Lotteries over Money and Risk
Economics 302 - Microeconomic Theory II: Strategic Behavior

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(with thanks to Anke Kessler)
Most Important Things to Learn

1. Computing the expected value, expected utility and certainty equivalent of a lottery over wealth.

2. The shape of the utility function for risk-averse, risk-neutral and risk-loving agents.

3. Computing the coefficient of absolute risk aversion (CARA) and the coefficient of relative risk aversion (CRRA).

4. Understand insurance.
Recall that an agent’s expected utility from a lottery $L$ with probability $p_1$ of outcome 1 yielding utility $u_1$, $p_2$ of outcome 2 yielding $u_2$, ..., $p_n$ of outcome $n$ yielding $u_n$ is:

$$p_1 u_1 + p_2 u_2 + ... + p_n u_n$$

From expected utility theory, we know that under some assumptions, it is possible to assign $u_i$ so that the above function represents the agent’s preferences.

But we didn’t say anything about how the $u_i$’s are assigned!

Now: look at lotteries over wealth, and make assumptions on the utility.

We will then return to asymmetric information and use what we learned to study moral hazard.
Definitions (I)

- Suppose you have utility $u(\cdot)$ over wealth and a lottery $L = (p_1, p_2, \ldots, p_n)$, where outcome $i$ is wealth $w_i$, for each $i = 1, \ldots, n$.

- The **expected value (EV)** of $L$ is:

  $$E[L] = \sum_{i=1}^{n} p_i w_i$$

- (Sometimes, we are instead interested in the expected **change** in wealth (relative to your original wealth) under a lottery, which may be a bet/project/lottery ticket/etc. I will refer to that as the expected value of the bet/project/lottery ticket/etc. But the expected value of a lottery will refer to the expected **total** wealth, as above.)

- Your **expected utility (EU)** from $L$ is:

  $$E[u(L)] = p_1 u(w_1) + p_2 u(w_2) + \ldots + p_n u(w_n)$$
Definitions (II)

- Your **certainty equivalent (CE) for** $L$ is:
  \[ w \text{ such that } E[u(L)] = u(w) \]
- You’re indifferent between $L$ and having the CE for sure.
- Your **risk premium for** $L$ is:
  \[ E[L] - CE(L) \]
- Exercise: You have utility over wealth $u(w) = \sqrt{w}$ and start with $50. Consider a bet with probability 0.5 of winning $14, and probability 0.5 of losing $14. Define the lottery associated with taking the bet. Find the EV of the lottery and of the bet. Find your EU, CE and risk premium for the lottery. Would you take the bet?
Attitudes toward Risk

- An individual is **risk-averse** if $CE(L) < E[L]$ (i.e. $E[u(L)] < u(E[L])$).
- An individual is **risk-neutral** if $CE(L) = E[L]$.
- An individual is **risk-loving** if $CE(L) > E[L]$.
- Sometimes, a lottery $L$ is given, and you can define risk attitude with respect to $L$.
- If a specific lottery is not given, then the definitions above apply to all lotteries. For example, an individual is risk-averse if for any lottery $L$ (where no outcome occurs with certainty), $CE(L) < E[L]$. 
Remember the following math definition: a function \( u(.) \) is (strictly) concave if \( u(tx + (1 - t)y) > tu(x) + (1 - t)u(y) \) for all \( t \in (0, 1) \) and \( x \neq y \).

If you view \( x \) and \( y \) as outcomes of a lottery \( L = (t, 1 - t) \) over wealth, this just says \( u(E[L]) > E[u(L)] \).

In fact, you can show that this implication also holds for \( L \)’s with more than two outcomes.

Therefore, risk aversion \( \Leftrightarrow u(.) \) is strictly concave.

If \( u(.) \) is twice-differentiable, which we will usually assume, then \( u'' < 0 \) almost everywhere.

Similarly, risk-neutral agents have linear \( u(.) \), and risk-loving agents have strictly convex \( u(.) \).
Many economic models assume that people are risk-averse. It is useful to quantify how risk-averse they are.

The coefficient of absolute risk-aversion (CARA) at wealth $w$ is defined as:

$$u''(w) \quad \text{denominator}\quad u'(w)$$

The $u'(w)$ in the denominator is for normalization: $ku(.)$ represents the same preferences for any $k > 0$, so CARA shouldn’t change with $k$.

Exercise: show that $u(w) = -e^{-aw}$ has constant CARA for any $a > 0$, and find the CARA.
Constant CARA implies that whether you’d take a bet doesn’t depend on your initial wealth.

Not very realistic! CARA should probably decrease as wealth increases.

The coefficient of relative risk-aversion (CRRA) at wealth $w$ is defined as:

$$\frac{-u''(w)}{u'(w)}$$

Exercise: show that $u(w) = w^\sigma$ has constant CRRA for any $\sigma \in (0, 1)$, and find the CRRA. Do the same for $u(w) = -w^\sigma$ for $\sigma < 0$, and $u(w) = \ln(w)$.

Constant-CRRA utility is widely used in financial economics.
Al has initial wealth $w_0$ and has utility over wealth $u(.)$. He is risk-averse, so $u(.)$ is strictly concave.

Accident happens with probability $p > 0$, and causes loss $L$.

Let $\pi$ denote the price of the insurance policy, called a premium. It must be paid regardless of whether the loss occurs.

Let $C$ be the policy’s payout when a loss occurs.

What is Al’s expected utility without the policy? With the policy?
Assume that the insurance company is risk-neutral. (Why does this assumption make sense?)

Let's figure out the Pareto efficient insurance policy.

Suppose Al carries risk (e.g. he is uninsured or only partially insured). How does his certainty equivalent compare to his expected wealth?

Can you think of a Pareto improvement involving Al and the insurance company?

When a risk-neutral and a risk-averse agent share risk, the risk-neutral agent should bear all of it, absent other considerations.
Choosing Quantity of Insurance

- Suppose that when a loss occurs, the insurance policy pays $sL$. (So $s = 0$ means no insurance, and $s = 1$ means full insurance.)

- Suppose $\pi = sK$: it would cost Al $K$ to full insure himself, but he can also choose to insure himself partially ($s < 1$) or even overinsure ($s > 1$).

- How much insurance ($s$) does Al choose?


- When $K = pL$, we say that the premium is actuarially fair: it is equal to the expected cost to the insurance firm.

- Would you expect (almost) actuarially fair premia in a competitive insurance industry? But are insurance industries likely to be perfectly competitive?