0^1 \int_0^y cxy\, dx\, dy = 1$

$\Rightarrow c \int_0^1 \left(\frac{x^2}{2} \Big|_0^y \right) y\, dy = 1$

$\Rightarrow \frac{c}{8} \left(y^4 \Big|_0^1 \right) = 1 \Rightarrow c = 8$

(b) $E(Z) = \iint xy\, f(x,y)\, dx\, dy$

$= \int_0^1 \int_0^y (xy)(8xy)\, dx\, dy$

$= 8 \int_0^1 \left(\frac{x^3}{3} \Big|_0^y \right) y^2\, dy$

$= \frac{8}{18} \left(y^6 \Big|_0^1 \right) = \frac{4}{9}$

6. IQ test scores of seventh grade girls in the western United States follow a normal distribution with a satndard deviation of 15 IQ points. A random sample of 10 seventh grade girls from a school district in western United States is taken and eac takes an IQ test. The sample mean IQ was found to be 102.

   (a) Construct a 95% confidence interval fro the average IQ test scores and interprete it.

   use $\hat{\mu} = 102$ as an estimate for μ. ← (b) If a second sample of size 5 is taken, what is the probability that the new sample mean is within the interval obtained in part (b)?

   (c) If 10 samples of size 5 are taken, what is the probability that at least one of the sample means is included in the interval obtained in part (b)?

$$X_1, \ldots, X_{10} \sim \text{Normal}(\mu, 15^2) \Rightarrow \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

(a) 95% CI for μ,

$$\bar{x} \pm z_{\alpha/2} \frac{15}{\sqrt{10}}$$

$$102 \pm 1.96 \frac{15}{\sqrt{10}} \Rightarrow (92.7, 111.3)$$

If we repeat sampling many times and construct a large number of CI's of the form given above we expect 95% of these CI's cover the true value of μ.

$$\text{(b)}\ p_0 = P(92.7 < \bar{X} < 111.3) \overset{\hat{\mu}=102}{=} P\left(\frac{92.7 - 102}{15/\sqrt{5}} < z < \frac{111.3 - 102}{15/\sqrt{5}}\right)$$

$$= P(-1.39 < z < 1.39)$$

$$= .9177 - (1 - .9177)$$

$$= .8354$$

(c) Y : # sample means falling into the given interval.

$Y \sim \text{binomial}(10, p_0)$ ⇝ $P(Y \geq 1) = 1 - P(Y = 0) = 1 - (1-p_0) = p_0$

$= .8354$

7. It is hypothesized that the proportion of red candies in small M&M's bags are equal. The number of red candies are counted fro two small M&M's bags. There are 31 candies in the first bag 7 of which are red and 32 candies in the second bag 10 of which are red. Test the above hypothesis at $\alpha = .05$.

$$\begin{cases} H_0: P_1 = P_2 \\ H_1: P_1 \neq P_2 \end{cases} \Rightarrow \begin{cases} H_0: P_1 - P_2 = 0 \\ H_1: P_1 - P_2 \neq 0 \end{cases} \qquad \begin{aligned} \hat{P}_1 &= \frac{7}{31} = .2258 \quad m = 31 \\ \hat{P}_2 &= \frac{10}{32} = .3125 \quad n = 32 \end{aligned}$$

Assuming $mP_1 > 5$, $m(1-P_1) > 5$ and $nP_2 > 5$, $n(1-P_2) > 5$,

we can conclude that,

$$\frac{\hat{P}_1 - \hat{P}_2 - (P_1 - P_2)}{\sqrt{\frac{P_1(1-P_1)}{m} + \frac{P_2(1-P_2)}{n}}} \sim N(0,1)$$

under H_0: $P_1 = P_2 = P \Rightarrow$ estimate p with $\tilde{p} = \frac{7+10}{31+32} = .27$

$$\text{P-value} = 2P(\hat{P}_1 - \hat{P}_2 \leq .2258 - .3125 \mid P_1 - P_2 = 0)$$

$$= 2P\left(Z \leq \frac{-.0867 - 0}{\sqrt{(.27)(.73)(\frac{1}{31} + \frac{1}{32})}}\right)$$

$$= 2P(Z \leq -.77)$$

$$= 2(1 - .7794) = .4412 > .05$$

There is not enough evidence to reject the hypothesis of equal proportions of red candies in two bags.